

ALL PRINCIPAL CONGRUENCE LINK GROUPS

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ABSTRACT. In this note we give the complete enumeration of all the principal congruence link complements in S^3 , thereby answering a question of W. Thurston.

1. INTRODUCTION

Let d be a square-free positive integer, let O_d denote the ring of integers in $\mathbb{Q}(\sqrt{-d})$, and h_d denote the class number of $\mathbb{Q}(\sqrt{-d})$.

Setting $Q_d = \mathbb{H}^3/\mathrm{PSL}(2, O_d)$ to be the Bianchi orbifold, it is well-known that Q_d is a finite volume hyperbolic orbifold with h_d cusps (see [MR03] Chapters 8 and 9 for example). A non-compact finite volume hyperbolic 3-manifold X is called *arithmetic* if X and Q_d are commensurable, that is to say they share a common finite sheeted cover (see [MR03] Chapter 8 for more on this).

An important class of arithmetic 3-manifolds consists of the *congruence* manifolds. Recall that a subgroup $\Gamma < \mathrm{PSL}(2, O_d)$ is called a *congruence subgroup* if there exists an ideal $I \subset O_d$ so that Γ contains the *principal congruence group*:

$$\Gamma(I) = \ker\{\mathrm{PSL}(2, O_d) \rightarrow \mathrm{PSL}(2, O_d/I)\},$$

where $\mathrm{PSL}(2, O_d/I) = \mathrm{SL}(2, O_d/I)/\{\pm \mathrm{Id}\}$. The largest ideal I for which $\Gamma(I) < \Gamma$ is called the *level* of Γ . A manifold $M = \mathbb{H}^3/\Gamma$ is called *congruence* (resp. *principal congruence*) if $\Gamma > \Gamma(I)$ (resp. $\Gamma = \Gamma(I)$) for some ideal I .

In an email to the first and third authors in 2009, W. Thurston asked the following question about principal congruence link complements:

“Although there are infinitely many arithmetic link complements, there are only finitely many that come from principal congruence subgroups. Some of the examples known seem to be among the most general (given their volume) for producing lots of exceptional manifolds by Dehn filling, so I’m curious about the complete list.”

In this note, we give the complete enumeration of all the principal congruence link complements in S^3 , together with their levels. Our main result is the following:

Theorem 1.1. *The following list of 48 pairs (d, I) describes all principal congruence subgroups $\Gamma(I) < \mathrm{PSL}(2, O_d)$ such that $\mathbb{H}^3/\Gamma(I)$ is a link complement in S^3 :*

- (1) $d = 1$: $I = \langle 2 \rangle, \langle 2 \pm i \rangle, \langle (1 \pm i)^3 \rangle, \langle 3 \rangle, \langle 3 \pm i \rangle, \langle 3 \pm 2i \rangle, \langle 4 \pm i \rangle$.
- (2) $d = 2$: $I = \langle 1 \pm \sqrt{-2} \rangle, \langle 2 \rangle, \langle 2 \pm \sqrt{-2} \rangle, \langle 1 \pm 2\sqrt{-2} \rangle, \langle 3 \pm \sqrt{-2} \rangle$.
- (3) $d = 3$: $I = \langle 2 \rangle, \langle 3 \rangle, \langle (5 \pm \sqrt{-3})/2 \rangle, \langle 3 \pm \sqrt{-3} \rangle, \langle (7 \pm \sqrt{-3})/2 \rangle, \langle 4 \pm \sqrt{-3} \rangle, \langle (9 \pm \sqrt{-3})/2 \rangle$.
- (4) $d = 5$: $I = \langle 3, (1 \pm \sqrt{-5}) \rangle$.
- (5) $d = 7$: $I = \langle (1 \pm \sqrt{-7})/2 \rangle, \langle 2 \rangle, \langle (3 \pm \sqrt{-7})/2 \rangle, \langle \pm\sqrt{-7} \rangle, \langle 1 \pm \sqrt{-7} \rangle, \langle (5 \pm \sqrt{-7})/2 \rangle, \langle 2 \pm \sqrt{-7} \rangle, \langle (7 \pm \sqrt{-7})/2 \rangle, \langle (1 \pm 3\sqrt{-7})/2 \rangle$.

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- (6) $d = 11$: $I = \langle (1 \pm \sqrt{-11})/2 \rangle, \langle (3 \pm \sqrt{-11})/2 \rangle, \langle (5 \pm \sqrt{-11})/2 \rangle$.
- (7) $d = 15$: $I = \langle 2, (1 \pm \sqrt{-15})/2 \rangle, \langle 3, (3 \pm \sqrt{-15})/2 \rangle, \langle (1 \pm \sqrt{-15})/2 \rangle, \langle 5, (5 \pm \sqrt{-15})/2 \rangle, \langle (3 \pm \sqrt{-15})/2 \rangle$.
- (8) $d = 19$: $I = \langle (1 \pm \sqrt{-19})/2 \rangle$.
- (9) $d = 23$: $I = \langle 2, (1 \pm \sqrt{-23})/2 \rangle, \langle 3, (1 \pm \sqrt{-23})/2 \rangle, \langle 4, (3 \pm \sqrt{-23})/2 \rangle$.
- (10) $d = 31$: $I = \langle 2, (1 \pm \sqrt{-31})/2 \rangle, \langle 4, (1 \pm \sqrt{-31})/2 \rangle, \langle 5, (3 \pm \sqrt{-31})/2 \rangle$.
- (11) $d = 47$: $I = \langle 2, (1 \pm \sqrt{-47})/2 \rangle, \langle 3, (1 \pm \sqrt{-47})/2 \rangle, \langle 4, (1 \pm \sqrt{-47})/2 \rangle$.
- (12) $d = 71$: $I = \langle 2, (1 \pm \sqrt{-71})/2 \rangle$.

As we will describe in §2 and 3, using previous work of the authors ([BR14], [BR17], [Goe15]), the proof of Theorem 1.1 can be reduced to establishing the following theorem:

Theorem 1.2. *When $d \in \{2, 7, 11\}$ the following list of pairs (d, I) determine principal congruence subgroups $\Gamma(I) < \text{PSL}(2, O_d)$ such that $\mathbb{H}^3/\Gamma(I)$ is a link complement in S^3 :*

- (1) $d = 2$: $I = \langle 1 \pm 2\sqrt{-2} \rangle, \langle 3 \pm \sqrt{-2} \rangle$.
- (2) $d = 7$: $I = \langle \pm\sqrt{-7} \rangle, \langle (5 \pm \sqrt{-7})/2 \rangle, \langle 2 \pm \sqrt{-7} \rangle, \langle (7 \pm \sqrt{-7})/2 \rangle, \langle (1 \pm 3\sqrt{-7})/2 \rangle$.
- (3) $d = 11$: $I = \langle (5 \pm \sqrt{-11})/2 \rangle$.

Furthermore, in the case when $d = 2$, $\Gamma(\langle 1 + 3\sqrt{-2} \rangle)$ is not a link group.

Theorem 1.2 will be deduced by combining previous work of the authors, as well as further applications of these techniques, together with Lemma 4.1, which deals with the case $(2, \langle 1 + 3\sqrt{-2} \rangle)$. In contrast to the other cases, this final case required finding an automatic structure for a certain group for which we used the program Monoid Automata Factory (MAF) [Wil17].

We finish the Introduction with some commentary. The proof of Theorem 1.2 is largely computational and builds upon (for the most part) the techniques developed in our previous independent work to deal with earlier cases. In the light of this, we have decided to present the work here essentially in “announcement form”, deferring the technical details (other than Lemma 4.1) including the Magma routines, SnapPy computations of homology for congruence manifolds and several new principal congruence link diagrams to a technical report [BGR18] which will be posted on the arXiv.

Moreover, rather than a collaboration, this paper (and the companion technical report) is the conclusion of overlapping efforts of the first and third authors and independently the second author. It was suggested to the authors by Ian Agol that since Theorem 1.1 was proved almost simultaneously, that a collaborative effort should be undertaken to describe the final solution. The main goal of this note is to provide a brief overview of previous work and summary of techniques that lead to Theorem 1.2. Detailed discussion of the computations will appear in [BGR18]. We refer the reader to [BR18] for further background, history and connections with other questions regarding the topology of congruence link complements.

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2. PRELIMINARIES AND TECHNIQUES

In this section, we review some earlier work that was used in [BR14], [BR17], and [Goe15] that produced a finite list of potential pairs (d, I) .

Note that if $I \subset O_d$ is an ideal and $\bar{I} \subset O_d$ the complex conjugate ideal, then $\Gamma(I)$ is a link group if and only if $\Gamma(\bar{I})$ is a link group — since complex conjugation induces an orientation-reversing involution of $\mathbb{H}^3/\Gamma(I)$. Hence it suffices to consider only one of the ideals I and \bar{I} as a candidate level for a link group.

2.1. Reducing to finitely many cases. Suppose that $L \subset S^3$ is a link with n components, and $M \cong S^3 \setminus L$. Abusing notation and identifying M with the link exterior, set $\iota : \partial M \rightarrow M$ to be the inclusion map. Now $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^n$ and $H_1(M; \mathbb{Z})/\iota_*(H_1(\partial M; \mathbb{Z}))$ is trivial (i.e. link complements have trivial cuspidal cohomology). Hence the solution to the Cuspidal Cohomology Problem (completed in [Vog85]), provides the following consequence for principal congruence link complements:

Theorem 2.1. *Suppose that $M = \mathbb{H}^3/\Gamma(I)$ is homeomorphic to a link complement and $M \rightarrow Q_d$. Then M , and hence Q_d , has trivial cuspidal cohomology, and so*

$$d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}.$$

First, note that if $\Gamma(I)$ is a link group, it must be a torsion-free subgroup of $\mathrm{PSL}(2, O_d)$. We can disregard the case when $I = O_d$, since the groups $\mathrm{PSL}(2, O_d)$ all contain elements of orders 2 and 3. It can then be easily checked that there are only 6 pairs (d, I) , up to complex conjugation with d as above, so that $\Gamma(I)$ contains a non-trivial element of finite order.

To pass from finitely many values of d to finitely many possible pairs (d, I) the norm of the ideal I needs to be bounded. To achieve this, we argue as follows. When the class number is 1, we can use the 6-Theorem of Agol [Ago00] and Lackenby [Lac00] to control which peripheral curves can produce S^3 by Dehn filling. When the class number is > 1 , an upper bound for the systole for a hyperbolic link complement in S^3 from [AR00] can be used. Since systole length grows with the norm of the ideal this provides the necessary control. Moreover, when the class number is 1, all ideals are principal and the argument above bounds the absolute value of a generator of the ideal, and when the class number is > 1 , this bounds the absolute value of *some* $x \in I$. Summarizing this discussion we obtain (see [BR14, Section 4.1] and [BR17, Lemma 4.1]):

Theorem 2.2. *If (d, I) determines a link complement, then there must be a non-trivial $x \in I$ with $|x| < 6$ (if $h_d = 1$), respectively, $|x|^2 < 39$ (if $h_d > 1$). In particular there are only finitely many pairs (d, I) .*

Theorem 2.2 reduces the classification of which principal congruence groups are link groups to a finite list, indeed there are 302 cases (up to complex conjugation and excluding the 6 cases that give groups containing elements of finite order).

2.2. Proving a case to not be a link complement. Assume that (d, I) is one of the finitely many pairs provided by Theorems 2.1 and 2.2 for which we need to decide that $M = \mathbb{H}^3/\Gamma(I)$ is, or is not, homeomorphic to a link complement. We first discuss the case of proving that M is not a link complement. Indeed, we will show something slightly stronger, namely that Γ is not generated by parabolic elements (equivalently its peripheral subgroups).

To describe how this is achieved, fix a collection of $\mathrm{PSL}(2, O_d)$ -inequivalent cusps c_i for $i = 1, \dots, h_d$, let P_i be the peripheral subgroup of $\mathrm{PSL}(2, O_d)$ fixing the cusp c_i , set $P_i(I) = P_i \cap \Gamma(I)$ to be the peripheral subgroup of $\Gamma(I)$ fixing c_i , and let $N_d(I)$ denote the normal closure in $\mathrm{PSL}(2, O_d)$ of $\{P_1(I), \dots, P_{h_d}(I)\}$. Note that $N_d(I) < \Gamma(I)$ since $\Gamma(I)$ is a normal subgroup of $\mathrm{PSL}(2, O_d)$. Both M and $\mathbb{H}^3/N_d(I)$ are covering spaces of the Bianchi orbifold Q_d with the covering groups given by $\mathrm{PSL}(2, O_d/I)$, and $\mathrm{PSL}(2, O_d)/N_d(I)$ respectively. Now $\Gamma(I)$ will be generated by parabolic

elements if and only if $|\mathrm{PSL}(2, O_d)/N_d(I)| = |\mathrm{PSL}(2, O_d/I)|$. Since $|\mathrm{PSL}(2, O_d/I)|$ can easily be computed from a factorization of I (see [BR17, Section 2.1]), this reduces the problem to determining $|\mathrm{PSL}(2, O_d)/N_d(I)|$.

For many small values of d , finite presentations of the group $\mathrm{PSL}(2, O_d)$ together with the matrices corresponding to the generators were computed by Swan [Swa71]. More recently, for the remaining values of d , Page [Pag15] computed such presentations (see [BR17]). If we add the words representing the generators of each $P_j(I)$ as relations to the finite presentation of $\mathrm{PSL}(2, O_d)$, we have a finite presentation for $\mathrm{PSL}(2, O_d)/N_d(I)$. From this finite presentation, we can use various techniques from computer algebra to compute (a lower bound for) the size of $\mathrm{PSL}(2, O_d)/N_d(I)$.

Often, however, it is sufficient (but not always easier) to compute or estimate the homology of M itself to prove that it is not a link complement. In fact, this suffices for all but the three cases $(1, 4 + 3\sqrt{-1})$, $(2, 1 + 3\sqrt{-2})$, and $(3, (11 + \sqrt{-3})/2)$ where $H_1(M; \mathbb{Z})/\iota_*(H_1(\partial M; \mathbb{Z}))$ is trivial. More recently, the second author wrote a computer program to compute and triangulate a Dirichlet domain for a Bianchi orbifold Q_d and construct covers of Q_d to generate a triangulation of the principal congruence manifold M , respectively, the congruence manifold associated to the upper unit-triangular matrices in $\mathrm{PSL}(2, O_d/I)$. Using this program and the fact that $H_1(M; \mathbb{Z})/\iota_*(H_1(\partial M; \mathbb{Z}))$ cannot vanish for a cover $M \rightarrow N$ of degree less than $|H_1(N; \mathbb{Z})/\iota_*(H_1(\partial N; \mathbb{Z}))|$, it is feasible to compute integral homology for enough congruence manifolds to rule out all but the aforementioned three cases. Further discussion of this program, as well as the output of these computations is contained in [BGR18].

2.3. Proving a case to be a link complement. By Perelman's resolution of the Geometrization Conjecture, to prove that $M = \mathbb{H}^3/\Gamma(I)$ is homeomorphic to a link complement in S^3 , it is sufficient to find Dehn fillings of the manifold M trivializing the fundamental group. Thus the task is to find a collection of slopes (essential simple closed curves), one from each cusp, so that killing these words trivializes the fundamental group. The first and third author did this by finding an explicit description of cusp subgroup $P_j(I)$ for $j = 1, \dots, h_d$. The second author used the computer program mentioned above to generate a triangulation of M and then used SnapPy to find the slopes (using techniques similar to those already described in [Goe15, Section 7.3.2]) for which SnapPy [CDGW17] could then prove that the fundamental group of the Dehn-filled manifold along those slopes is trivial. Further details are to be found in [BGR18].

3. THE REMAINING CASES

We now review how our previous work using the methods of §2 reduces the classification of principal congruence link groups to the cases in Theorem 1.2.

In the case of $h_d > 1$, the complete list of the 16 pairs (d, I) corresponding to principal congruence link complements was determined in [BR17]. The possible values of d are $d \in \{5, 15, 23, 31, 47, 71\}$ with the levels shown in Theorem 1.1.

Concerning the case when $h_d = 1$ (i.e. $d \in \{1, 2, 3, 7, 11, 19\}$), certain examples already existed in the literature (see [BR14]), and using the 6-Theorem as described in §2 to restrict the possible levels, we subsequently gave 9 new examples of principal congruence link groups in [BR14]. This brought the total known when $h_d = 1$ to be 18:

- (1) $d = 1$: $I = \langle 2 \rangle, \langle 2 \pm i \rangle, \langle (1 \pm i)^3 \rangle, \langle 3 \rangle$.
- (2) $d = 2$: $I = \langle 1 \pm \sqrt{-2} \rangle, \langle 2 \rangle, \langle 2 \pm 2\sqrt{-2} \rangle$.
- (3) $d = 3$: $I = \langle 2 \rangle, \langle 3 \rangle, \langle (5 \pm \sqrt{-3})/2 \rangle, \langle 3 \pm \sqrt{-3} \rangle$.
- (4) $d = 7$: $I = \langle (1 \pm \sqrt{-7})/2 \rangle, \langle 2 \rangle, \langle (3 \pm \sqrt{-7})/2 \rangle, \langle 1 \pm \sqrt{-7} \rangle$.
- (5) $d = 11$: $I = \langle (1 \pm \sqrt{-11})/2 \rangle, \langle (3 \pm \sqrt{-11})/2 \rangle$.
- (6) $d = 19$: $I = \langle (1 \pm \sqrt{-19})/2 \rangle$.

Moreover, in the cases $d = 1, 3$, as well as identifying the cases described above, in [Goe15] the second author determined the complete list of pairs (d, I) that yield link groups; namely those above, together with:

- (1) $d = 1$: $I = \langle 3 \pm i \rangle, \langle 3 \pm 2i \rangle, \langle 4 \pm i \rangle$.
(2) $d = 3$: $I = \langle (7 \pm \sqrt{-3})/2 \rangle, \langle 4 \pm \sqrt{-3} \rangle, \langle (9 \pm \sqrt{-3})/2 \rangle$.

The upshot of these combined works is that 40 pairs (d, I) were determined that yield principal congruence link groups, and using a combination of the techniques described in §2, all remaining cases were eliminated except for the 8 pairs (d, I) stated in Theorem 1.2, and $(2, \langle 1 + 3\sqrt{-2} \rangle)$.

In Table 1 we provide some additional information associated to the 8 cases to be shown to be link groups: in the second, third, and fourth columns of Table 1, we list x a generator of the ideal being considered, its norm and the order of $\mathrm{PSL}(2, O_d)/\Gamma(I)$.

TABLE 1. The 8 cases in which we still need to prove that $\Gamma(\langle x \rangle)$ is a link group.

d	x	$N(\langle x \rangle)$	$ \mathrm{PSL}(2, O_d/\langle x \rangle) $	Number of cusps
2	$1 + 2\sqrt{-2}$	9	324	36
2	$3 + \sqrt{-2}$	11	660	60
7	$\sqrt{-7}$	7	168	24
7	$(5 + \sqrt{-7})/2$	8	192	24
7	$2 + \sqrt{-7}$	11	660	60
7	$(7 + \sqrt{-7})/2$	14	1008	72
7	$(1 + 3\sqrt{-7})/2$	16	1536	96
11	$(5 + \sqrt{-11})/2$	9	324	36

Details of the computations establishing that these do indeed give link groups are provided in [BGR18].

4. THE FINAL CASE

As mentioned in the Introduction, the final case required a technique different from the other cases to prove the finitely presented group $G = \mathrm{PSL}(2, O_d)/N_d(I)$ to be large enough. For the other cases, this could be shown in Magma [BCP97] or GAP [GAP18] either by computing $|G|$ itself or the Abelianization of G or a suitable subgroup of G . The necessary methods for this are based on Todd-Coxeter coset enumeration, Reidemeister-Schreier rewriting, and Smith form.

However, the final case was solved using the Monoid Automata Factory (MAF) which is a successor of the GAP package kbmag. We are very grateful to Alun Williams who helped us implement this.

Lemma 4.1. *The principal congruence manifold $\mathbb{H}^3/\Gamma(\langle 2, 1 + 3\sqrt{-2} \rangle)$ is not homeomorphic to a link complement in S^3 .*

Proof. From [Swa71], we have the following presentation for

$$\mathrm{PSL}(2, O_2) = \langle a, t, u \mid a^2 = (ta)^3 = (au^{-1}au)^2 = tut^{-1}u^{-1} = 1 \rangle$$

where

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad u = \begin{pmatrix} 1 & \sqrt{-2} \\ 0 & 1 \end{pmatrix}.$$

Following §2.2, we obtain t^6u^{-1} and t^{19} as the two parabolic elements that we take to normally generate $N_2(\langle 1 + 3\sqrt{-2} \rangle)$, giving us the following presentation

$$G = \frac{\mathrm{PSL}(2, O_2)}{N_2(\langle 1 + 3\sqrt{-2} \rangle)} = \langle a, t, u \mid a^2 = (ta)^3 = (au^{-1}au)^2 = tut^{-1}u^{-1} = t^6u^{-1} = t^{19} = 1 \rangle.$$

We give this presentation of G to MAF [Wil17] in form of a file `myGroup`:

```
_RWS := rec(
  isRWS := true,
  generatorOrder := [_g1, _g2, _g3, _g4, _g5],
  inverses :=      [_g1, _g3, _g2, _g5, _g4],
  ordering := "shortlex",
  equations := [
    [_g2*_g1*_g2, _g1*_g3*_g1],
    [_g1*_g5*_g1*_g4, _g5*_g1*_g4*_g1],
    [_g2*_g4, _g4*_g2],
    [_g2^4, _g4*_g3^2],
    [_g2^10, _g3^9] ]);
```

and then call (which takes about 2 hours of time on a MacBook pro with a 2.6Ghz Intel Core i5):

```
$ automata -no-kb myGroup      # find automatic structure
$ gpaxioms myGroup            # verify automatic structure for correct group
[...]
Checking relation _g2*_g4=_g4*_g2
Checking relation _g2^4=_g4*_g3^2
Checking relation _g2^10=_g3^9
Axiom check succeeded.
$ fsaccount myGroup.wa        # count words accepted by word acceptor
The accepted language is infinite
```

Since `automata` always finds a word acceptor automaton that accepts exactly one word for any group element, this proves G to be infinite. From the discussion in §2.2, we deduce that $\Gamma(\langle 1 + 3\sqrt{-2} \rangle)$ is not a link group. \square

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