ALL PRINCIPAL CONGRUENCE LINK GROUPS

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Abstract. In this paper we give a complete enumeration of all the principal congruence link complements in S^3, thereby answering a question of W. Thurston.

1. Introduction

Let d be a square-free positive integer, let O_d denote the ring of integers in $\mathbb{Q}(\sqrt{-d})$, and let Q_d denote the Bianchi orbifold $\mathbb{H}^3/\mathrm{PSL}(2, O_d)$.

As is well-known Q_d is a finite volume hyperbolic orbifold with h_d cusps, where h_d is the class number of $\mathbb{Q}(\sqrt{-d})$ (see [MR03] Chapters 8 and 9 for example). A non-compact finite volume hyperbolic 3-manifold X is called *arithmetic* if X and Q_d are commensurable, that is to say they share a common finite sheeted cover (see [MR03] Chapter 8 for more on this).

An important class of arithmetic 3-manifolds consists of the *congruence* manifolds. Recall that a subgroup $\Gamma < \mathrm{PSL}(2, O_d)$ is called a *congruence subgroup* if there exists an ideal $I \subset O_d$ so that Γ contains the *principal congruence group*:

$$\Gamma(I) = \ker\{\mathrm{PSL}(2, O_d) \to \mathrm{PSL}(2, O_d/I)\},$$

where $\mathrm{PSL}(2, O_d/I) = \mathrm{SL}(2, O_d/I)/\{\pm \mathrm{Id}\}$. The largest ideal I for which $\Gamma(I) < \Gamma$ is called the *level* of Γ. A manifold $M = \mathbb{H}^3/\Gamma$ is called *congruence* (resp. *principal congruence*) if $\Gamma > \Gamma(I)$ (resp. $\Gamma = \Gamma(I)$) for some ideal I.

In an email to the first and third authors in 2009, W. Thurston asked the following question about principal congruence link complements:

"Although there are infinitely many arithmetic link complements, there are only finitely many that come from principal congruence subgroups. Some of the examples known seem to be among the most general (given their volume) for producing lots of exceptional manifolds by Dehn filling, so I'm curious about the complete list."

In this paper, we give a complete enumeration of all the principal congruence link complements in S^3, together with their levels. Our main result is the following:

Theorem 1.1. *The following list of* 48 *pairs* (d, I) *describes all principal congruence subgroups* $\Gamma(I) < \mathrm{PSL}(2, O_d)$ *such that* $\mathbb{H}^3/\Gamma(I)$ *is a link complement in* S^3:

(1) $d = 1$**:** $I = \langle\ 2\ \rangle$, $\langle\ 2 \pm i\ \rangle$, $\langle\ (1 \pm i)^3\ \rangle$, $\langle\ 3\ \rangle$, $\langle\ 3 \pm i\ \rangle$, $\langle\ 3 \pm 2i\ \rangle$, $\langle\ 4 \pm i\ \rangle$.
(2) $d = 2$**:** $I = \langle\ 1 \pm \sqrt{-2}\ \rangle$, $\langle\ 2\ \rangle$, $\langle\ 2 \pm \sqrt{-2}\ \rangle$, $\langle\ 1 \pm 2\sqrt{-2}\ \rangle$, $\langle\ 3 \pm \sqrt{-2}\ \rangle$.
(3) $d = 3$**:** $I = \langle\ 2\ \rangle$, $\langle\ 3\ \rangle$, $\langle\ (5 \pm \sqrt{-3})/2\ \rangle$, $\langle\ 3 \pm \sqrt{-3}\ \rangle$, $\langle\ (7 \pm \sqrt{-3})/2\ \rangle$, $\langle\ 4 \pm \sqrt{-3}\ \rangle$, $\langle\ (9 \pm \sqrt{-3})/2\ \rangle$.
(4) $d = 5$**:** $I = \langle\ 3, (1 \pm \sqrt{-5})\ \rangle$.
(5) $d = 7$**:** $I = \langle\ (1 \pm \sqrt{-7})/2\ \rangle$, $\langle\ 2\ \rangle$, $\langle\ (3 \pm \sqrt{-7})/2\ \rangle$, $\langle\ \pm\sqrt{-7}\ \rangle$, $\langle\ 1 \pm \sqrt{-7}\ \rangle$, $\langle\ (-5 \pm \sqrt{-7})/2\ \rangle$, $\langle\ 2 \pm \sqrt{-7}\ \rangle$, $\langle\ (7 \pm \sqrt{-7})/2\ \rangle$, $\langle\ (1 \pm 3\sqrt{-7})/2\ \rangle$.

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(6) $d = 11$: $I = \langle\ 2, (1 \pm \sqrt{-11})/2\ \rangle$, $\langle\ (3 \pm \sqrt{-11})/2\ \rangle$, $\langle\ (5 \pm \sqrt{-11})/2\ \rangle$.
(7) $d = 15$: $I = \langle\ 2, (1 \pm \sqrt{-15})/2\ \rangle$, $\langle\ 3, (3 \pm \sqrt{-15})/2\ \rangle$, $\langle\ 4, (1 \pm \sqrt{-15})/2\ \rangle$, $\langle\ 5, (5 \pm \sqrt{-15})/2\ \rangle$, $\langle\ 6, (-3 \pm \sqrt{-15})/2\ \rangle$.
(8) $d = 19$: $I = \langle\ (1 \pm \sqrt{-19})/2\ \rangle$.
(9) $d = 23$: $I = \langle\ 2, (1 \pm \sqrt{-23})/2\ \rangle$, $\langle\ 3, (1 \pm \sqrt{-23})/2\ \rangle$, $\langle\ 4, (-3 \pm \sqrt{-23})/2\ \rangle$.
(10) $d = 31$: $I = \langle\ 2, (1 \pm \sqrt{-31})/2\ \rangle$, $\langle\ 4, (1 \pm \sqrt{-31})/2\ \rangle$, $\langle\ 5, (3 \pm \sqrt{-31})/2\ \rangle$.
(11) $d = 47$: $I = \langle\ 2, (1 \pm \sqrt{-47})/2\ \rangle$, $\langle\ 3, (1 \pm \sqrt{-47})/2\ \rangle$, $\langle\ 4, (1 \pm \sqrt{-47})/2\ \rangle$.
(12) $d = 71$: $I = \langle\ 2, (1 \pm \sqrt{-71})/2\ \rangle$.

As we describe in §2, using previous work of the authors, the proof of Theorem 1.1 is reduced to the following theorem:

Theorem 1.2. *When $d \in \{2, 7, 11\}$ the following list of pairs (d, I) determine principal congruence subgroups $\Gamma(I) < \mathrm{PSL}(2, O_d)$ such that $\mathbb{H}^3/\Gamma(I)$ is a link complement in S^3:*

(1) $d = 2$: $I = \langle\ 1 \pm 2\sqrt{-2}\ \rangle$, $\langle\ 3 \pm \sqrt{-2}\ \rangle$.
(2) $d = 7$: $I = \langle\ \pm\sqrt{-7}\ \rangle$, $\langle\ (-5 \pm \sqrt{-7})/2\ \rangle$, $\langle\ 2 \pm \sqrt{-7}\ \rangle$, $\langle\ (7 \pm \sqrt{-7})/2\ \rangle$, $\langle\ (1 \pm 3\sqrt{-7})/2\ \rangle$.
(3) $d = 11$: $I = \langle\ (5 \pm \sqrt{-11})/2\ \rangle$.

Furthermore $\Gamma(\langle\ 1 + 3\sqrt{-2}\ \rangle)$ is not a link group.

For all d with $h_d = 1$, Appendix A conveniently summarizes in diagrammatic form those $x \in O_d$ for which $\Gamma(\langle\ x\ \rangle)$ is a link group (note that Theorem 1.1 shows that when $h_d > 1$, there is no link group associated to a pair (d, I) where I is a principal ideal). In addition, each diagram is annotated with the reasons why the remaining values of x with $|x| < 6$ do not yield link groups. Appendix B provides some "new" examples of links whose complements are principal congruence manifolds.

We finish the Introduction with some commentary. Rather than a collaboration, this paper is the conclusion of overlapping efforts of the first and third authors and independently the second author. It was suggested to the authors by Ian Agol that since Theorem 1.1 was proved almost simultaneously, that a single paper should be written describing the solution. As such, we assume some familiarity with the methods of [BR14] and [BR17] on the one hand, and [Goe11] and [Goe15] on the other. The main goal of this paper is to describe a combination of the tools used in finishing off the most stubborn cases that remained from previous work [BR14], [BR17], [Goe11] and [Goe15] (see Theorem 1.2 and §2). We refer the reader to [BR18] for background, history and connections with other questions regarding the topology of congruence link complements.

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2. The remaining cases

We begin by recapping some of the previous, separate work of the authors that reduce to the short list of 9 cases of (d, I) shown in Theorem 1.2 which need to be either shown to be a link group or not. To that end, in the case of $h_d > 1$, the complete list of the 16 pairs (d, I) corresponding to principal congruence link complements was determined in [BR17]. The possible values of d are $d \in \{5, 15, 23, 31, 47, 71\}$ with the levels shown in Theorem 1.1.

Concerning the case when $h_d = 1$, the search for possible levels of principal congruence link groups is aided by the following from [BR14] and [Goe11]: *if* $I = \langle\, x \,\rangle \subset O_d$ *and* $\Gamma(I)$ *is a link group then* $|x| < 6$. Using this, in [BR14] we gave 9 new examples of principal congruence link groups, bringing the total known to 18:

(1) $d = 1$**:** $I = \langle\, 2 \,\rangle, \langle\, 2 \pm i \,\rangle, \langle\, (1 \pm i)^3 \,\rangle, \langle\, 3 \,\rangle$.
(2) $d = 2$**:** $I = \langle\, 1 \pm \sqrt{-2} \,\rangle, \langle\, 2 \,\rangle, \langle\, 2 \pm 2\sqrt{-2} \,\rangle$.
(3) $d = 3$**:** $I = \langle\, 2 \,\rangle, \langle\, 3 \,\rangle, \langle\, (5 \pm \sqrt{-3})/2 \,\rangle, \langle\, 3 \pm \sqrt{-3} \,\rangle$.
(4) $d = 7$**:** $I = \langle\, (1 \pm \sqrt{-7})/2 \,\rangle, \langle\, 2 \,\rangle, \langle\, (3 \pm \sqrt{-7})/2 \,\rangle, \langle\, 1 + \pm\sqrt{-7} \,\rangle$.
(5) $d = 11$**:** $I = \langle\, (1 \pm \sqrt{-11})/2 \,\rangle, \langle\, (3 \pm \sqrt{-11})/2 \,\rangle$.
(6) $d = 19$**:** $I = \langle\, (1 \pm \sqrt{-19})/2 \,\rangle$.

Moreover, in the cases $d = 1, 3$, as well as identifying the cases described above, in [Goe15] the second author determines the complete list of pairs (d, I) that yield link groups; namely those above, together with:

(1) $d = 1$**:** $I = \langle\, 3 \pm i \,\rangle, \langle\, 3 \pm 2i \,\rangle, \langle\, 4 \pm i \,\rangle$.
(2) $d = 3$**:** $I = \langle\, (7 \pm \sqrt{-3})/2 \,\rangle, \langle\, 4 \pm \sqrt{-3} \,\rangle, \langle\, (9 \pm \sqrt{-3})/2 \,\rangle$.

The upshot of these works is that 40 pairs (d, I) were determined that yield principal congruence link groups, and using a combination of techniques (including use of Magma [BCP97] and the comment above on the norm of a generator of the principal ideal), all the remaining cases were eliminated except for the 8 that correspond to the remaining levels in Theorem 1.2 that will be shown to be link groups, together with the group $\Gamma(\langle\, 1 + 3\sqrt{-2} \,\rangle)$ which will be shown not to be a link group.

In Table 1 we provide some additional information associated to the 8 cases to be shown to be link groups that will be helpful in what follows: in the second, third, and fourth columns of Table 1, we list x a generator of the ideal being considered, its norm N and the order O of $\mathrm{PSL}(2, O_d)/\Gamma(I)$.

Table 1. The 8 cases in which we still need to prove that $\Gamma(\langle\, x \,\rangle)$ is a link group.

d	x	$N(\langle\, x \,\rangle)$	$\lvert\mathrm{PSL}(2, O_d/\langle\, x \,\rangle)\rvert$	Number of cusps
2	$1 + 2\sqrt{-2}$	9	324	36
2	$3 + \sqrt{-2}$	11	660	60
7	$\sqrt{-7}$	7	168	24
7	$(5 + \sqrt{-7})/2$	8	192	24
7	$2 + \sqrt{-7}$	11	660	60
7	$(7 + \sqrt{-7})/2$	14	1008	72
7	$(1 + 3\sqrt{-7})/2$	16	1536	96
11	$(5 + \sqrt{-11})/2$	9	324	36

Note that if $I = \langle\, x \,\rangle \subset O_d$ is a principal ideal for which $\Gamma(I)$ is a link group, and $\overline{I} = \langle\, \overline{x} \,\rangle$ the complex conjugate ideal, then $\Gamma(\overline{I})$ is also a link group — since complex conjugation induces an orientation-reversing involution of $\mathbb{H}^3/\Gamma(I)$. Hence it suffices to consider only one of the ideals as a candidate level for a link group.

In §3 we describe how these 9 cases are handled.

3. Proof of Theorem 1.1

In the subsections below we indicate how the cases in Theorem 1.2 are handled. We do this by illustrating various examples of how the methods of the first and third author can be used to handle some cases, and then how the methods of the second author handle some other cases. We emphasize that both methods identify all remaining cases of link groups, but we only choose to include a sample of each. The case that is not a link group is handled in §3.3.

3.1. **The methods of** [BR14] **and** [BR17]**.** We recall the following. Let $\Gamma \leq \mathrm{PSL}(2, O_d)$ be a finite index subgroup. From Theorem 1.2 we can assume that $d \neq 1, 3$, then

- A *cusp*, $[c]$, of Γ is a Γ-orbit of points in $\mathbb{P}^1(\mathbb{Q}(\sqrt{-d}))$
- A *peripheral subgroup* of Γ for $[c]$ is a maximal parabolic subgroup, $P_x < \Gamma$, fixing $x \in [c]$. Note that if $y \in [c]$, then P_x and P_y are conjugate; hence a peripheral subgroup for $[c]$ is determined up to conjugacy.
- A *set of peripheral subgroups for* Γ is the choice of one peripheral subgroup for each cusp of Γ.

We will use the term *cusp* to mean $[c]$, a choice of point x in $[c]$, as well as the end of $\mathbb{H}^3/\Gamma$ corresponding to $[c]$. Which one is meant should be clear from the context. Note that since $d \neq 1, 3$, each peripheral subgroup is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

To prove Theorem 1.2, we use the methods of [BR14] and [BR17], and in particular the following useful property: $\mathbb{H}^3/\Gamma(I) \cong S^3 \setminus L$ if and only if $\Gamma(I)$ can be trivialized by setting one parabolic element from each cusp of $\Gamma(I)$ equal to 1. As in [BR14] and [BR17], Magma [BCP97] can be used to check this.

We now provide two illustrative examples on how the methods of [BR14] and [BR17] are used to show that the groups $(2, \langle\ 1 + 2\sqrt{-2}\ \rangle)$ and $(7, \langle\ (1 + 3\sqrt{-7})/2\ \rangle)$ are respectively a 36 component link group and a 96 component link group. The remaining 6 groups are handled similarly. This is done as per the above property by finding a set of parabolic elements, one for each cusp of $\Gamma(I)$, such that trivializing these elements trivializes $\Gamma(I)$.

One can calculate the cusps of $\Gamma(I)$ for each of the 8 groups above as in [BR14] and [BR17].

The case $(2, \langle\ 1 + 2\sqrt{-2}\ \rangle)$**:** Let $I = \langle\ 1 + 2\sqrt{-2}\ \rangle$ and note that $N(I) = 9$. From [Swa71] we have the following presentation for $\mathrm{PSL}(2, O_2)$:

$$\mathrm{PSL}(2, O_2) = \langle\ a, t, u \mid a^2 = (ta)^3 = (au^{-1}au)^2 = 1, [t, u] = 1\ \rangle$$

where $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 1 & \sqrt{-2} \\ 0 & 1 \end{pmatrix}$, and $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now $\Gamma(I)$ has 36 cusps, since $\mathrm{PSL}(2, O_2)/\Gamma(I)$ has order 324 and the image of $P_\infty(I) = \langle\ t^9, t^{-4}u\ \rangle$ in $\mathrm{PSL}(2, O_2)/\Gamma(I)$ is of order 9.

In the following Magma routine, $G = \mathrm{PSL}(2, O_2)$, $H = P_\infty(I) = \langle\ t^9, t^{-4}u\ \rangle$, $N = \langle\langle\ H\ \rangle\rangle$ (the normal closure of H), and Q denotes the quotient of $\Gamma(I)$ by the normal closure of 36 parabolic elements (one from each cusp of $\Gamma(I)$). Magma calculates that $Q = \langle\ 1\ \rangle$; hence $\Gamma(\langle\ 1 + 2\sqrt{-2}\ \rangle)$ is indeed a 36 component link group.

```
G<a,t,u>:=Group<a,t,u|a^2,(t*a)^3,(a*u^-1*a*u)^2,(t,u)>;

H:=sub<G|t^-4*u,t^9>;
J:=NormalClosure(G,H);
N:=Rewrite(G,J);
Index(G,N);
324
\\
Q:=quo<N|t^-4*u,
a*(t*u^2)*a,

t*a*(t*u^2)*a*t^-1,
a*t*a*(t^-4*u)*a*t^-1*a,
```

```
t^2*a*(t*u^2)*a*t^-2,
a*t^2*a*(t*u^2)*a*t^-2*a,

t^-2*a*(t*u^2)*a*t^2,
a*t^-2*a*(t^-4*u)*a*t^2*a,

t^3*a*(t*u^2)*a*t^-3,
a*t^3*a*(t^-4*u)*a*t^-3*a,

t^-3*a*(t^-4*u)*a*t^3,
a*t^-3*a*(t*u^2)*a*t^3*a,

t^4*a*(t*u^2)*a*t^-4,
a*t^4*a*(t^-4*u)*a*t^-4*a,

t^-4*a*(t*u^2)*a*t^4,
a*t^-4*a*(t*u^2)*a*t^4*a,

t*a*t^-2*a*(t^-4*u)*a*t^2*a*t^-1,
a*t*a*t^-2*a*(t^-4*u)*a*t^2*a*t^-1*a,

t^-2*a*t^-2*a*(t*u^2)*a*t^2*a*t^2,
a*t^-2*a*t^-2*a*(t*u^2)*a*t^2*a*t^2*a,

t^2*a*t^-2*a*(t^-4*u)*a*t^2*a*t^-2,
a*t^2*a*t^-2*a*(t^-4*u)*a*t^2*a*t^-2*a,

t^-3*a*t^-2*a*(t*u^2)*a*t^2*a*t^3,
a*t^-3*a*t^-2*a*(t^-4*u)*a*t^2*a*t^3*a,

t^3*a*t^-2*a*(t*u^2)*a*t^2*a*t^-3,
a*t^3*a*t^-2*a*(t*u^2)*a*t^2*a*t^-3*a,

t^4*a*t^-2*a*(t^-4*u)*a*t^2*a*t^-4,
a*t^4*a*t^-2*a*(t*u^2)*a*t^2*a*t^-4*a,

t*a*t^4*a*t^-1*(t*u^2)*t*a*t^-4*a*t^-1,
a*t*a*t^4*a*t^-1*(t^-4*u)*t*a*t^-4*a*t^-1*a,

t^-1*a*t^-4*a*t*(t*u^2)*t^-1*a*t^4*a*t,
a*t^-1*a*t^-4*a*t*(t^-4*u)*t^-1*a*t^4*a*t*a,

t*a*t^-4*a*t^-1*(t^-4*u)*t*a*t^4*a*t^-1,
a*t*a*t^-4*a*t^-1*(t*u^2)*t*a*t^4*a*t^-1*a,

t^2*a*t^-4*a*t^-1*(t^-4*u)*t*a*t^4*a*t^-2,
a*t^2*a*t^-4*a*t^-1*(t^5*u)*t*a*t^4*a*t^-2*a>;
```

```
Q:=ReduceGenerators(Q);
Order(Q);
1
\\
```

The case $(7, \langle\ (1+3\sqrt{-7})/2\ \rangle)$**:** Let $I = \langle\ (1+3\sqrt{-7})/2\ \rangle$ and note that $N(I) = 16$. From [Swa71] we have the following presentation for $\mathrm{PSL}(2, O_7)$:

$$\mathrm{PSL}(2, O_7) = \langle\ a, t, u \mid a^2 = (ta)^3 = (atu^{-1}au)^2 = 1, [t, u] = 1\ \rangle$$

where $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 1 & (1+\sqrt{-7})/2 \\ 0 & 1 \end{pmatrix}$, and $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now $\Gamma(I)$ has 96 cusps, since $\mathrm{PSL}(2, O_7)/\Gamma(I)$ has order 1536 and the image of $P_\infty(I) = \langle\ t^{16}, t^5u\ \rangle$ in $\mathrm{PSL}(2, O_7)/\Gamma(I)$ is of order 16.

In the following Magma routine, $G = \mathrm{PSL}(2, O_7)$, $H = P_\infty(I) = \langle\ t^{16}, t^5u\ \rangle$, $N = \langle\langle\ H\ \rangle\rangle$ (the normal closure of H), and Q denotes the quotient of $\Gamma(I)$ by the normal closure of 96 parabolic elements (one from each cusp of $\Gamma(I)$). Magma calculates that $Q = \langle\ 1\ \rangle$; hence $\Gamma(\langle\ (1+3\sqrt{-7})/2\ \rangle)$ is indeed a 96 component link group.

```
G<a,t,u>:=Group<a,t,u|a^2,(t*a)^3,(a*t*u^-1*a*u)^2,(t,u)>;
H:=sub<G|t^5*u,t^16>;
J:=NormalClosure(G,H);
Index(G,J);
1536
\\

N:=Rewrite(G,J:Simplify:=false);

h:=t*a;

Q:=quo<N| t^-7*u^5,
h*(t^-1*u^3)*h^-1,
h^-1*(t^-1*u^3)*h,

t^-1*a*(t^-6*u^2)*a*t,
h*t^-1*a*(t^-1*u^3)*a*t*h^-1,
h^-1*t^-1*a*(t^-1*u^3)*a*t*h,


t^-2*a*(t^-1*u^3)*a*t^2,
h*t^-2*a*(t^-1*u^3)*a*t^2*h^-1,
h^-1*t^-2*a*(t^-1*u^3)*a*t^2*h,


t^3*a*(t^-1*u^3)*a*t^-3,
h*t^3*a*(t^-1*u^3)*a*t^-3*h^-1,
h^-1*t^3*a*(t^-1*u^3)*a*t^-3*h,


t^-3*a*(t^-1*u^3)*a*t^3,
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h*t^-3*a*(t^-1*u^3)*a*t^3*h^-1,
h^-1*t^-3*a*(t^-1*u^3)*a*t^3*h,

t^4*a*(t^-1*u^3)*a*t^-4,
h*t^4*a*(t^-6*u^2)*a*t^-4*h^-1,
h^-1*t^4*a*(t^-1*u^3)*a*t^-4*h,

t^-4*a*(t^-1*u^3)*a*t^4,
h*t^-4*a*(t^-1*u^3)*a*t^4*h^-1,
h^-1*t^-4*a*(t^-1*u^3)*a*t^4*h,

t^5*a*(t^-1*u^3)*a*t^-5,
h*t^5*a*(t^-1*u^3)*a*t^-5*h^-1,
h^-1*t^5*a*(t^-1*u^3)*a*t^-5*h,

t^-5*a*(t^-1*u^3)*a*t^5,
h*t^-5*a*(t^-1*u^3)*a*t^5*h^-1,
h^-1*t^-5*a*(t^-1*u^3)*a*t^5*h,

t^6*a*(t^-1*u^3)*a*t^-6,
h*t^6*a*(t^-1*u^3)*a*t^-6*h^-1,
h^-1*t^6*a*(t^-1*u^3)*a*t^-6*h,

t^-6*a*(t^-1*u^3)*a*t^6,
h*t^-6*a*(t^-1*u^3)*a*t^6*h^-1,
h^-1*t^-6*a*(t^-1*u^3)*a*t^6*h,

t^7*a*(t^-1*u^3)*a*t^-7,
h*t^7*a*(t^-1*u^3)*a*t^-7*h^-1,
h^-1*t^7*a*(t^-1*u^3)*a*t^-7*h,

t^-7*a*(t^-1*u^3)*a*t^7,
h*t^-7*a*(t^-1*u^3)*a*t^7*h^-1,
h^-1*t^-7*a*(t^-1*u^3)*a*t^7*h,

t^8*a*(t^-1*u^3)*a*t^-8,
h*t^8*a*(t^-1*u^3)*a*t^-8*h^-1,
h^-1*t^8*a*(t^-1*u^3)*a*t^-8*h,
```

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t^-1*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t,
h*t^-1*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t*h^-1,
h^-1*t^-1*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t*h,

t^-2*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t^2,
h*t^-2*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t^2*h^-1,
h^-1*t^-2*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t^2*h,

t^3*a*t^-2*a*(t^-1*u^3)*a*t^2*a*t^-3,
h*t^3*a*t^-2*a*(t^-1*u^3)*a*t^2*a*t^-3*h^-1,
h^-1*t^3*a*t^-2*a*(t^-1*u^3)*a*t^2*a*t^-3*h,

t^-3*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t^3,
h*t^-3*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t^3*h^-1,
h^-1*t^-3*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t^3*h,

t^-1*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t,
h*t^-1*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t*h^-1,
h^-1*t^-1*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t*h,

t^-1*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t,
h*t^-1*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t*h^-1,
h^-1*t^-1*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t*h,

t^-2*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t^2,
h*t^-2*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t^2*h^-1,
h^-1*t^-2*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t^2*h,

t^2*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t^-2,
h*t^2*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t^-2*h^-1,
h^-1*t^2*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t^-2*h,

t^-2*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^2,
h*t^-2*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^2*h^-1,
h^-1*t^-2*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^2*h,

t^3*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^-3,
h*t^3*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^-3*h^-1,
h^-1*t^3*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^-3*h,
```

```
t^3*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t^-3,
h*t^3*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t^-3*h^-1,
h^-1*t^3*a*t^-3*a*(t^-1*u^3)*a*t^3*a*t^-3*h,


t^-3*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^3,
h*t^-3*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^3*h^-1,
h^-1*t^-3*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^3*h,


t^-4*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^4,
h*t^-4*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^4*h^-1,
h^-1*t^-4*a*t^3*a*(t^-1*u^3)*a*t^-3*a*t^4*h,


t^-1*a*t^4*a*(t^-1*u^3)*a*t^-4*a*t,
h*t^-1*a*t^4*a*(t^-1*u^3)*a*t^-4*a*t*h^-1,
h^-1*t^-1*a*t^4*a*(t^-1*u^3)*a*t^-4*a*t*h,


t^-2*a*t^-4*a*(t^-1*u^3)*a*t^4*a*t^2,
h*t^-2*a*t^-4*a*(t^-1*u^3)*a*t^4*a*t^2*h^-1,
h^-1*t^-2*a*t^-4*a*(t^-1*u^3)*a*t^4*a*t^2*h,


t^-1*a*t^5*a*(t^-7*u^5)*a*t^-5*a*t,
h*t^-1*a*t^5*a*(t^-1*u^3)*a*t^-5*a*t*h^-1,
h^-1*t^-1*a*t^5*a*(t^-1*u^3)*a*t^-5*a*t*h,


t^-2*a*t^5*a*(t^-1*u^3)*a*t^-5*a*t^2,
h*t^-2*a*t^5*a*(t^-7*u^5)*a*t^-5*a*t^2*h^-1,
h^-1*t^-2*a*t^5*a*(t^-1*u^3)*a*t^-5*a*t^2*h,


t^2*a*t^-3*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t^3*a*t^-2,
h*t^2*a*t^-3*a*t^2*a*(t^-1*u^3)*a*t^-2*a*t^3*a*t^-2*h^-1,
h^-1*t^2*a*t^-3*a*t^2*a*(t^-7*u^5)*a*t^-2*a*t^3*a*t^-2*h>;

Q:=ReduceGenerators(Q);
Order(Q);
1
\\
```

We remark that due to the complexity of this case (96 cusps) we needed to deactivate the `simplify` subroutine of the `rewrite` routine in order to get Magma to calculate without timing out.

3.2. **The methods of** [Goe11] **and** [Goe15]**.** In [Goe11] and [Goe15], the second author used the geometry of regular tessellations arising from fundamental polyhedra associated to the actions of

$\mathrm{PGL}(2, O_1)$ and $\mathrm{PGL}(2, O_3)$ to completely enumerate the principal congruence links when $d = 1, 3$. Using (less symmetric) fundamental domains of the Bianchi orbifolds Q_d, a similar approach is taken to construct a triangulation of the principal congruence manifold M given a (d, I). We will construct triangulations for all cases (d, I) in Theorem 1.2, or more generally, all cases where $h_d = 1$ and $I = \langle\ x\ \rangle$ where $|x| < 6$. For most of these cases (in particular, for all in Table 1), we can use SnapPy [CDGW17] to determine whether the principal congruence manifold M for (d, I) is a link complement: either we can find one peripheral curve per cusp such that Dehn-filling M along these curves trivializes the fundamental group or we can verify that $H_1(M, \mathbb{Z})/H_1(\partial M, \mathbb{Z})$ is non-trivial (summarized in Appendix A). Of the three remaining cases, $(1, 4 + 3\sqrt{-1})$ and $(3, (11 + \sqrt{-3})/2)$ were already covered in [Goe15] and $(2, 1 + 3\sqrt{-2})$ is accounted for in §3.3.

The programs we wrote to construct M require SageMath [Sag17] and are available at [Goe17]: `generatePrincipalCongruenceManifold.py` generates a Regina [BBP17] triangulation of M using a fundamental domain for Q_d which can be produced with `generateBianchiOrbifold.py`. For each principal congruence link complement M in Theorem 1.1, we have stored the peripheral curves trivializing the fundamental group as meridians on a SnapPy triangulation of M in the directory `LinkComplementCertificates`.

We briefly describe how these programs work. Further details can be found in the source code as comments.

3.2.1. *Fundamental domains for Bianchi groups.* For the construction of principal congruence manifolds, we need the combinatorics of a fundamental polyhedron P for the Bianchi group $\mathrm{PSL}(2, O_d)$ together with the following information for each face f of P:

- another face f' of P called the mate face
- the mating matrix $g_f \in \mathrm{PSL}(2, O_d)$ such that $g_f f' = f$
- for each (finite or ideal) vertex v of f the corresponding vertex v' of f' with $g_f v' = v$.

For simplicity, we triangulate P by taking the barycentric subdivision. We index the vertices of the resulting simplices such that vertex i of a simplex corresponds to the center of an i-cell of P. This results in a triangulation where the gluing permutations are always the identity. We can then store the extra information by assigning to each simplex its "mate" simplex and the mating matrix $g_f \in \mathrm{PSL}(2, O_d)$ that takes face 3 of the simplex to face 3 of its mate.

3.2.2. *Constructing principal congruence manifolds.* Let I be an ideal in O_d. The goal is to construct a triangulation of the principal congruence manifold $\mathbb{H}^3/\Gamma(I)$ using copies of the triangulated fundamental polyhedron P of Q_d and its associated information as described in §3.2.1. We label each copy by P_m where $m \in \mathrm{PSL}(2, O_d/I)$. We use the following algorithm:

(1) Start with a "base" copy P_{Id}.
(2) While there is a copy P_m with an unglued face f:
   (a) Compute $m' = mg_f \in \mathrm{PSL}(2, O_d/I)$.
   (b) If there is no copy $P_{m'}$ yet, create one.
   (c) Glue face f of P_m to the mate face f' of $P_{m'}$ such that the vertices are matching as described in the information about the fundamental polyhedron.

SnapPy can remove the finite vertices of the resulting triangulation to obtain an ideal triangulation.

When labeling the polyhedron in a computer implementation, we need a method giving a canonical matrix with coefficients in O_d to represent an element $m \in \mathrm{PSL}(2, O_d/I)$. For this, we need a procedure to reduce a representative in O_d of an element in O_d/I to a canonical representative.

Given a 2-vector v, the reduced form of v with respect to the vectors v_1 and v_2 is the element in $v + \mathbb{Z}v_1 + \mathbb{Z}v_2$ in the parallelogram spanned by v_1 and v_2. In other words, v is reduced with respect to v_1 and v_2 if $v \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^{-1}$ has coordinates in $[0, 1)$. Let us associate the vector (a, b) to an element $a+b\sqrt{-d}\in O_d$. Let v_1 and v_2 be two vectors that span I as a lattice. We can then reduce a representative in O_d by reducing the associated vector (a,b) by v_1 and v_2.

It is left to find such v_1 and v_2 given generators $x_1,\dots,x_k\in O_d$ of the ideal. As a lattice, I is spanned by the vectors associated to $x_1, x_1\omega_d,\dots,x_k,x_k\omega_d$ where $\omega_d=(1+\sqrt{-d})/2$ (if $d\equiv 3$ mod 4) or $\sqrt{-d}$ (otherwise). We need a procedure that takes such a set of vectors and returns two vectors spanning the same lattice as the input vectors. By iterating, it suffices to have a method that produces two vectors spanning the same lattice as three given vectors. This can be done by repeatedly reducing one vector by the other two vectors until one of them is zero.

3.2.3. *Dirichlet domains.* We use a Dirichlet domain for the Bianchi group $\mathrm{PSL}(2,O_d)$ as a fundamental domain. Recall that given a base point p_0 in hyperbolic space and a sample of matrices $m\in\mathrm{PSL}(2,O_d)$, we obtain a candidate polyhedron P by intersecting the half spaces associated to the matrices. Here, we associate to a matrix m the half space containing p_0 that is limited by the plane bisecting p_0 and the image of p_0 under the action of m. Once we have the candidate polyhedron P, we can try to obtain the information described in §3.2.1: g_f is the matrix that the plane containing a face f of P was associated to, the mate face f' of f is identified by having matrix $g_{f'}={g_f}^{-1}$ and we can try to find the corresponding v' of f' for each vertex v of f. If we succeed, we have verified that P is a fundamental domain for a (possibly trivial) cover of the Bianchi orbifold Q_d.

If we do this in the Klein model, the equations for the associated planes turn out to have rational coefficients (up to a constant scaling). Our primary focus here is on the observations yielding to rational coefficients and we refer the reader to [Pag15] and [Ril83] as well as the SnapPy source code for further details about the construction of Dirichlet domains. In particular, we do not discuss how to verify that our samples of matrices was large enough so that the g_f generate $\mathrm{PSL}(2,O_d)$ and that P covers Q_d trivially (we can check this using volume since the volume of the Bianchi orbifolds is well-known).

It is convient to let p_0 be the origin 0 in the Klein or Poincaré ball model. Unfortunately, there are matrices in $\mathrm{PSL}(2,O_d)$ that fix the origin. But we can pick a suitable matrix $l\in\mathrm{PSL}(2,\mathbb{Q}(\sqrt{-d}))$ and let $m'=l^{-1}ml$ instead of $m\in\mathrm{PSL}(2,O_d)$ act on $\mathbb{H}^3$ or B^3.

3.2.4. *Poincaré extension for the Poincaré ball.* Let $\mathbf{H}$ denote Hamilton's quaternions and $\mathbb{H}^3=\{z+tj : z\in\mathbb{C}, t>0\}\subset\mathbf{H}$ and $B^3=\{x+yj+zk : x^2+y^2+z^2<1\}$ be the upper half space, respectively, Poincaré ball model of hyperbolic 3-space. There is an action of suitable 2×2 matrices with quaternions as coefficients on $\mathbf{H}\cup\{\infty\}$ given by

$$T:\begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto\left(w\mapsto(aw+b)\cdot(cw+d)^{-1}\right).$$

If $m\in\mathrm{PSL}(2,\mathbb{C})$, then $T(m)_{|\mathbb{H}^3}$ is an isometry of $\mathbb{H}^3$. Furthermore, letting

$$m_{\mathbb{H}^3\to B^3}=\begin{pmatrix} 1 & 1\\ j & -j\end{pmatrix}\quad\text{and}\quad m_{B^3\to\mathbb{H}^3}=\begin{pmatrix} 1 & -j\\ 1 & j\end{pmatrix},$$

$T(m_{\mathbb{H}^3\to B^3})$ and $T(m_{B^3\to\mathbb{H}^3})$ convert between $\mathbb{H}^3$ and B^3. Thus, $T(m_{\mathbb{H}^3\to B^3}\cdot m\cdot m_{B^3\to\mathbb{H}^3})_{|B^3}$ is the isometry of the Poincaré ball model B^3 corresponding to m.

3.2.5. *Hyperbolic midpoint and conversion to Klein model.* It is convenient to work in the Klein model since hyperbolic half spaces become Euclidean half spaces (intersected with the unit ball).

When converting between the Klein and the Poincaré ball model, we do so such that the origin and the boundary of the unit ball are fixed.

Lemma 3.1. *Let p be the point in the Poincaré ball model with coordinates (x_p,y_p,z_p). The result of taking the hyperbolic midpoint between p and the origin and then converting that midpoint to the*

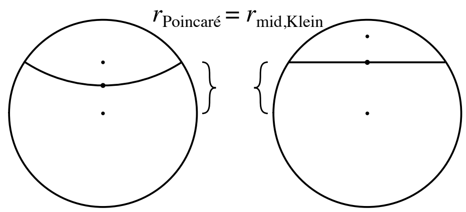


FIGURE 1. Taking the midpoint in the Poincaré model and converting it to the Klein model gives the same Euclidean point.

Klein model also has coordinates (x_p, y_p, z_p)*, see Figure 1. Thus, the plane bisecting* p *and the origin has equation* $x_p x + y_p y + z_p z = x_p^2 + y_p^2 + z_p^2$ *in the Klein model.*

Proof. Let r_{Klein} and $r_{\text{Poincaré}}$ be the Euclidean distance of the origin to a point in the Klein model, respectively, the corresponding point in the Poincaré ball model. We have

$$r_{\text{Poincaré}} = \frac{r_{\text{Klein}}}{1 + \sqrt{1 - r_{\text{Klein}}^2}}.$$

Note that this is the same relationship we have between the Euclidean distance $r_{\text{Poincaré}}$ of a point in Poincaré ball model and $r_{\text{mid,Poincaré}}$ of the hyperbolic midpoint between that point and the origin:

$$r_{\text{mid,Poincaré}} = \frac{r_{\text{Poincaré}}}{1 + \sqrt{1 - r_{\text{Poincaré}}}}.$$

Thus, we have $r_{\text{mid,Klein}} = r_{\text{Poincaré}}$. $\square$

3.2.6. *Rational plane equation.*

Lemma 3.2. *Let* $m \in \mathrm{PSL}(2, \mathbb{Q}(\sqrt{-d}))$*. Let* (x_p, y_p, z_p) *be the coordinates of the image of the origin* 0 *in the Poincaré ball model* B^3 *under the action of* m*. Then,* $x_p, y_p \in \mathbb{Q}$ *and* $z_p \in \sqrt{d}\mathbb{Q}$*. Thus, in the Klein model, the equation for the plane associated to* m *has rational coefficients when replacing* z *by* $\sqrt{d}z'$ *in Lemma 3.1.*

Proof. Let

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The image of the origin in B^3 is given by $(x_p, y_p, z_p) = T(m_{\mathbb{H}^3 \to B^3} \cdot m \cdot m_{B^3 \to \mathbb{H}^3})(0)$. Note that the origin in B^3 corresponds to j in $\mathbb{H}^3$ and a standard calculation gives:

$$T(m)(z + tj) = \left((az + b)(\overline{cz + d}) + a\bar{c}t^2 + tj\right) / |c(z + tj) + d|^2, \tag{1}$$

so

$$T(m)(j) = (b\bar{d} + a\bar{c} + j)/|cj + d|^2 \in \mathbb{Q} + i\sqrt{d}\mathbb{Q} + j\mathbb{Q}.$$

Applying the conversion $T(m_{\mathbb{H}^3 \to B^3})$ now gives the result. $\square$

Unfortunately, we do need to deal with a further quadratic extension of $\mathbb{Q}(\sqrt{d})$ when verifying the correspondences between the vertices v and v' of a face f and its mate face f'.

3.3. **The final case.** To eliminate the group $\Gamma(\langle\, 1 + 3\sqrt{-2}\,\rangle)$ we show that it is not generated by its parabolic elements. This will be the case if the following group ($B_2(\langle\, 1+3\sqrt{-2}\,\rangle)$ in the notation of [BR14] and [BR17])

$$G \cong \langle\, a, t, u \mid a^2 = (ta)^3 = (au^{-1}au)^2 = 1, [t, u] = 1, tu^3, t^{19} \,\rangle$$

is of order greater than $|\mathrm{PSL}(2, O_2)/\Gamma(\langle\, 1 + 3\sqrt{-2}\,\rangle)|$. Indeed, we will show that G is infinite.

In previous work such computations could be handled by Magma [BCP97]. However, in this case, more sophisticated computer programs were needed, in particular the program Monoid Automata Factory (MAF) which is a successor of the GAP package kbmag, and we could not have implemented this without the assistance of Alun Williams, who we wish to sincerely thank for his help.

Lemma 3.3. *G is infinite. Thus, the principal congruence manifold $M = \mathbb{H}^3/\Gamma(\langle\, 1+3\sqrt{-2}\,\rangle)$ is not a link complement.*

Proof. From §3.1 we have the following presentation for

$$\mathrm{PSL}(2, O_2) = \langle a, t, u | a^2 = (ta)^3 = (au^{-1}au)^2 = tut^{-1}u^{-1} = 1\rangle$$

with the matrices for a, t and u also give in §3.1. As in previous considerations, if M were a link complement, then $\pi_1(M)$ would be generated by parabolic elements which are all $\mathrm{PSL}(2, O_2)$-conjugates of products of tu^3 and t^{19}. Thus, the covering group associated to $M \to Q_2$ would be given by the above G.

However, as we now briefly describe, using MAF [Wil17], G can be proved to be infinite. For this, we write the following presentation of G into a file `myGroup`:

```
_RWS := rec(
    isRWS := true,
    generatorOrder := [_g1,_g2,_g3,_g4,_g5],
    inverses :=       [_g1,_g3,_g2,_g5,_g4],
    ordering := "shortlex",
    equations := [
        [_g2*_g1*_g2,_g1*_g3*_g1],
        [_g1*_g5*_g1*_g4,_g5*_g1*_g4*_g1],
        [_g2*_g4,_g4*_g2],
        [_g2^4,_g4*_g3^2],
        [_g2^10,_g3^9]
    ]
);
```

and then call (which takes about 2 hours of time on a MacBook pro with a 2.6Ghz Intel Core i5)

```
$ automata -no-kb myGroup
$ gpaxioms myGroup
[...]
Checking relation _g1*_g5*_g1*_g4=_g5*_g1*_g4*_g1
Checking relation _g2*_g4=_g4*_g2
Checking relation _g2^4=_g4*_g3^2
Checking relation _g2^10=_g3^9
Axiom check succeeded.
$ fsacount myGroup.wa
The accepted language is infinite
```

The first command finds an automatic structure and the second command verifies that the automatic structure found is indeed for the group G. `automata` always find as word acceptor automaton that accepts only one word for any group element. Thus, G is infinite since the automaton `myGroup.wa` accepts infinitely many words. □

Appendix A. Diagrams for class number 1

Figures 2, 3, 4, 5, 6 and 7 show all cases (d, I) with $h_d = 1$ and $I = \langle\ x\ \rangle$ where $|x| < 6$. We only show one value among all x yielding the same ideal $\langle\ x\ \rangle$ up to complex conjugation. For each case, we list the number of cusps of the principal congruence manifold M and whether it is a link complement. In case it is not, we either list $H_1(M,\mathbb{Z})/H_1(\partial M,\mathbb{Z})$ (computed using Lemma 9.6 in [Goe15]) or reference a proof that M is not a link complement. We also give a reference to a figure of a link if it is known in a particular case.

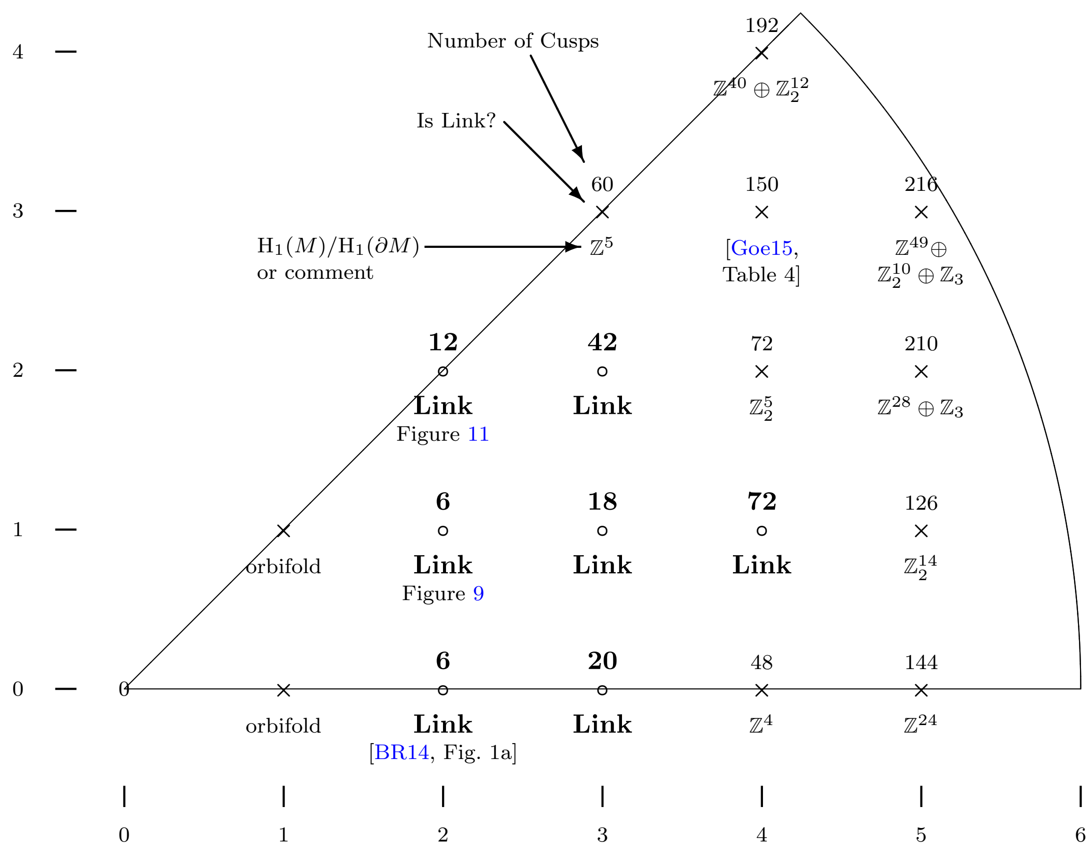
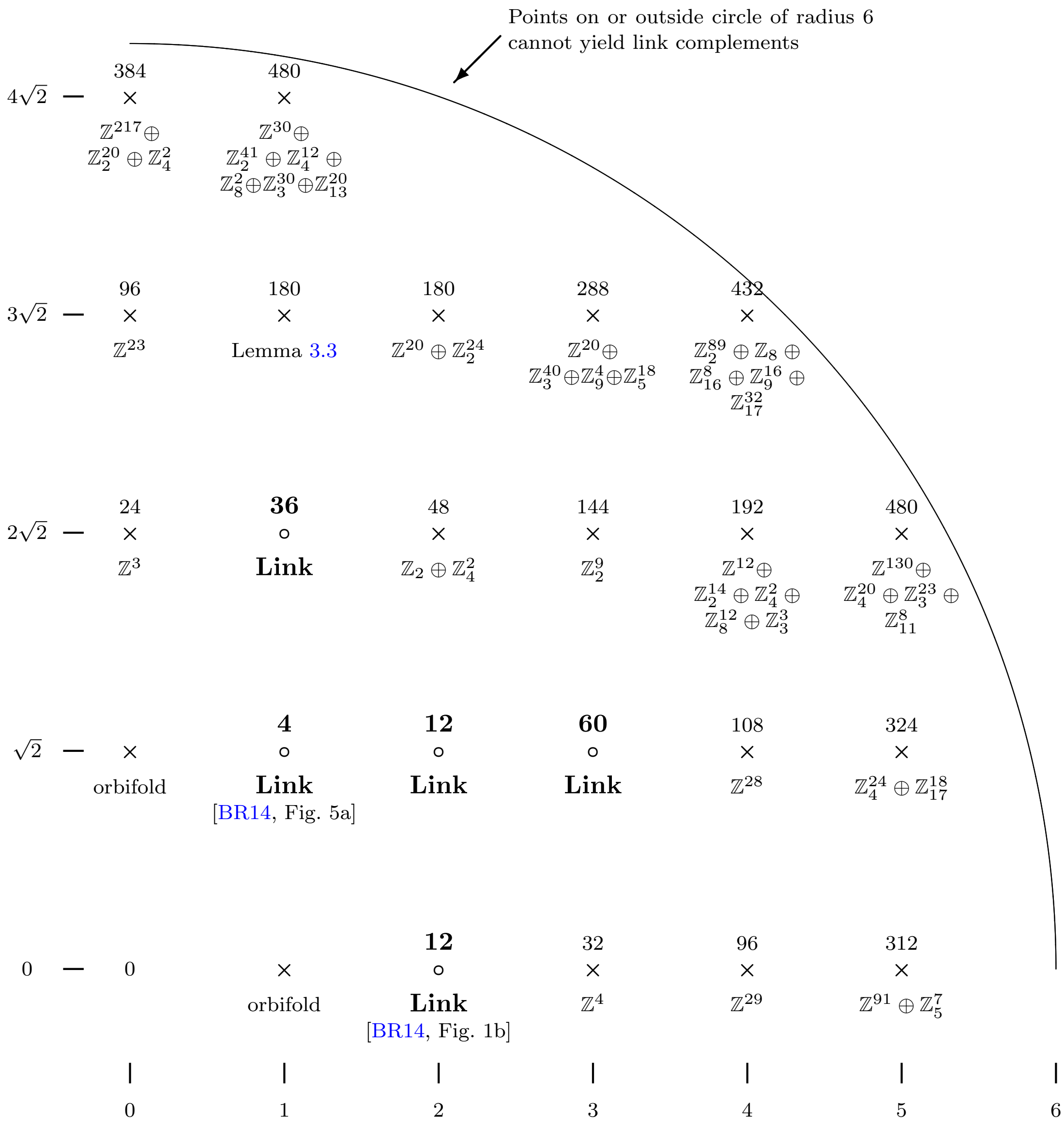


Figure 2. $d = 1$.



Points on or outside circle of radius 6
cannot yield link complements
$4\sqrt{2}$
384
$\mathbb{Z}^{217}\oplus$
$\mathbb{Z}_2^{20}\oplus\mathbb{Z}_4^2$
480
$\mathbb{Z}^{30}\oplus$
$\mathbb{Z}_2^{41}\oplus\mathbb{Z}_4^{12}\oplus$
$\mathbb{Z}_8^2\oplus\mathbb{Z}_3^{30}\oplus\mathbb{Z}_{13}^{20}$
$3\sqrt{2}$
96
$\mathbb{Z}^{23}$
180
Lemma 3.3
180
$\mathbb{Z}^{20}\oplus\mathbb{Z}_2^{24}$
288
$\mathbb{Z}^{20}\oplus$
$\mathbb{Z}_3^{40}\oplus\mathbb{Z}_9^4\oplus\mathbb{Z}_5^{18}$
432
$\mathbb{Z}_2^{89}\oplus\mathbb{Z}_8\oplus$
$\mathbb{Z}_{16}^8\oplus\mathbb{Z}_9^{16}\oplus$
$\mathbb{Z}_{17}^{32}$
$2\sqrt{2}$
24
$\mathbb{Z}^3$
36
Link
48
$\mathbb{Z}_2\oplus\mathbb{Z}_4^2$
144
$\mathbb{Z}_2^9$
192
$\mathbb{Z}^{12}\oplus$
$\mathbb{Z}_2^{14}\oplus\mathbb{Z}_4^2\oplus$
$\mathbb{Z}_8^{12}\oplus\mathbb{Z}_3^3$
480
$\mathbb{Z}^{130}\oplus$
$\mathbb{Z}_4^{20}\oplus\mathbb{Z}_3^{23}\oplus$
$\mathbb{Z}_{11}^8$
$\sqrt{2}$
orbifold
4
Link
[BR14, Fig. 5a]
12
Link
60
Link
108
$\mathbb{Z}^{28}$
324
$\mathbb{Z}_4^{24}\oplus\mathbb{Z}_{17}^{18}$
0
0
orbifold
12
Link
[BR14, Fig. 1b]
32
$\mathbb{Z}^4$
96
$\mathbb{Z}^{29}$
312
$\mathbb{Z}^{91}\oplus\mathbb{Z}_5^7$
0
1
2
3
4
5
6

FIGURE 3. $d = 2$.

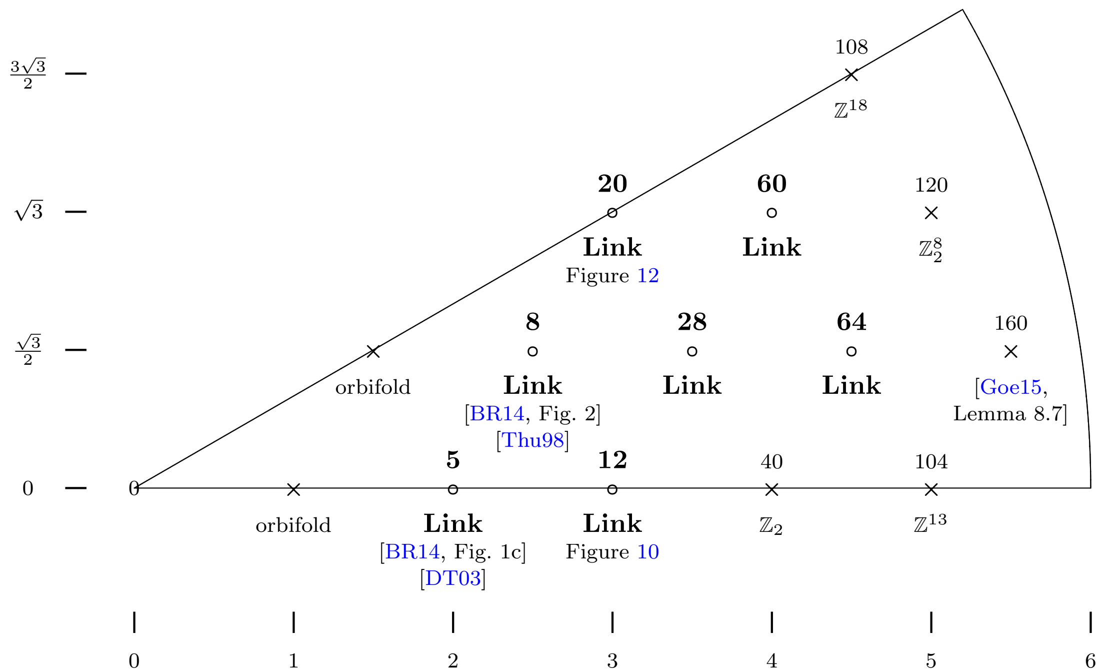

$\frac{3\sqrt{3}}{2}$
$\sqrt{3}$
$\frac{\sqrt{3}}{2}$
0
108
$\mathbb{Z}^{18}$
20
Link
Figure 12
60
Link
120
$\mathbb{Z}_2^8$
orbifold
8
Link
[BR14, Fig. 2]
[Thu98]
28
Link
64
Link
160
[Goe15, Lemma 8.7]
0
orbifold
5
Link
[BR14, Fig. 1c]
[DT03]
12
Link
Figure 10
40
$\mathbb{Z}_2$
104
$\mathbb{Z}^{13}$
0
1
2
3
4
5
6

FIGURE 4. $d = 3$.

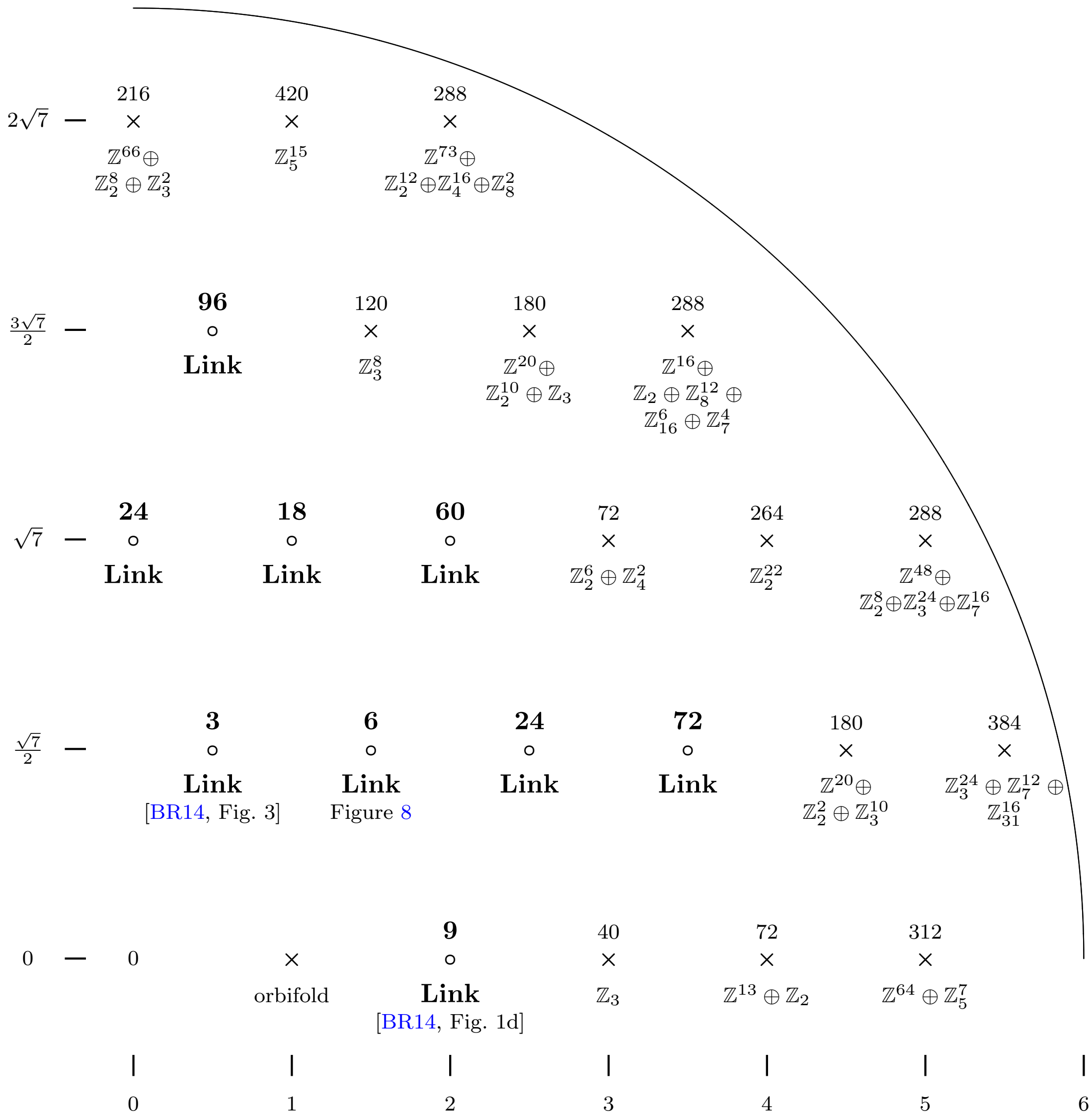

$2\sqrt{7}$
216
$\mathbb{Z}^{66}\oplus$ $\mathbb{Z}_2^8 \oplus \mathbb{Z}_3^2$
420
$\mathbb{Z}_5^{15}$
288
$\mathbb{Z}^{73}\oplus$ $\mathbb{Z}_2^{12}\oplus\mathbb{Z}_4^{16}\oplus\mathbb{Z}_8^2$
$\frac{3\sqrt{7}}{2}$
96 **Link**
120
$\mathbb{Z}_3^8$
180
$\mathbb{Z}^{20}\oplus$ $\mathbb{Z}_2^{10} \oplus \mathbb{Z}_3$
288
$\mathbb{Z}^{16}\oplus$ $\mathbb{Z}_2 \oplus \mathbb{Z}_8^{12} \oplus$ $\mathbb{Z}_{16}^6 \oplus \mathbb{Z}_7^4$
$\sqrt{7}$
24 **Link**
18 **Link**
60 **Link**
72
$\mathbb{Z}_2^6 \oplus \mathbb{Z}_4^2$
264
$\mathbb{Z}_2^{22}$
288
$\mathbb{Z}^{48}\oplus$ $\mathbb{Z}_2^8\oplus\mathbb{Z}_3^{24}\oplus\mathbb{Z}_7^{16}$
$\frac{\sqrt{7}}{2}$
3 **Link** [BR14, Fig. 3]
6 **Link** Figure 8
24 **Link**
72 **Link**
180
$\mathbb{Z}^{20}\oplus$ $\mathbb{Z}_2^2 \oplus \mathbb{Z}_3^{10}$
384
$\mathbb{Z}_3^{24} \oplus \mathbb{Z}_7^{12} \oplus$ $\mathbb{Z}_{31}^{16}$
0
0
orbifold
9 **Link** [BR14, Fig. 1d]
40
$\mathbb{Z}_3$
72
$\mathbb{Z}^{13} \oplus \mathbb{Z}_2$
312
$\mathbb{Z}^{64} \oplus \mathbb{Z}_5^7$
0
1
2
3
4
5
6

FIGURE 5. $d = 7$.

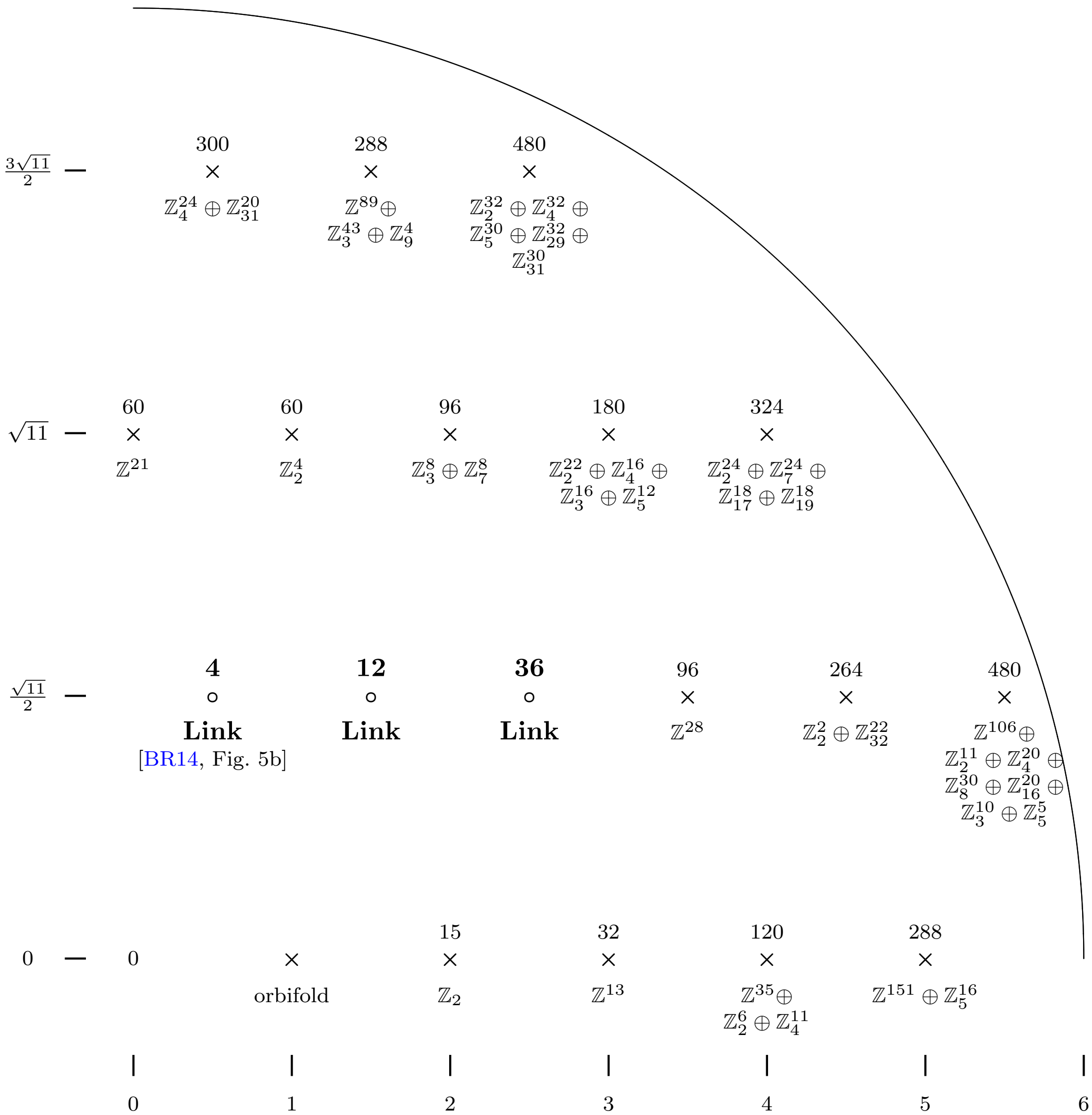
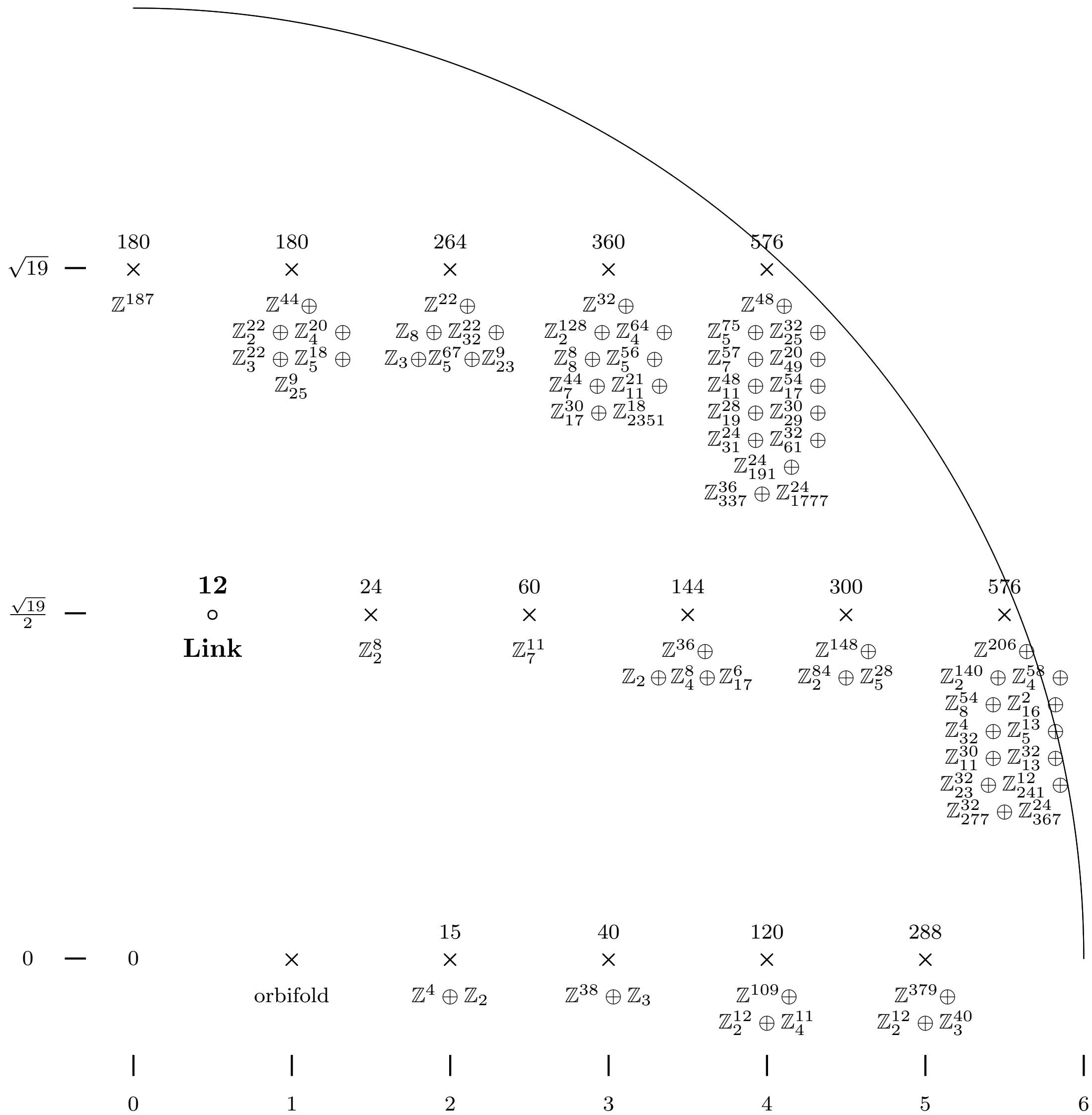

$\frac{3\sqrt{11}}{2}$
300
$\mathbb{Z}_4^{24} \oplus \mathbb{Z}_{31}^{20}$
288
$\mathbb{Z}^{89} \oplus \mathbb{Z}_3^{43} \oplus \mathbb{Z}_9^4$
480
$\mathbb{Z}_2^{32} \oplus \mathbb{Z}_4^{32} \oplus \mathbb{Z}_5^{30} \oplus \mathbb{Z}_{29}^{32} \oplus \mathbb{Z}_{31}^{30}$
$\sqrt{11}$
60
$\mathbb{Z}^{21}$
60
$\mathbb{Z}_2^4$
96
$\mathbb{Z}_3^8 \oplus \mathbb{Z}_7^8$
180
$\mathbb{Z}_2^{22} \oplus \mathbb{Z}_4^{16} \oplus \mathbb{Z}_3^{16} \oplus \mathbb{Z}_5^{12}$
324
$\mathbb{Z}_2^{24} \oplus \mathbb{Z}_7^{24} \oplus \mathbb{Z}_{17}^{18} \oplus \mathbb{Z}_{19}^{18}$
$\frac{\sqrt{11}}{2}$
4
Link
[BR14, Fig. 5b]
12
Link
36
Link
96
$\mathbb{Z}^{28}$
264
$\mathbb{Z}_2^2 \oplus \mathbb{Z}_{32}^{22}$
480
$\mathbb{Z}^{106} \oplus \mathbb{Z}_2^{11} \oplus \mathbb{Z}_4^{20} \oplus \mathbb{Z}_8^{30} \oplus \mathbb{Z}_{16}^{20} \oplus \mathbb{Z}_3^{10} \oplus \mathbb{Z}_5^5$
0
0
orbifold
15
$\mathbb{Z}_2$
32
$\mathbb{Z}^{13}$
120
$\mathbb{Z}^{35} \oplus \mathbb{Z}_2^6 \oplus \mathbb{Z}_4^{11}$
288
$\mathbb{Z}^{151} \oplus \mathbb{Z}_5^{16}$
0
1
2
3
4
5
6

FIGURE 6. $d = 11$.



$\sqrt{19}$
180
$\mathbb{Z}^{187}$
180
$\mathbb{Z}^{44} \oplus$
$\mathbb{Z}_2^{22} \oplus \mathbb{Z}_4^{20} \oplus$
$\mathbb{Z}_3^{22} \oplus \mathbb{Z}_5^{18} \oplus$
$\mathbb{Z}_{25}^{9}$
264
$\mathbb{Z}^{22} \oplus$
$\mathbb{Z}_8 \oplus \mathbb{Z}_{32}^{22} \oplus$
$\mathbb{Z}_3 \oplus \mathbb{Z}_5^{67} \oplus \mathbb{Z}_{23}^{9}$
360
$\mathbb{Z}^{32} \oplus$
$\mathbb{Z}_2^{128} \oplus \mathbb{Z}_4^{64} \oplus$
$\mathbb{Z}_8^{8} \oplus \mathbb{Z}_5^{56} \oplus$
$\mathbb{Z}_7^{44} \oplus \mathbb{Z}_{11}^{21} \oplus$
$\mathbb{Z}_{17}^{30} \oplus \mathbb{Z}_{2351}^{18}$
576
$\mathbb{Z}^{48} \oplus$
$\mathbb{Z}_5^{75} \oplus \mathbb{Z}_{25}^{32} \oplus$
$\mathbb{Z}_7^{57} \oplus \mathbb{Z}_{49}^{20} \oplus$
$\mathbb{Z}_{11}^{48} \oplus \mathbb{Z}_{17}^{54} \oplus$
$\mathbb{Z}_{19}^{28} \oplus \mathbb{Z}_{29}^{30} \oplus$
$\mathbb{Z}_{31}^{24} \oplus \mathbb{Z}_{61}^{32} \oplus$
$\mathbb{Z}_{191}^{24} \oplus$
$\mathbb{Z}_{337}^{36} \oplus \mathbb{Z}_{1777}^{24}$
$\frac{\sqrt{19}}{2}$
12
Link
24
$\mathbb{Z}_2^{8}$
60
$\mathbb{Z}_7^{11}$
144
$\mathbb{Z}^{36} \oplus$
$\mathbb{Z}_2 \oplus \mathbb{Z}_4^{8} \oplus \mathbb{Z}_{17}^{6}$
300
$\mathbb{Z}^{148} \oplus$
$\mathbb{Z}_2^{84} \oplus \mathbb{Z}_5^{28}$
576
$\mathbb{Z}^{206} \oplus$
$\mathbb{Z}_2^{140} \oplus \mathbb{Z}_4^{58} \oplus$
$\mathbb{Z}_8^{54} \oplus \mathbb{Z}_{16}^{2} \oplus$
$\mathbb{Z}_{32}^{4} \oplus \mathbb{Z}_5^{13} \oplus$
$\mathbb{Z}_{11}^{30} \oplus \mathbb{Z}_{13}^{32} \oplus$
$\mathbb{Z}_{23}^{32} \oplus \mathbb{Z}_{241}^{12} \oplus$
$\mathbb{Z}_{277}^{32} \oplus \mathbb{Z}_{367}^{24}$
0
0
orbifold
15
$\mathbb{Z}^4 \oplus \mathbb{Z}_2$
40
$\mathbb{Z}^{38} \oplus \mathbb{Z}_3$
120
$\mathbb{Z}^{109} \oplus$
$\mathbb{Z}_2^{12} \oplus \mathbb{Z}_4^{11}$
288
$\mathbb{Z}^{379} \oplus$
$\mathbb{Z}_2^{12} \oplus \mathbb{Z}_3^{40}$
0
1
2
3
4
5
6

FIGURE 7. $d = 19$.

Appendix B. New link diagrams

Figures 8, 9, 10, 11 and 12 show previously unpublished principal congruence links. A dot in these figures indicates the line perpendicular to the paper plane connecting to $\infty \in S^3$. Link diagrams for all known principal congruence links are available at [Goe17] in `Links`.

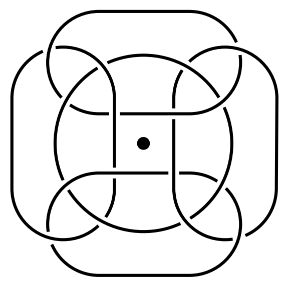


Figure 8. $\frac{3+\sqrt{-7}}{2}$.

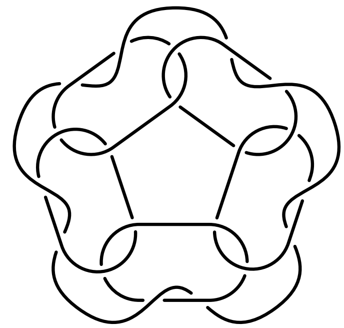


Figure 9. $2+\sqrt{-1}$.

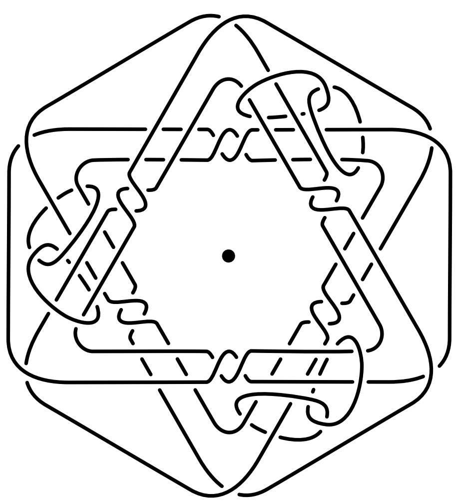


Figure 10. $3+0\sqrt{-3}$ [Goe11] .

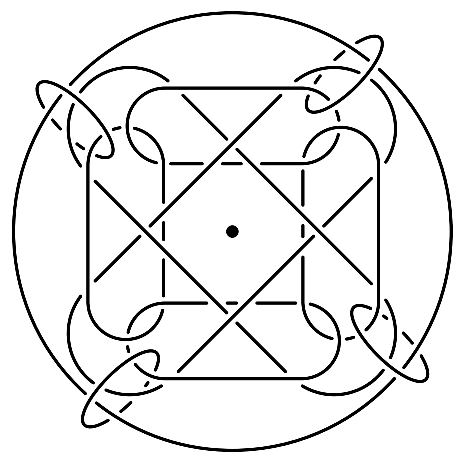


Figure 11. $2+2\sqrt{-1}$.

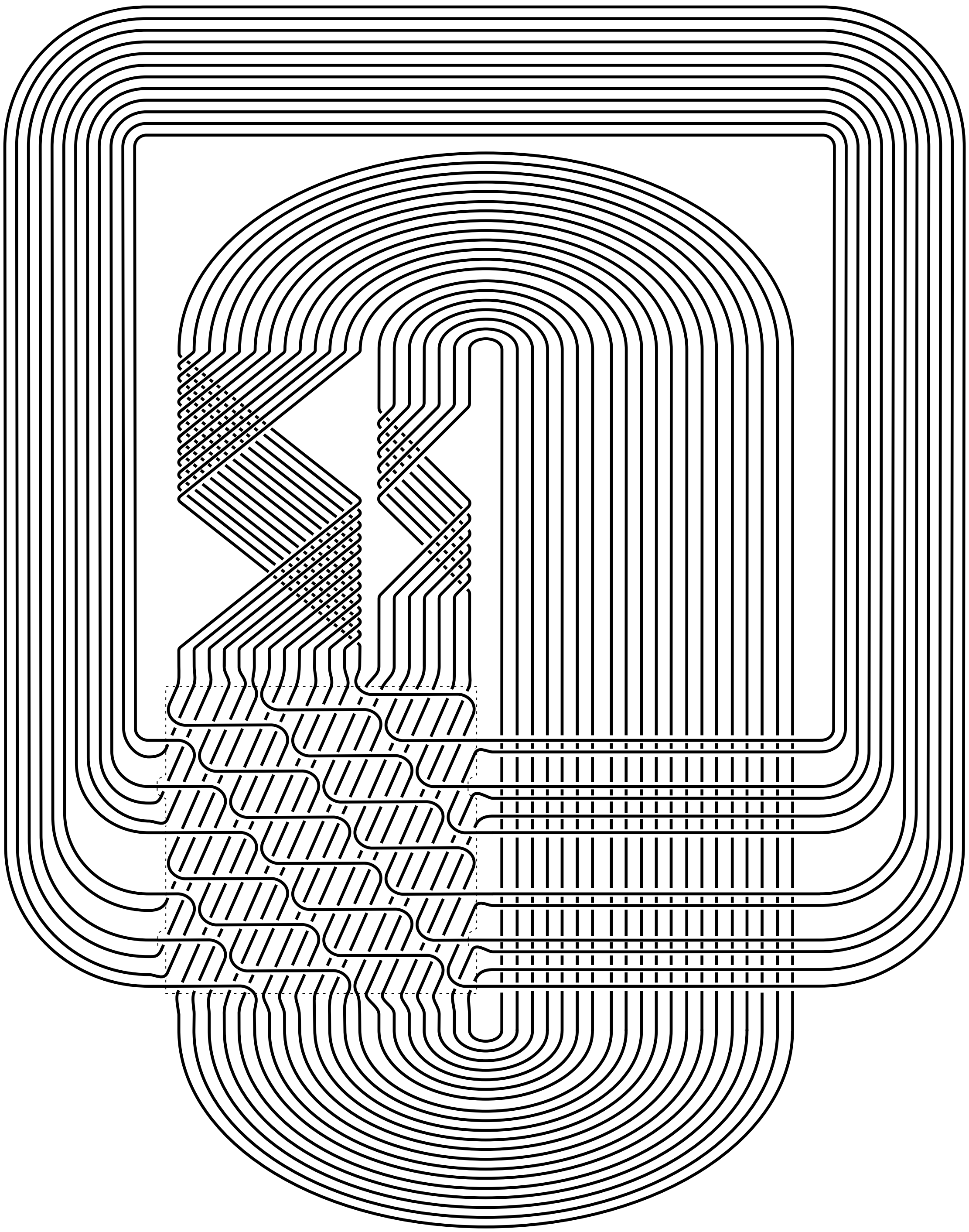


FIGURE 12. $3 + \sqrt{-3}$ [Goe11] .

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