Math 542: Homework 8

1. Use the Riemann-Hurwitz Theorem to show that $\mathbf{H}^2/\Delta(2,3,7)$ is the minimal volume orientable hyperbolic 2-orbifold.
2. Complete the proof that the minimal volume arithmetic hyperbolic 3-orbifold Q covering an orbifold of the form $\mathbf{H}^3/\Gamma^1_{\mathcal{O}}$ is the orbifold from class (arising from the quaternion algebra B/k where $k = \mathbf{Q}(\sqrt{3-2\sqrt{5}})$ and B is unramified at all finite places). You need to rule out fields of degree 2, 3 and 4. The small discriminants are: in degree 4, -275, -283, and in degree 3, -23, -31, -44.
3. Compute the volume of $\mathbf{H}^3/\mathrm{PSL}(2,\mathrm{O}_2)$.
4. Let $k = \mathbf{Q}(\theta)$ where $\theta = \sqrt{(1-\sqrt{5})/2}$.

(a) Prove that $\{1, \theta, \theta^2, \theta^3\}$ is an integral basis for R_k.

(b) Prove that the class number of k is 1.

(c) Prove that $[R^*_{k,+} : R^{*2}_k] = 2$ be finding units with the correct sign at the real embeddings as done in class (you can assume that $[R^*_k : R^{*2}_k] = 8$).

(d) Compute the minimal volume of an orbifold in the commensurability class defined by B/k which is unramified at all finite places.

5. Let $k = \mathbf{Q}(\sqrt{-7})$.

(a) Prove that the ideal $2R_k$ splits as a product of two prime ideals of norm 2. Denote these by $\mathcal{P}_1$ and $\mathcal{P}_2$.

(b) Let B/k be the quaternion algebra ramified at $\mathcal{P}_1$ and $\mathcal{P}_2$, and $\mathcal{O}$ a maximal order. Compute the volume of $\mathbf{H}^3/\Gamma^1_{\mathcal{O}}$.

6. Let p be an odd prime, and B_p be the quaternion algebra over $\mathbf{Q}$ ramified at exactly 2 and p. Let $\mathcal{O}_p$ be a maximal order of B_p.

(a) Compute the volume of $\mathbf{H}^2/\Gamma^1_{\mathcal{O}_p}$.

(b) Given that for $p = 3$, $\Gamma^1_{\mathcal{O}_3}$ has signature $(g_3; 2, 2, 3, 3; 0)$. Compute g_3 using the Riemann-Hurwitz Theorem.