NILPOTENT COMPLETIONS OF GROUPS, GROTHENDIECK PAIRS, AND FOUR PROBLEMS OF BAUMSLAG

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ABSTRACT. Two groups are said to have the same nilpotent genus if they have the same nilpotent quotients. We answer four questions of Baumslag concerning nilpotent completions. (i) There exists a pair of finitely generated, residually torsion-free-nilpotent groups of the same nilpotent genus such that one is finitely presented and the other is not. (ii) There exists a pair of finitely presented, residually torsion-free-nilpotent groups of the same nilpotent genus such that one has a solvable conjugacy problem and the other does not. (iii) There exists a pair of finitely generated, residually torsion-free-nilpotent groups of the same nilpotent genus such that one has finitely generated second homology $H_2(-,\mathbb{Z})$ and the other does not. (iv) A non-trivial normal subgroup of infinite index in a finitely generated parafree group cannot be finitely generated. In proving this last result, we establish that the first L^2-Betti number of a finitely generated parafree group in the same nilpotent genus as a free group of rank r is $r-1$. It follows that the reduced C^*-algebra of the group is simple if $r \geq 2$, and that a version of the Freiheitssatz holds for parafree groups.

1. INTRODUCTION

If each finite subset of a group Γ injects into some nilpotent (or finite) quotient of Γ, then it is reasonable to expect that one will be able to detect many properties of Γ from the totality of its nilpotent (or finite) quotients. Attempts to lend precision to this observation, and to test its limitations, have surfaced repeatedly in the study of discrete and profinite groups over the last forty years, and there has been a particular resurgence of interest recently, marked by several notable breakthroughs. We take up this theme here, with a focus on the nilpotent completions of residually torsion-free-nilpotent groups.

We begin by recalling some terminology. A group Γ is said to be *residually nilpotent* (resp. *residually torsion-free-nilpotent*) if for each non-trivial $\gamma \in \Gamma$ there exists a nilpotent group (resp. torsion-free-nilpotent group) Q and a homomorphism $\phi : \Gamma \to Q$ with $\phi(\gamma) \neq 1$. Thus Γ is residually nilpotent if and only if $\bigcap \Gamma_n = 1$, where Γ_n is the n-th term of the *lower central series* of Γ, defined inductively by setting $\Gamma_1 = \Gamma$ and defining $\Gamma_{n+1} = \langle [x,y] : x \in \Gamma_n, y \in \Gamma \rangle$.

We say that two residually nilpotent groups Γ and Λ have the same *nilpotent genus*[1] if they have the same lower central series quotients; i.e. $\Gamma/\Gamma_c \cong \Lambda/\Lambda_c$ for all $c \geq 1$.

Residually nilpotent groups with the same nilpotent genus as a free group are termed *parafree*. In [5] Gilbert Baumslag surveyed the state of the art concerning groups of the same nilpotent genus with particular emphasis on the nature of parafree groups. He concludes by listing a number of open problems that are of particular importance in the field. We will address Problems 2, 4 and 6 from his list [5], two of which he raised again in [6] where he emphasised the importance of the third of the problems described below.

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[1] Baumslag uses the simpler term "genus", but this conflicts with the usage of the term in the study of profinite groups, and since we study different completions it seems best to be more precise here.

Problem 1.1. *Does there exist a pair of finitely generated, residually torsion-free-nilpotent groups of the same nilpotent genus such that one is finitely presented and the other is not?*

Problem 1.2. *Does there exist a pair of finitely presented residually torsion-free-nilpotent groups of the same nilpotent genus such that one has a solvable conjugacy problem and the other does not?*

Problem 1.3. *Does there exist a pair of finitely generated, residually torsion-free-nilpotent groups of the same nilpotent genus such that one has finitely generated second homology $H_2(-,\mathbb{Z})$ and the other does not?*

Problem 1.4. *Let G be a finitely generated parafree group and let $N < G$ be a finitely generated, non-trivial, normal subgroup. Must N be of finite index in G?*

In [5] Baumslag gives some evidence to suggest that pairs of groups as in Problem 1.2 might exist; he had earlier proved that there exist finitely presented residually torsion-free nilpotent groups with unsolvable conjugacy problem [4]. Baumslag also proves a partial result in connection with Problem 1.4 (see Theorem 7 of [5]). Note that Problem 1.4 is well known to have a positive answer for free groups.

In this paper we answer these questions. Concerning the possible divergence in behaviour between groups in the same nilpotent genus, we prove:

Theorem A. (1) *There exist pairs of finitely generated, residually torsion-free-nilpotent groups $H \hookrightarrow D$ of the same nilpotent genus such that D is finitely presented and H is not.*
(2) *There exist pairs of finitely presented, residually torsion-free-nilpotent groups $P \hookrightarrow \Gamma$ of the same nilpotent genus such that Γ has a solvable conjugacy problem and P does not.*
(3) *There exist pairs of finitely generated, residually torsion-free-nilpotent groups $N \hookrightarrow \Gamma$ of the same nilpotent genus such that $H_2(\Gamma,\mathbb{Z})$ is finitely generated but $H_2(N,\mathbb{Z})$ is not.*

The following result strengthens items (1) and (3) of the above theorem.

Theorem B. *There exist pairs of finitely generated residually torsion-free-nilpotent groups $N \hookrightarrow \Gamma$ that have the same nilpotent genus and the same profinite completion, but Γ is finitely presented while $H_2(N,\mathbb{Q})$ is infinite dimensional.*

In the above theorems, the pairs of discrete groups that we construct are such that the inclusion map induces isomorphisms of both the profinite and pro-nilpotent completions. These results emphasise how divergent the behaviour can be within a nilpotent genus. Our solution to Problem 1.4, in contrast, establishes a commonality among parafree groups, and further commonalities are established in Section 8.

Theorem C. *Let G be a finitely generated parafree group and let $N < G$ be a non-trivial normal subgroup. If N is finitely generated, then G/N is finite.*

Theorem C is proved in Section 7. As with our other results, the proof exploits the theory of profinite groups. It also relies on results concerning the L^2-Betti numbers of discrete groups. A key observation here is that the first L^2-Betti number of a finitely presented residually torsion-free-nilpotent group is an invariant of its pro-p completion for an arbitrary prime p (see Corollary 7.6). Theorem C will emerge as a special case of a result concerning L^2-Betti numbers for dense subgroups of free pro-p groups (Theorem 7.7).

In Section 8 we discuss some implications of our results. The non-vanishing of the first L^2-Betti number for a parafree group in the same nilpotent genus as a free group of rank $r \geq 2$ is shown to have several important consequences, two of which are gathered in the following theorem (the second part of which was originally proved by Baumslag [3]). We also apply our results to the study of homology boundary links and prove that a non-free parafree group cannot be isomorphic to a lattice in a connected Lie group.

Theorem D. *Let G be a finitely generated parafree group in the same nilpotent genus as a free group of rank $r \geq 2$. Then the reduced C^*-algebra of G is simple, and every generating set for G contains an r-element subset that generates a free subgroup of rank r.*

The first part of the paper is organised as follows. In Section 2 we recall some basic properties about profinite and pro-nilpotent completions of discrete groups and about the correspondence between the subgroup structure of the discrete group and that of its various completions. In Section 3 we describe criteria for constructing distinct groups of the same nilpotent genus. In particular we prove that if a map of finitely generated discrete groups $P \hookrightarrow \Gamma$ induces an isomorphism of profinite completions, then P and Γ have the same nilpotent genus. (If ones assumes merely that $\widehat{P} \cong \widehat{\Gamma}$, then the nilpotent genus need not be the same.) Our proof of Theorem A involves the construction of carefully crafted pairs of residually torsion-free-nilpotent groups $u : P \hookrightarrow \Gamma$ such that $\widehat{u} : \widehat{P} \to \widehat{\Gamma}$ is an isomorphism. Theorem A(1) is proved in Section 5, and Theorem A(2) is proved in Section 4. The proof of Theorem B is more elaborate: it involves the construction of a finitely presented group with particular properties that may be of independent interest (Proposition 6.5).

In relation to Theorem A(1), we draw the reader's attention to a recent paper of Alexander Lubotzky [36] in which he constructs pairs of groups that have isomorphic profinite completions but have finiteness lengths that can be chosen arbitrarily. These examples are S-arithmetic groups over function fields; they do not have infinite proper quotients and therefore are not residually torsion-free nilpotent, but they are residually finite-nilpotent.

Commenting on an earlier version of this manuscript, Chuck Miller pointed out that a non-constructive solution to Problem 1.2 is implicit in his work with Baumslag on the isomorphism problem for residually torsion-free nilpotent groups [8]; see Remark 4.3. He also suggested an alternative proof of Theorem B; see Remark 6.6. We thank him for these comments. We also thank an anonymous referee whose comments improved Section 3.

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2. Profinite and Pro-Nilpotent Completions

Let Γ be a finitely generated group. If one orders the normal subgroups of finite index $N < \Gamma$ by reverse inclusion, then the quotients Γ/N form an inverse system whose inverse limit

$$\widehat{\Gamma} = \varprojlim \Gamma/N$$

is the *profinite completion* of Γ. Similarly, the *pro-nilpotent completion*, denoted $\widehat{\Gamma}_{\rm nil}$ is the inverse limit of the nilpotent quotients of Γ, and the *pro-(finite nilpotent)* completion, denoted $\widehat{\Gamma}_{\rm fn}$, is the inverse limit of the finite nilpotent quotients of Γ. Given a prime p, the the *pro-p completion* $\widehat{\Gamma}_p$ is the inverse limit of the finite p-group quotients of Γ.

Remarks 2.1. (1) The inverse limit topology on $\widehat{\Gamma}$, on $\widehat{\Gamma}_p$, and on $\widehat{\Gamma}_{\rm fn}$ makes them *compact* topological groups. The induced topologies on Γ are called the profinite, pro-p and *pro-(finite nilpotent)* topologies, respectively.

(2) We do not view $\widehat{\Gamma}_{\rm nil}$ as a topological group, and the absence of a useful topology makes it a less interesting object.

(3) By construction, $\widehat{\Gamma}$ (resp. $\widehat{\Gamma}_{\rm nil}$) is residually finite (resp. nilpotent), while $\widehat{\Gamma}_{\rm fn}$ is residually finite-nilpotent, and $\widehat{\Gamma}_p$ is residually p.

(4) The natural homomorphism $\Gamma \to \widehat{\Gamma}$ is injective if and only if Γ is residually finite, while the natural maps $\Gamma \to \widehat{\Gamma}_{\rm nil}$ and $\Gamma \to \widehat{\Gamma}_{\rm fn}$ are injective if and only if Γ is residually nilpotent.

(5) $\Gamma \to \widehat{\Gamma}_{\rm nil}$ is an isomorphism if and only if Γ is nilpotent.

(6) The quotients Γ/Γ_c of Γ by the terms of its lower central series are cofinal in the system of all nilpotent quotients, so one can equally define $\widehat{\Gamma}_{\rm nil}$ to be the inverse limit of these.
(7) If $\widehat{\Gamma}_{\rm nil} \cong \widehat{\Lambda}_{\rm nil}$ then $\widehat{\Gamma}_{\rm fn} \cong \widehat{\Lambda}_{\rm fn}$, but the converse is false in general since there exist finitely generated non-isomorphic nilpotent groups that have the same finite quotients [48].
(8) Less obviously, if Γ and Λ are finitely generated and have the same nilpotent genus then $\widehat{\Gamma}_{\rm fn} \cong \widehat{\Lambda}_{\rm fn}$. To see this note that having the same nilpotent genus is equivalent to the statement that Γ and Λ have the same nilpotent quotients, in particular the same finite nilpotent quotients. The conclusion $\widehat{\Gamma}_{\rm fn} \cong \widehat{\Lambda}_{\rm fn}$ follows from Theorem 3.2.7 of [49].
(9) Finite p-groups are nilpotent, so for every prime p there is a natural epimorphism $\widehat{\Gamma}_{\rm fn} \twoheadrightarrow \widehat{\Gamma}_p$. Finitely generated groups of the same nilpotent genus have isomorphic pro-p completions for all primes p.
(10) Every homomorphism of discrete groups $u : \Gamma \to \Lambda$ induces maps $\widehat{u} : \widehat{\Gamma} \to \widehat{\Lambda}$ and $\widehat{u}_{(p)} : \widehat{\Gamma}_p \to \widehat{\Lambda}_p$ and $\widehat{u}_{\rm nil} : \widehat{\Gamma}_{\rm nil} \to \widehat{\Lambda}_{\rm nil}$ and $\widehat{u}_{\rm fn} : \widehat{\Gamma}_{\rm fn} \to \widehat{\Lambda}_{\rm fn}$.

The image of the canonical map $\Gamma \to \widehat{\Gamma}$ is dense regardless of whether Γ is residually finite or not, so the restriction to Γ of any continuous epimomorphism from $\widehat{\Gamma}$ to a finite group is onto. A deep theorem of Nikolov and Segal [45] implies that if Γ is finitely generated then *every* homomorphism from $\widehat{\Gamma}$ to a finite group is continuous. And the universal property of $\widehat{\Gamma}$ ensures that every homomorphism from Γ to a finite group extends uniquely to $\widehat{\Gamma}$. Thus we have the following basic result in which $\mathrm{Hom}(\Gamma, Q)$ denotes the set of homomorphisms from the group Γ to the group Q, and $\mathrm{Epi}(\Gamma, Q)$ denotes the set of epimorphisms. If one replaces $\widehat{\Gamma}$ by $\widehat{\Gamma}_{\rm nil}$ or $\widehat{\Gamma}_{\rm fn}$ in the following lemma, one obtains bijections for finite nilpotent groups Q; and for $\widehat{\Gamma}_p$ one obtains bijections when Q is a finite p-group.

Lemma 2.2. *Let Γ be a finitely generated group and let $\iota : \Gamma \to \widehat{\Gamma}$ be the natural map to its profinite completion. Then, for every finite group Q, the map $\mathrm{Hom}(\widehat{\Gamma}, Q) \to \mathrm{Hom}(\Gamma, Q)$ defined by $g \mapsto g \circ \iota$ is a bijection, and this restricts to a bijection $\mathrm{Epi}(\widehat{\Gamma}, Q) \to \mathrm{Epi}(\Gamma, Q)$.*

Closely related to this, we have the following basic but important fact relating the subgroup structures of Γ and $\widehat{\Gamma}_{\rm fn}$ and $\widehat{\Gamma}_p$ (see [49] Proposition 3.2.2, and note that the argument is valid for other profinite completions).

Notation. Given a subset X of a pro-finite group G, we write $\overline{X}$ to denote the closure of X in G.

Proposition 2.3. *Let $\mathcal{C}$ be the class of finite nilpotent groups or finite p-groups (for a fixed prime p). If Γ is a finitely generated discrete group which is residually $\mathcal{C}$, then there is a one-to-one correspondence between the set $\mathcal{X}$ of subgroups of Γ that are open in the pro-$\mathcal{C}$ topology on Γ, and the set $\mathcal{Y}$ of all open subgroups in the pro-$\mathcal{C}$ completion of Γ. Identifying Γ with its image in the completion, this correspondence is given by:*

- *For $H \in \mathcal{X}$, $H \mapsto \overline{H}$.*
- *For $Y \in \mathcal{Y}$, $Y \mapsto Y \cap \Gamma$.*

If $H, K \in \mathcal{X}$ and $K < H$ then $[H : K] = [\overline{H} : \overline{K}]$. Moreover, $K \lhd H$ if and only if $\overline{K} \lhd \overline{H}$, and $\overline{H}/\overline{K} \cong H/K$.

Corollary 2.4. *Let Γ be a finitely generated group that is residually nilpotent, and for each $d \in \mathbb{N}$ let $M(d)$ denote the intersection of all the normal subgroups $\Delta \lhd \Gamma$ of index $\leq d$ such that Γ/Δ is nilpotent. Let $\overline{M(d)}$ be the closure of $M(d)$ in $\widehat{\Gamma}_{\rm fn}$. Then $\overline{M(d)}$ is the intersection of all the normal subgroups of index $\leq d$ in $\widehat{\Gamma}_{\rm fn}$, and hence $\bigcap_d \overline{M(d)} = 1$.*

Proof. We begin with a preliminary remark. It is proved in [2] that for a finitely generated group Γ, *every* subgroup of finite index in $\widehat{\Gamma}_{\rm fn}$ is open, and likewise in $\widehat{\Gamma}_p$. (See [45] for the case of a general profinite group.) Thus, by Proposition 2.3, every normal subgroup of index d in $\widehat{\Gamma}_{\rm fn}$ is the closure of a subgroup $K \lhd \Gamma$ such that Γ/K is nilpotent and $|\Gamma/K| = d$, and every subgroup of index $d = p^k$ in $\widehat{\Gamma}_p$ is the closure of a subgroup of index p^k in Γ.

In the light of this remark, it suffices to show that if $K_1, K_2 < \Gamma$ are normal and $Q_1 = \Gamma/K_1$ and $Q_2 = \Gamma/K_2$ are finite and nilpotent, then $\overline{K_1 \cap K_2} = \overline{K_1} \cap \overline{K_2}$. But $\overline{K_1 \cap K_2}$ is the kernel of the extension of $\Gamma \to Q_1 \times Q_2$ to $\widehat{\Gamma}_{\rm fn}$, while $\overline{K}_1 \times \overline{K}_2$ is the kernel of the map $\widehat{\Gamma}_{\rm fn} \to Q_1 \times Q_2$ that one gets by extending each of $\Gamma \to Q_i$ and then taking the direct product. These maps coincide on Γ, which is dense, and are therefore equal. □

The same argument establishes:

Corollary 2.5. *Let p be a prime and let Γ be a finitely generated group that is residually p. For each $d \in \mathbb{N}$ let $N(d)$ denote the intersection of all the normal subgroups $\Delta \lhd \Gamma$ of p-power index less than d. Let $\overline{N(d)}$ be the closure of $N(d)$ in $\widehat{\Gamma}_{\rm fn}$. Then $\overline{N(d)}$ is the intersection of all the normal subgroups of index $\leq d$ in $\widehat{\Gamma}_{\rm fn}$, and hence $\bigcap_d \overline{N(d)} = 1$.*

Remark 2.6. A key feature of the subgroups $M(d)$ [resp. $N(d)$] is that they form a fundamental system of open neighbourhoods of $1 \in \Gamma$ defining the pro-(finite nilpotent) [resp. pro-p] topology. If we had merely taken an exhausting sequence of normal subgroups in Γ, then we would not have been able to conclude that the intersection in $\widehat{\Gamma}_{\rm fn}$ [resp. $\widehat{\Gamma}_p$] of their closures was trivial.

2.1. **Subgroups of p-Power Index.** An advantage of the class of p-groups over the class of finite nilpotent groups is that the former is closed under group extensions whereas the latter is not. The importance of this fact from our point of view is that it means that the induced topology on normal subgroups of p-power index $\Lambda < \Gamma$ behaves well. The following is a consequence of Lemma 3.1.4(a) of [49] (in the case where $\mathcal{C}$ is the class of finite p-groups).

Lemma 2.7. *Let p be a prime and let Γ be a finitely generated group that is residually-p. If $\Lambda < \Gamma$ is a normal subgroup of p-power index, then the natural map $\widehat{\Lambda}_p \to \overline{\Lambda} < \widehat{\Gamma}_p$ is an isomorphism.*

2.2. **Betti numbers.** The first Betti number of a finitely generated group is

$$b_1(\Gamma) = \dim_{\mathbb{Q}}\ [(\Gamma/[\Gamma,\Gamma]) \otimes_{\mathbb{Z}} \mathbb{Q}].$$

Given any prime p, one can detect $b_1(\Gamma)$ in the p-group quotients of Γ, since it is the *greatest integer b such that Γ surjects $(\mathbb{Z}/p^k\mathbb{Z})^b$ for every $k \in \mathbb{N}$*. We exploit this observation as follows:

Lemma 2.8. *Let Λ and Γ be finitely generated groups and let p be a prime. If Λ is isomorphic to a dense subgroup of $\widehat{\Gamma}_p$, then $b_1(\Lambda) \geq b_1(\Gamma)$.*

Proof. For every finite p-group A, each epimorphism $\widehat{\Gamma} \to A$ will restrict to an epimorphism on both Γ and Λ (since by density Λ cannot be contained in a proper closed subgroup). But the resulting map $\text{Epi}(\widehat{\Gamma}, A) \to \text{Epi}(\Lambda, A)$ need not be surjective, in contrast to Lemma 2.2. Thus if Γ surjects $(\mathbb{Z}/p^k\mathbb{Z})^b$ then so does Λ (but perhaps not vice versa). □

Corollary 2.9. *If Λ and Γ are finitely generated and $\widehat{\Lambda}_p \cong \widehat{\Gamma}_p$, then $b_1(\Lambda) = b_1(\Gamma)$.*

3. Criteria for pro-nilpotent equivalence

The proof of the following proposition uses the Lyndon-Hochschild-Serre (LHS) spectral sequence, which is explained on page 171 of [22]. Given a short exact sequence of groups $1 \to N \to G \to Q \to 1$, the LHS spectral sequence calculates the homology of G in terms of the homology of N and of Q. The terms on the E^2 page of the spectral sequence are $E^2_{pq} = H_p(Q, H_q(N,\mathbb{Z}))$, where the action of Q on $H_*(N,\mathbb{Z})$ is induced by the action of G on N by conjugation. The terms on the diagonal $p+q=n$ of the E^∞ page are the composition factors of a series for $H_n(G,\mathbb{Z})$.

Proposition 3.1. *Let $1 \to N \overset{u}{\to} \Gamma \to Q \to 1$ be a short exact sequence of groups and let $u_c : N/N_c \to \Gamma/\Gamma_c$ be the homomorphism induced by $u : N \hookrightarrow \Gamma$. Suppose that N is finitely generated, that Q has no non-trivial finite quotients, and that $H_2(Q,\mathbb{Z}) = 0$. Then u_c is an isomorphism for all $c \geq 1$, and hence $\widehat{u}_{\rm nil} : \widehat{N}_{\rm nil} \to \widehat{\Gamma}_{\rm nil}$ is an isomorphism. In particular, if Γ is residually nilpotent then N and Γ have the same nilpotent genus.*

Proof. From the lower left corner of the LHS spectral sequence one obtains the following 5-term exact sequence (see [24] page 328):

$$H_2(Q,\mathbb{Z}) \to H_0(Q, H_1(N,\mathbb{Z})) \to H_1(\Gamma,\mathbb{Z}) \to H_1(Q,\mathbb{Z}) \to 0.$$

We have assumed that $H_2(Q,\mathbb{Z}) = H_1(Q,\mathbb{Z}) = 0$, so the second arrow $H_0(Q, H_1(N,\mathbb{Z})) \to H_1(\Gamma,\mathbb{Z})$ is an isomorphism.

By definition, $H_0(Q, H_1(N,\mathbb{Z}))$ is the quotient of $H_1(N,\mathbb{Z})$ by the action of Q. But since $H := H_1(N,\mathbb{Z})$ is a finitely generated abelian group, $\mathrm{Aut}(H)$ is residually finite, whereas Q has no non-trivial finite quotients. Thus, the action of Q on H is trivial and the composition $H_1(N,\mathbb{Z}) \to H_0(Q, H_1(N,\mathbb{Z})) \to H_1(\Gamma,\mathbb{Z})$ (which is the map on $H_1(-,\mathbb{Z})$ induced by $u : N \to \Gamma$) is an isomorphism. Moreover, as Q is perfect and $H_1(N,\mathbb{Z})$ is a trivial Q-module, $H_1(Q, H_1(N,\mathbb{Z})) = 0$. And since $E^2_{2,0} = H_2(Q, H_0(N,\mathbb{Z})) = H_2(Q,\mathbb{Z})$ is also assumed to be zero, the spectral sequence has only one term $E^2_{p,q}$ with $p + q = 2$ that might be non-zero, namely $E^2_{0,2} = H_0(Q, H_2(N,\mathbb{Z}))$. It follows that the composition $H_2(N,\mathbb{Z}) \to H_0(Q, H_2(N,\mathbb{Z})) \to E^\infty_{0,2} \to H_2(\Gamma,\mathbb{Z})$ (which is the map on $H_2(-,\mathbb{Z})$ induced by $u : N \to \Gamma$) is an epimorphism.

Stallings [51] proved that if a homomorphism of groups $u : N \to \Gamma$ induces an isomorphism on $H_1(-,\mathbb{Z})$ and an epimorphism on $H_2(-,\mathbb{Z})$, then $u_c : N/N_c \to \Gamma/\Gamma_c$ is an isomorphism for all $c \geq 1$. □

We remind the reader that the *fibre product* $P < \Gamma \times \Gamma$ associated to an epimorphism of groups $p : \Gamma \twoheadrightarrow Q$ is the subgroup $P = \{(x, y) \mid p(x) = p(y)\}$.

Corollary 3.2. *Under the hypotheses of Proposition 3.1, the inclusion $P \hookrightarrow \Gamma \times \Gamma$ of the fibre product induces an isomorphism $P/P_c \to \Gamma/\Gamma_c \times \Gamma/\Gamma_c$ for every $c \geq 1$, and hence an isomorphism $\widehat{P}_{\rm nil} \to \widehat{\Gamma}_{\rm nil} \times \widehat{\Gamma}_{\rm nil}$.*

Proof. We have inclusions $N \times N \overset{i}{\to} P \overset{j}{\to} \Gamma \times \Gamma$. The quotient of P by $N \times N$ is Q and the quotient of $\Gamma \times \Gamma$ by $N \times N$ is $Q \times Q$. The Künneth formula tells us that $H_2(Q \times Q,\mathbb{Z}) = 0$, and it is obvious that $Q \times Q$ has no finite quotients. Thus, by the proposition, both i and $j \circ i$ induce isomorphisms modulo any term of the lower central series, and therefore j does as well. □

3.1. **Grothendieck Pairs and Pro-Nilpotent completions.** The criterion established in Proposition 3.1 will be enough for the demands of Theorems A and B but we should point out that (with different finiteness assumptions) one can prove a stronger result, namely that $N \hookrightarrow \Gamma$ and $P \hookrightarrow \Gamma \times \Gamma$ induce isomorphisms of profinite completions. The purpose of this subsection is to expand on this remark and explain why the statement about profinite completions is indeed stronger (cf. Remark 3.5).

Let Γ be a residually finite group and let $u : P \hookrightarrow \Gamma$ be the inclusion of a subgroup P. Then $(\Gamma, P)_u$ is called a *Grothendieck Pair* if the induced homomorphism $\widehat{u} : \widehat{P} \to \widehat{\Gamma}$ is an isomorphism but u is not. (When no confusion is likely to arise, it is usual to write (Γ, P) rather than $(\Gamma, P)_u$.) Grothendieck [28] asked about the existence of such pairs of finitely presented groups and the first such pairs were constructed by Bridson and Grunewald in [17]. The analogous problem for finitely generated groups had been settled earlier by Platonov and Tavgen [46]. Both constructions rely on versions of the following result (cf. [46], [17] Theorem 5.1 and [14]).

Proposition 3.3. *Let $1 \to N \to \Gamma \to Q \to 1$ be a short exact sequence of groups with Γ finitely generated and let P be the associated fibre product. Suppose that $Q \neq 1$ is finitely presented, has no proper subgroups of finite index, and $H_2(Q, \mathbb{Z}) = 0$. Then*

(1) *$(\Gamma \times \Gamma, P)$ is a Grothendieck pair;*
(2) *if N is finitely generated then (Γ, N) is a Grothendieck pair.*

Grothendieck pairs give rise to pro-nilpotent equivalences:

Proposition 3.4. *Let $u : P \hookrightarrow \Gamma$ be a pair of finitely generated, residually finite groups, and for each $c \geq 1$, let $u_c : P/P_c \to \Gamma/\Gamma_c$ be the induced homomorphism. If $(\Gamma, P)_u$ is a Grothendieck pair, then u_c is an isomorphism for all $c \geq 1$, and hence $\widehat{u}_{\rm nil} : \widehat{P}_{\rm nil} \to \widehat{\Gamma}_{\rm nil}$ is an isomorphism. In particular, if P and Γ are residually nilpotent, then they have the same nilpotent genus.*

Before presenting the proof of this proposition we make a remark and establish a lemma.

Remark 3.5. In this proposition, it is vital that the isomorphism between $\widehat{P}$ and $\widehat{\Gamma}$ is induced by a map $P \to \Gamma$ of discrete groups. For example, as we remarked earlier, there exist finitely generated nilpotent groups that are not isomorphic but have the same profinite completion. A nilpotent group is its own pro-nilpotent completion, so these examples have the same profinite genus but different nilpotent genera.

Lemma 3.6. *Let $u : P \hookrightarrow \Gamma$ be a pair of finitely generated, residually finite groups. If $(\Gamma, P)_u$ is a Grothendieck pair, then for every finite group G, the map $q \mapsto q \circ u$ defines a bijection $\mathrm{Epi}(\Gamma, G) \to \mathrm{Epi}(P, G)$.*

Proof. As in Lemma 2.2, by restricting homomorphisms $\widehat{\Gamma} \to G$ to $\Gamma < \widehat{\Gamma}$ we obtain a bijection $\mathrm{Epi}(\widehat{\Gamma}, G) \to \mathrm{Epi}(\Gamma, G)$. Similarly, there is a bijection $\mathrm{Epi}(\widehat{P}, G) \to \mathrm{Epi}(P, G)$. And the isomorphism $\widehat{u}$ induces a bijection $\mathrm{Epi}(\widehat{\Gamma}, G) \to \mathrm{Epi}(\widehat{P}, G)$. Given $q \in \mathrm{Epi}(\Gamma, G)$, the map $q \mapsto q \circ u$ described in Lemma 2.2 completes the commutative square

$$\begin{array}{ccc} \mathrm{Epi}(\Gamma, G) & \longrightarrow & \mathrm{Epi}(P, G) \\ \uparrow & & \uparrow \\ \mathrm{Epi}(\widehat{\Gamma}, G) & \longrightarrow & \mathrm{Epi}(\widehat{P}, G) \end{array}$$

and hence is a bijection. □

Proof of Proposition 3.4: If G is nilpotent of class c, and H is any group, then every homomorphism from H to G factors uniquely through H/H_c and hence there is a natural bijection $\mathrm{Epi}(H/H_c, G) \to \mathrm{Epi}(H, G)$. By combining two such bijections with the epimorphism $q \mapsto q \circ u$ of Lemma 3.6,

$$\mathrm{Epi}(\Gamma/\Gamma_c, G) \to \mathrm{Epi}(\Gamma, G) \to \mathrm{Epi}(P, G) \to \mathrm{Epi}(P/P_c, G),$$

we see that if G is finite, then $q \mapsto q \circ u_c$ defines a bijection $\mathrm{Epi}(\Gamma/\Gamma_c, G) \to \mathrm{Epi}(P/P_c, G)$.

Finitely generated nilpotent groups are residually finite, so for every $c > 0$ and every non-trivial element $\gamma \in P/P_c$, there is an epimorphism $\pi : P/P_c \to G$ to a finite (nilpotent) group such that $\pi(\gamma) \neq 1$. The preceding argument provides $q \in \mathrm{Epi}(\Gamma/\Gamma_c, G)$ such that $\pi = q \circ u_c$, whence $u_c(\gamma) \neq 1$. Thus u_c is injective.

u_c is also surjective, for if $\gamma \in \Gamma/\Gamma_c$ were not in the image then using the subgroup separability of nilpotent groups [41], we would have an epimorphism $q : \Gamma/\Gamma_c \to G$ to some finite group such that $q(\gamma) \notin q \circ u_c(P/P_c)$, contradicting the fact that $q \circ u_c$ is an epimorphism. □

4. Conjugacy Problem: A Solution to Problem 1.2

Recall that a finitely generated group G is said to have a *solvable conjugacy problem* if there is an algorithm that, given any pair of words in the generators, can correctly determine whether or not these words define conjugate elements of the group. If no such algorithm exists then one says that the group has an unsolvable conjugacy problem.

In the light of Proposition 3.4, the following theorem settles Problem 1.2 (cf. Remark 4.3). This theorem will be proved by combining the techniques of [15] with recent advances in the understanding of non-positively curved cube complexes and right-angled Artin groups (RAAGs). We remind the reader that a RAAG is a group with a finite presentation of the form

$$A = \langle a_1, \ldots, a_n \mid [a_i, a_j] = 1 \forall (i,j) \in E \rangle.$$

Much is known about such groups. For our purposes here, their most important feature is that they are residually torsion-free-nilpotent ([26] Theorem 2.1).

Theorem 4.1. *There exist pairs of finitely presented, residually torsion-free-nilpotent groups $u : P \hookrightarrow \Gamma$ such that u induces isomorphisms of pro-nilpotent and profinite completions, but Γ has a solvable conjugacy problem while P has an unsolvable conjugacy problem.*

Proof. Let $1 \to N \to H \xrightarrow{p} Q \to 1$ be a short exact sequence of groups and let $P < H \times H$ be the associated fibre product. The 1-2-3 Theorem [7] states that if N is finitely generated, H is finitely presented, and Q has a classifying space with a finite 3-skeleton, then the P is finitely presented [7].

A further result from [7] states that if H is torsion-free and hyperbolic, and Q has an unsolvable word problem, then $P < H \times H$ will have an unsolvable conjugacy problem. On the other hand, the conjugacy problem in $H \times H$ is always solvable in polynomial time (cf. [15] §1.5).

Using ideas from [25], it is proved in [15] that there exist groups Q with no finite quotients that are of type F_3, have $H_2(Q, \mathbb{Z}) = 0$ and are such that the word problem in Q is unsolvable.

Combining the conclusion of the preceding three paragraphs with Proposition 3.3(1) (or Proposition 3.1 in the pro-nilpotent case), we see that the theorem would be proved if, given a finitely presented group Q with the properties described above, we could construct a short exact sequence $1 \to N \to H \to Q \to 1$ with N finitely generated and with H hyperbolic and residually torsion-free-nilpotent: for then, $P \hookrightarrow H \times H$ would be the required pair of groups.

In [50] Rips describes an algorithm that, given a finite presentation of any group Q will construct a short exact sequence $1 \to K \to H \xrightarrow{p} Q \to 1$ where K is finitely generated and H is hyperbolic. There are many refinements of Rips's construction in the literature. Haglund and Wise [32] proved a version in which H is constructed to be *virtually special*. By definition, a virtually special group H has a subgroup of finite index $H_0 < H$ that is a subgroup of a RAAG [32], and as remarked upon above, RAAGs are residually torsion-free-nilpotent by [26]. Consider the short exact sequence $1 \to K \cap H_0 \to H_0 \to p(H_0) \to 1$. If we take Q to be as in third paragraph of the proof, then $p(H_0) = Q$, because Q has no proper subgroups of finite index. And $N = K \cap H_0$, being of finite index in K, is finitely generated. Thus we have constructed a short exact sequence $1 \to N \to H_0 \to Q \to 1$ of the required form. □

Remarks 4.2. (1) In the preceding proof we quoted Haglund and Wise's version of the Rips construction; this was used to ensure that the group H was virtually special and hence virtually residually torsion-free-nilpotent. Less directly, it follows from recent work of Agol [1] and Wise [53] that the groups produced by the original Rips construction [50] are also virtually special.

(2) If one is willing to reduce the finiteness properties in the above theorem then one can simplify the proof: it is easy to prove that if Q has an unsolvable word problem and $p : H \to Q$ is an epimorphism from a hyperbolic group with $\ker p$ finitely generated, then the conjugacy problem in $\ker p$ is unsolvable. So if $1 \to N \to H \to Q \to 1$ is constructed as in the above

proof, then (H, N) is a Grothendieck pair of groups in the same nilpotent genus, but H is hyperbolic and N has an unsolvable conjugacy problem.

Remark 4.3. Commenting on an earlier version of this manuscript, Chuck Miller pointed out to us that his work with Baumslag on the isomorphism problem for residually torsion-free nilpotent groups [8] contains an implicit solution to Problem 1.1. To prove the main theorem in that paper, the authors construct a sequence of finite presentations for residually torsion-free nilpotent groups G_w indexed by words in the generators of an auxillary group H that has an unsolvable word problem. G_w is isomorphic to G_1 if and only if $w = 1$ in H. If $w \neq 1$ in H then G_w has unsolvable conjugacy problem, whereas G_1 has a solvable conjugacy problem. It follows that some G_w with $w \neq 1$ must have the same nilpotent genus as G_1, for if not then one could decide which G_w were isomorphic to G_1 be running the following algorithm.

Given a finite presentation of Γ_w, use the solution to the isomorphism problem for nilpotent groups to test if the quotients of G_w by the terms of its lower central series are isomorphic to the corresponding quotients of G_1. (Explicit presentations for these quotients are obtained by simply adding basic commutators to the given presentations of G_w and G_1.) If the nilpotent genus of G_w is different from that of G_1, this process will eventually find non-isomorphic quotients. At the same time, search naively for an isomorphism from G_w to G_1. These parallel processes will together determine whether or not G_w is isomorphic to G_1 *unless* G_w and G_1 are non-isomorphic groups of the same nilpotent genus.

Notice that although this argument settles Problem 1.2, it does not prove Theorem A(2): it does not construct an explicit pair of groups G_1 and $G_{w \neq 1}$ with the same nilpotent quotients, it just proves that such pairs exist. Moreover, there is no homomorphism between the discrete groups inducing the isomorphism of pro-nilpotent completions.

5. Finite Presentation: A Solution to Problem 1.1

We present two constructions settling Problem 1.1. Both rely on the existence of finitely presented infinite groups Q with $H_2(Q, \mathbb{Z}) = 0$ that have no non-trivial finite quotients. A general method for constructing such groups is described in [17]. The first such group was discovered by Graham Higman:

$$Q = \langle a, b, c, d \mid bab^{-1} = a^2,\ cbc^{-1} = b^2,\ dcd^{-1} = c^2,\ ada^{-1} = d^2 \rangle.$$

5.1. **Subdirect products of free groups.**

Finitely generated free groups are residually torsion-free nilpotent [42], and hence so are subgroups of their direct products. Thus the following proposition resolves Problem 1.1.

Proposition 5.1. *Let Q be a finitely presented infinite group with $H_2(Q, \mathbb{Z}) = 0$ that has no finite quotients. Let F be a finitely generated free group, let $F \to Q$ be a surjection, and let $P < F \times F$ be the associated fibre product. Then:*

(1) *P is finitely generated but not finitely presented;*
(2) *$P \hookrightarrow F \times F$ induces an isomorphism of pro-nilpotent completions.*

Proof. Let F be the free group on $\{x_1, \ldots, x_n\}$. Since Q is finitely presented, the kernel of $F \to Q$ is the normal closure of a finite set $\{r_1, \ldots, r_m\}$. It is easy to check that $P < F \times F$ is generated by $\{(x_1, x_1), \ldots, (x_n, x_n), (r_1, 1), \ldots, (r_m, 1)\}$. But Grunewald [30] proves that P is finitely presentable if and only if Q is finite. Assertion (2) is a special case of Corollary 3.2. □

Remark 5.2. Platonov and Tavgen [46] proved that $P \hookrightarrow F \times F$ also induces an isomorphism of profinite completions.

5.2. **A solution via the Rips construction.** Again, let Q be a finitely presented infinite group with $H_2(Q,\mathbb{Z})=0$ that has no non-trivial finite quotients. In the proof of Theorem 4.1 we used a version of the Rips construction to obtain a short exact sequence $1\to N\to H\to Q\to 1$ with H hyperbolic (in particular finitely presented) and residually torsion-free nilpotent, and with N finitely generated and not free. A further property of the Rips construction is that H is small cancellation, in particular of cohomological dimension 2. Bieri [10] proved that a finitely presented normal subgroup of infinite index in a group of cohomological dimension 2 must be free. Thus if Q is infinite then N is not finitely presented. But Proposition 3.1 tells us that N and Γ have the same nilpotent genus (and Proposition 3.3 tells us that (N,Γ) is a Grothendieck pair).

6. Nilpotent Genus and the Schur Multiplier: Problem 1.3

In the previous section we settled the question of whether there exist finitely generated residually torsion-free-nilpotent groups of the same genus such that one is finitely presented and the other was not. In [6] Baumslag laid particular emphasis on a homological variant of this question: is there is a pair of finitely generated residually torsion-free-nilpotent groups of the same genus such that the Schur multiplier $H_2(G,\mathbb{Z})$ of one is finitely generated, while that of the other group is not? We shall settle this question by proving the following theorem. Here, and in what follows, we take homology with coefficients in $\mathbb{Q}$. When we discuss the *dimension* of a homology group, we mean its dimension as $\mathbb{Q}$-vector space. Note that if $H_n(G,\mathbb{Z})$ is finitely generated then $H_n(G,\mathbb{Q})$ is finite dimensional.

Theorem 6.1. *There exists a pair of finitely generated residually torsion-free-nilpotent groups $N\hookrightarrow\Gamma$ that have the same nilpotent genus and the same profinite completion, but Γ is finitely presented while $H_2(N,\mathbb{Q})$ is infinite dimensional.*

Our proof of the above theorem draws on the ideas in previous sections, augmenting them with a spectral sequence argument to control the homology of N. The proof also relies on the construction of a group with particular properties that we regard as having independent interest – see Proposition 6.5.

6.1. **A spectral sequence argument.**

Proposition 6.2. *Let $1\to N\to G\to Q\to 1$ be a short exact sequence of finitely generated groups. If $H_3(G,\mathbb{Q})$ is finite dimensional but $H_3(Q,\mathbb{Q})$ is infinite dimensional, then $H_2(N,\mathbb{Q})$ is infinite dimensional.*

Proof. As in the proof of Proposition 3.1 we use the LHS spectral sequence to calculate the homology of G in terms of N and Q. The terms on the E^2 page of the spectral sequence are $E^2_{pq}=H_p(Q,H_q(N,\mathbb{Q}))$.

$H_3(Q,H_0(N,\mathbb{Q}))$ is infinite dimensional and $H_1(Q,H_1(N,\mathbb{Q}))$ is finite dimensional, so E^3_{30}, which is the kernel of $d_2:E^2_{30}\to E^2_{11}$, is infinite dimensional.

The kernel of $d_3:E^3_{30}\to E^3_{02}$ is E^∞_{30}, which is a section of $H_3(G,\mathbb{Q})$ and hence is finite dimensional. So the image of the map to E^3_{02} is infinite dimensional. But E^3_{02} is a quotient of $H_0(Q,H_2(N,\mathbb{Q}))$, which in turn is a quotient of $H_2(N,\mathbb{Q})$. Thus $H_2(N,\mathbb{Q})$ is infinite dimensional. □

6.2. **A designer group.** Recall that a group A is termed *acyclic* (over $\mathbb{Z}$) if $H_i(A,\mathbb{Z})=0$ for all $i\geq 1$. The Higman group described in Section 5 was the first example of a finitely presented acyclic group with no proper subgroups of finite index. Further examples were constructed in [17], including, for each integer $p\geq 3$,

$$\langle a_1,a_2,b_1,b_2 \mid a_1^{-1}a_2^pa_1a_2^{-p-1},\ b_1^{-1}b_2^pb_1b_2^{-p-1},\ a_1^{-1}[b_2,b_1^{-1}b_2b_1],\ b_1^{-1}[a_2,a_1^{-1}a_2a_1]\rangle.$$

Let A be one of the above groups. The salient features of A are that it is finitely presented, acyclic over $\mathbb{Z}$, has no finite quotients, contains a 2-generator free group, F say, and is torsion-free (indeed it has a 2-dimensional classifying space $K(A,1)$, cf. [17] p.364). Let $\Delta=(A\times A)*_S(A\times A)$

be the double of $A \times A$ along $S < F \times F$, where S is the first Stallings-Bieri group, i.e. the kernel of a homomorphism $F \times F \to \mathbb{Z}$ whose restriction to each of the factors is surjective. The key features of S are that it is finitely generated but $H_2(S, \mathbb{Q})$ is infinite dimensional (see [51], or [18] pp. 482-485).

Lemma 6.3. *Δ is torsion-free, finitely presented, has no non-trivial finite quotients, and $H_3(\Delta, \mathbb{Q})$ is infinite dimensional.*

Proof. The amalgamated free product of two finitely presented groups along a finitely generated subgroup is finitely presented, so Δ is finitely presented. And an amalgam of torsion-free groups is torsion-free. The four visible copies of A generate Δ, and these all have trivial image in any finite quotient, so Δ has no non-trivial finite quotients. We calculate $H_3(\Delta, \mathbb{Q})$ using the Mayer-Vietoris sequence (omitting the coefficient module $\mathbb{Q}$ from the notation):

$$\dots H_3(A \times A) \oplus H_3(A \times A) \to H_3(\Delta) \to H_2(S) \to H_2(A \times A) \oplus H_2(A \times A) \to \dots$$

As A is acyclic, so is $A \times A$, by the Künneth formula. Hence $H_3(\Delta, \mathbb{Q}) \cong H_2(S, \mathbb{Q})$ is infinite dimensional. □

Recall that a group G is termed *super-perfect* if $H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0$. Proposition 3.3 explains our interest in this condition. Δ is perfect but it is not super-perfect.

Lemma 6.4. $H_2(\Delta, \mathbb{Z}) \cong H_1(S, \mathbb{Z}) \cong \mathbb{Z}^3$.

Proof. A slight variant of the above Mayer-Vietoris argument shows that $H_2(\Delta, \mathbb{Z}) \cong H_1(S, \mathbb{Z})$.

The first homology of S can be calculated using Theorem A of [21]: if $G \leq F \times F$ is a subdirect product and we write $F_1 = F \times 1$ and $L = G \cap F_1$, then

$$H_1(G, \mathbb{Z}) \cong H_1(F, \mathbb{Z}) \oplus H_2(F_1/L, \mathbb{Z}) \oplus C,$$

where $C = \ker(H_1(F_1, \mathbb{Z}) \to H_1(F_1/L, \mathbb{Z}))$.

In our case, $G = S$ and $F_1/L = \mathbb{Z}$, so $H_2(F_1/L, \mathbb{Z}) = 0$ and C is cyclic. (Recall that $F \cong F_1$ is free of rank 2.) □

To obtain a super-perfect group, we pass to the *universal central extension* $\widetilde{\Delta}$.

Proposition 6.5. *$\tilde{\Delta}$ is torsion-free, finitely presented, super-perfect, has no non-trivial finite quotients, and $H_3(\tilde{\Delta}, \mathbb{Q})$ is infinite dimensional.*

Proof. The standard theory of universal central extensions (see [44], Chapter 5) tells us that $\tilde{\Delta}$ is perfect and that there is a short exact sequence

$$1 \to H_2(\Delta, \mathbb{Z}) \to \tilde{\Delta} \to \Delta \to 1.$$

Since Δ and $H_2(\Delta, \mathbb{Z})$ are finitely presented and torsion-free, so is $\tilde{\Delta}$. Since Δ has no non-trivial finite quotients, $H_2(\Delta, \mathbb{Z})$ would have to map onto any finite quotient of $\tilde{\Delta}$, which means that all such quotients are abelian. Since $\tilde{\Delta}$ is perfect, it follows that it has no non-trivial finite quotients (cf. [17], p.369).

To see that $H_3(\tilde{\Delta}, \mathbb{Q})$ is infinite dimensional, we consider the LHS spectral sequence associated to the above short exact sequence. As Δ is finitely presented and $K := H_2(\Delta, \mathbb{Z})$ is of type FP_∞, all of the groups in the first three columns, $E^2_{pq} = H_p(\Delta, H_q(K, \mathbb{Q}))$ with $0 \leq p \leq 2$, are finite dimensional. On the other hand, $E^2_{30} = H_3(\Delta, \mathbb{Q})$ is infinite dimensional. Therefore $E^3_{30} = \ker(\mathrm{E}^2_{30} \to \mathrm{E}^2_{11})$ and $E^\infty_{30} = E^4_{30} = \ker(\mathrm{E}^3_{30} \to \mathrm{E}^3_{02})$ are infinite dimensional. But E^∞_{30} is a quotient of $H_3(\tilde{\Delta}, \mathbb{Q})$, so $H_3(\tilde{\Delta}, \mathbb{Q})$ is also infinite dimensional. □

6.3. **Proof of Theorem 6.1.** We have constructed a group $\tilde{\Delta}$ that is super-perfect, finitely presented and has $H_3(\tilde{\Delta}, \mathbb{Q})$ infinite dimensional. By applying a suitable version of the Rips construction to $\tilde{\Delta}$ (as in the proof of Theorem 4.1), we obtain a short exact sequence

$$1 \to N \to H \to \tilde{\Delta} \to 1$$

with N finitely generated and H is a 2-dimensional hyperbolic group that is virtually special. Passing to a subgroup of finite index $H_0 < H$ and replacing N by $N \cap H_0$, we may assume that H is a subgroup of a RAAG, and hence is residually torsion-free-nilpotent. Propositions 3.1 and 3.3(2) tell us that $N \to H$ induces an isomorphism of pro-nilpotent and profinite completions. Proposition 6.2 tells us that $H_2(N, \mathbb{Q})$ is infinite dimensional. $\square$

Remark 6.6. We outline an alternative proof of Theorem 6.1 suggested by Chuck Miller. Let A be a group satisfying the conditions of Section 6.2. We fix a finite presentation $A = \langle X \mid R \rangle$ then augment if by adding a new generator x_0 and a new relation $x_0 = 1$. Let F be the free group on $X \cup \{x_0\}$, let $F \to A$ be the natural epimorphism, and let $P < F \times F$ be the associated fibre product. The results in Section 3 tell us that P and $F \times F$ have the same nilpotent quotients. Because $(x_0, 1) \in P$ is part of a free basis for $F \times \{1\}$, one can express P as an HNN extension with stable letter x_0 and amalgamated subgroup $P \cap (\{1\} \times F)$. Then, as in the proof of Theorem 4 of [43], one can use the Mayer-Vietoris sequence for the HNN extension to prove that $H_2(P, \mathbb{Z})$ maps onto the first homology of the infinitely generated free group $\ker(F \to A)$.

7. Normal subgroups of parafree groups

We settled Baumslag's first three questions by constructing groups of a somewhat pathological nature that lie in the same nilpotent genus as well-behaved groups. Our solution to Problem 1.4 is of an entirely different nature: the point here is to prove that parafree groups share a significant property with free groups. Correspondingly, the nature of the mathematics that we shall draw on is entirely different.

Our proof of the following theorem relies on a mix of L^2-Betti numbers and profinite group theory that we first employed in our paper [16] with M. Conder.

Theorem 7.1. *(=Theorem C) If Γ is a finitely generated parafree group, then every finitely generated non-trivial normal subgroup of Γ is of finite index.*

7.1. **L^2-Betti numbers.** The standard reference for this material is Lück's treatise [38]. In what follows $b_1(X)$ denotes the usual first Betti number of a group, and $b_1^{(2)}$ denotes the first L^2-Betti number. We shall not recall the definition of the L^2-Betti number as it does not inform our arguments. In the case of finitely presented groups, one can use *Lück's Approximation Theorem* [37] to give a surrogate definition of $b_1^{(2)}$: Suppose that Γ is *finitely presented* and let

$$\Gamma = N_1 > N_2 > \ldots > N_m > \ldots,$$

be a sequence of normal subgroups, each of finite index in Γ, with $\bigcap_m N_m = 1$; Lück proves that

$$\lim_{m\to\infty} \frac{b_1(N_m)}{[\Gamma : N_m]} = b_1^{(2)}(\Gamma).$$

Example 7.2. Let F be a free group of rank r. Euler characteristic tells us that a subgroup of index d in F is free of rank $d(r-1)+1$, so by Lück's Theorem $b_1^{(2)}(F_r) = r-1$. A similar calculation shows that if Σ is the fundamental group of a closed surface of genus g, then $b_1^{(2)}(\Sigma) = 2g-2$.

If one assumes only that the group Γ is *finitely generated*, then one does not know if the above limit exists, and when it does exist one does not know if it is independent of the chosen tower of subgroups. This is a problem in the context of Theorem 7.1 because we do not know if finitely generated parafree groups are finitely presentable. Thus we appeal instead to the weaker form of Lück's approximation theorem established for finitely generated groups by Lück and Osin [39].

Theorem 7.3. *If Γ is a finitely generated residually finite group and (N_m) is a sequence of finite-index normal subgroups with $\bigcap_m N_m = 1$, then*

$$\limsup_{m\to\infty} \frac{b_1(N_m)}{[\Gamma : N_m]} \leq b_1^{(2)}(\Gamma).$$

Another result about L^2-Betti numbers that we will make use of is the following theorem of Gaboriau (see [27] Theorem 6.8).

Theorem 7.4. *Suppose that*

$$1 \to N \to \Gamma \to \Lambda \to 1$$

is an exact sequence of groups where N and Λ are infinite. If $b_1^{(2)}(N) < \infty$, then $b_1^{(2)}(\Gamma) = 0$.

7.2. **L^2-Betti numbers of dense subgroups.** The key step in the proof of Theorem 7.1 is the following result (cf. [16] Proposition 3.2). Recall that we write $\widehat{\Gamma}_p$ to denote the pro-p completion of a group Γ, and $\overline{H}$ to denote the closure of a subgroup in $\widehat{\Gamma}_p$.

Proposition 7.5. *Let Γ be a finitely generated group and let F be a finitely presented group that is residually-p for some prime p. Suppose that there is an injection $\Gamma \hookrightarrow \widehat{F}_p$ and that $\overline{\Gamma} = \widehat{F}_p$. Then $b_1^{(2)}(\Gamma) \geq b_1^{(2)}(F)$.*

Proof. For each positive integer d let $N(d) < F$ be the intersection of all normal subgroups of p-power index at most d in F. Let $L(d) = \Gamma \cap \overline{N(d)} < \widehat{F}_p$. We saw in Corollary 2.5 that $\bigcap_d \overline{N}(d) = 1$, hence $\bigcap_d L(d) = 1$. Since Γ and F are both dense in $\widehat{F}_p$, the restriction of $\widehat{F}_p \to \widehat{F}_p/\overline{N(d)}$ to each of these subgroups is surjective, and therefore

$$[\Gamma : L(d)] = [\widehat{F}_p : \overline{N(d)}] = [F : N(d)].$$

$L(d)$ is dense in $\overline{N(d)}$ and in Lemma 2.7 we saw that $\widehat{N(d)}_p \cong \overline{N(d)}$, so Lemma 2.8 implies that $b_1(L(d)) \geq b_1(N(d))$. We use the towers $(L(d))$ in Γ and $(N(d))$ in F to compare L^2-betti numbers, applying Theorem 7.3 to the finitely generated group Γ and Lück's Approximation Theorem to the finitely presented group Γ:

$$b_1^{(2)}(\Gamma) \geq \limsup_{d\to\infty} \frac{b_1(L(d))}{[\Gamma : L(d)]} \geq \limsup_{d\to\infty} \frac{b_1(N(d))}{[F : N(d)]} = \lim_{d\to\infty} \frac{b_1(N(d))}{[F : N(d)]} = b_1^{(2)}(F)$$

as required. □

Corollary 7.6. *Let Λ and Γ be finitely presented groups that are residually-p for some prime p. If $\widehat{\Gamma}_p \cong \widehat{\Lambda}_p$ then $b_1^{(2)}(\Gamma) = b_1^{(2)}(\Lambda)$.*

By combining Proposition 7.5 with Theorem 7.4 we deduce:

Theorem 7.7. *Let Γ be a finitely generated group and let $N \lhd \Gamma$ be a non-trivial normal subgroup. Let F be a finitely presented group that is residually-p for some prime p and suppose that there is an injection $\Gamma \hookrightarrow \widehat{F}_p$ with dense image. If $b_1^{(2)}(F) > 0$, then either $b_1^{(2)}(N) = \infty$ or else $|\Gamma/N| < \infty$. In particular, if N is finitely generated then it is of finite index.*

Proof of Theorem 7.1. Suppose now that Γ is a finitely generated parafree group, with the same nilpotent genus as the free group F, say. Finitely generated groups with the same nilpotent genus have isomorphic pro-p completions for every prime p (cf. Remark 2.1(9)), so $\widehat{\Gamma}_p \cong \widehat{F}_p$.

If F is cyclic, then it is easy to see that Γ must also be cyclic, so the conclusion of Theorem 7.1 holds. Therefore we assume that F has rank $r > 1$.

Γ is residually-p for all primes p. Indeed, since Γ is residually nilpotent, every non-trivial element of Γ has a non-trivial image in $\Gamma/\Gamma_c \cong F/F_c$ for some term Γ_c of the lower central series, and the free nilpotent group F/F_c is residually-p for all primes p (see [29]). Combining this observation with the conclusion of the first paragraph, we see that the natural map from Γ to $\widehat{\Gamma}_p \cong \widehat{F}_p$ is injective. In Example 7.2 we showed that $b_1^{(2)}(F_r) = (r-1) > 0$. Thus we are in the situation of Theorem 7.7 and the proof is complete. □

8. Applications

We close with some further consequences of the arguments in the preceding section, and a discussion of related matters.

8.1. **Parafree groups are C^*-simple.** For ease of exposition, it will be convenient to introduce the following definition. A group Γ is said to be *parafree of pf-rank r* if Γ is residually nilpotent and is in the same nilpotent genus as a free group of rank r. As special cases of Proposition 7.5 and Corollary 7.6 we have:

Corollary 8.1. *If Γ is a finitely generated parafree group of pf-rank r, then*

$$b_1^{(2)}(\Gamma) \geq r - 1 = b_1(\Gamma) - 1,$$

with equality if Γ is finitely presented.

A group Γ is *C^*-simple* if its *reduced C^*-algebra* $C^*_\lambda(\Gamma)$ is simple as a complex algebra (i.e. has no proper two-sided ideals). By definition, the *reduced C^*-algebra* $C^*_\lambda(\Gamma)$ is the norm closure of the image of the complex group algebra $\mathbb{C}[\Gamma]$ under the left-regular representation $\lambda_\Gamma : \mathbb{C}[\Gamma] \to \mathcal{L}(\ell^2(\Gamma))$ defined for $\gamma \in \Gamma$ by $(\lambda_\Gamma(\gamma)\xi)(x) = \xi(\gamma^{-1}x)$ for all $x \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. A group Γ is C^*-simple if and only if any unitary representation of Γ which is weakly contained in λ_Γ is weakly equivalent to λ_Γ. We refer the reader to [33] for a thorough account of the groups that were known to be *C^*-simple* by 2006. The subsequent work of Peterson and Thom [40] augments this knowledge.

An important early result in the field is the proof by Powers [47] that non-abelian free-groups are C^*-simple. In contexts where one is able to adapt the Powers argument, one also expects the canonical trace to be the only normalized trace on $C^*_\lambda(\Gamma)$ (cf. Appendix to [19]). By definition, a linear form τ on $C^*_\lambda(\Gamma)$ is a *normalised trace* if $\tau(1) = 1$ and $\tau(U^*U) \geq 0$, $\tau(UV) = \tau(VU)$ for all $U, V \in C^*_\lambda(\Gamma)$. The *canonical trace* is uniquely defined by

$$\tau_{\text{can}}\left(\sum_{f\in F} z_f \lambda_\Gamma(f)\right) = z_e$$

for every *finite* sum $\sum_{f\in F} z_f \lambda_\Gamma(f)$ where $z_f \in \mathbb{C}$ and $F \subset \Gamma$ contains 1.

Corollary 8.2. *Let Γ be a finitely generated group. If Γ is parafree of pf-rank $r \geq 2$, then the reduced group C^*-algebra $C^*_\lambda(\Gamma)$ is simple and carries a unique normalised trace.*

Proof. Corollary 8.1 tells us that $b_1^{(2)}(\Gamma) \neq 0$. By definition, Γ is residually torsion-free nilpotent, and therefore it satisfies condition ($\star$) of [40], Section 4, i.e. every non-trivial element of $\mathbb{Z}[\Gamma]$ acts without kernel on $\ell^2(\Gamma)$. With these facts in hand, the present result follows immediately from Corollary 4.6 of [40] (which in turn relies on [9]). □

8.2. **A Freiheitssatz for parafree groups.** Section 4 of Peterson and Thom [40] contains a number of other results concerning the structure of finitely presented groups that satisfy their condition ($\star$) and have non-zero $b_1^{(2)}$. In the context of parafree groups, the following consequence of our results and [40] has a particular appeal as it it gives a new proof of a result of Baumslag [3] (Theorem 4.1).

Corollary 8.3. *Let Γ be a finitely generated group that is parafree of pf-rank $r \geq 2$. Then every generating set $S \subset \Gamma$ has an r-element subset $T \subset S$ such that the subgroup of G generated by T is free of rank r.*

Proof. With Corollary 8.1 in hand, we can apply Corollary 4.7 of [40] where the conclusion of the corollary is shown to hold for any finitely generated group Γ with $b_1^{(2)}(\Gamma) > r-2$ that satisfies ($\star$). □

This result strengthens greatly Magnus's observation that an r-generator group of parafree rank r must be free of rank r. Also, the proof shows that if Γ is an $(r+1)$-generator group of parafree rank r (not necessarily finitely presented) then $b_1^{(2)}(\Gamma) = r-1$ (cf. Corollary 8.1).

Remark 8.4. The *parafree conjecture* posits that $H_2(G,\mathbb{Z}) = 0$ for every finitely generated parafree group G. If this were proved, then Corollary 8.3 would be a consequence of a theorem of Stallings [52] which states that if G is a finitely generated group with $H_2(G,\mathbb{Z}) = 0$ and $H_1(G,\mathbb{Z}) = \mathbb{Z}^r$ then any $Y \subset G$ that is independent in $H_1(G,\mathbb{Q})$ freely generates a free subgroup.

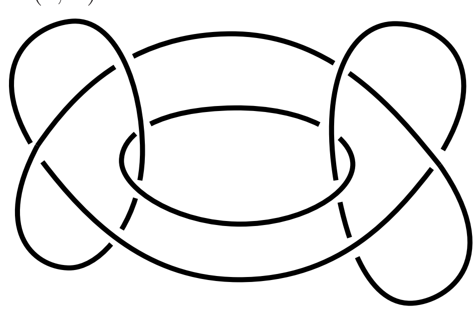
8.3. **Parafree groups and lattices.**

Corollary 8.5. *Let Γ be a parafree group of pf rank $r \geq 2$ that is not free. Then Γ is not isomorphic to a lattice in a connected Lie group.*

Proof. If Γ were a lattice then it would be finitely generated, so $b_1^{(2)}(\Gamma) \neq 0$ by Corollary 8.1. It is shown in [35] that if Γ is a lattice in a connected Lie group and $b_1^{(2)}(\Gamma) \neq 0$, then Γ is commensurable with a lattice in $\mathrm{PSL}(2,\mathbb{R})$; i.e. the group is virtually free or virtually the fundamental group of a closed orientable surface of genus at least 2. Now Γ is torsion-free, with torsion-free abelianization, so in fact, in this case, Γ is free or the fundamental group Σ_g of a closed orientable surface of genus $g \geq 2$. The former is ruled out by assumption, and the latter is ruled out by the observation that $b_1^{(2)}(\Sigma_g) = 2g-2 = b_1(\Sigma_g) - 2$; see Example 7.2. (Alternatively, it is straightforward to construct a finite nilpotent quotient of a free group of rank $2g$ that cannot be a quotient of the genus g surface group.) □

8.4. **Homology boundary links.** In contrast to Corollary 8.5, we give an example of a lattice in $\mathrm{PSL}(2,\mathbb{C})$ that *does* have the same lower central series as a free group. However, the lattice is not residually nilpotent.

Example 8.6. Consider the link L shown below. The complement of this link is hyperbolic, $S^3 \smallsetminus L \cong \mathbb{H}^3/\Gamma$ where $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is the torsion-free non-uniform lattice denoted A2 in [23].



From [23], we have the presentation

$$\Gamma = \langle u, v, z, l \mid [u,l] = 1, uzu^{-1} = v^{-1}zvz, l = v^{-1}uzu^{-1}vz \rangle,$$

where u is a meridian for the unknotted component and l is a longitude for u. The other peripheral subgroup (i.e. the one corresponding to the square knot component) is $\langle uzu^{-1}, uv^2uv^{-1}\rangle$, with meridian uv^2uv^{-1} and longitude uzu^{-1}. The two meridians generate $H_1(\Gamma,\mathbb{Z}) = \mathbb{Z}^2$.

By setting $z = 1$ we obtain a homomorphism from Γ onto $\langle u, v, l \mid l = 1\rangle$; i.e. the free group of rank 2. This homomorphism $\Gamma \to F_2$ induces an isomorphism $H_1G \to H_1F_2$ and a surjection on higher homology groups (since $H_k(F_2,\mathbb{Z}) = 0$ for all $k \geq 2$). An application of Stallings' Theorem [52] now establishes that Γ has the same lower central series quotients as F_2.

Note that the homomorphism $\Gamma \to F_2$ can be realised topologically by performing 0-surgery on both the unknotted component (forcing $l = 1$) and the knotted component (forcing $uzu^{-1} = 1$, hence $z = 1$). The closed manifold resulting from these surgeries is a connected sum of two copies of $\mathbb{S}^2 \times \mathbb{S}^1$.

Corollary 8.5 (alternatively, Remark 8.7) tells us that Γ is not residually nilpotent. In fact, it is not difficult to see that the longitudes described above lie in the intersection of the terms in the lower central series of Γ. $\square$

Remark 8.7. An anonymous referee pointed out to us that the lattice described above exemplifies the following more general phenomenon. Let F be a free group, let G be a finitely generated group that is not free, and suppose that there is an epimorphism $\pi : G \to F$ that induces an isomorphism $\overline{\pi} : H_1(G,\mathbb{Z}) \to H_1(F,\mathbb{Z})$. Then G is not residually nilpotent. To see that this is the case, note that since F is free there is a homomorphism $\sigma : F \to G$ such $\pi \circ \sigma = \mathrm{id}_F$. The map that σ induces on abelianisations is the inverse of $\overline{\pi}$; in particular $\sigma(F)$ generates $G/[G,G]$. But if a set generates a nilpotent group modulo the commutator subgroup, then it generates the nilpotent group itself. It follows that $\sigma \circ \pi$ induces an isomorphism $F/F_c \to G/G_c$ for all $c > 0$. Therefore, $\ker \pi$ (which is assumed to be non-trivial) is contained in the intersection of the terms of the lower central series of G.

The link depicted above is an example of a homology boundary link. Recall that a link $L \subset S^3$ of $m \geq 2$ components is called a *homology boundary link* if there exists an epimorphism $h : \pi_1(S^3\setminus L) \to F$ where F is a free group of rank m. By Alexander duality, $H_1(\pi_1(S^3\smallsetminus L),\mathbb{Z}) \cong \mathbb{Z}^m$. Thus h induces an isomorphism $H_1(\pi_1(S^3 \smallsetminus L),\mathbb{Z}) \cong H_1(F_m,\mathbb{Z})$, and arguing as above we conclude that (with the exception of trivial links) the fundamental groups of homology boundary links are never residually nilpotent. On the other hand, standard results guarantee that all lattices in $\mathrm{PSL}(2,\mathbb{C})$ are *virtually* residually p for all but a finite number of primes p (and hence virtually residually nilpotent). Recent work of Agol [1] implies that these lattices are also virtually residually torsion-free-nilpotent.

8.5. **Pro-p goodness.** Homology boundary links also provide interesting examples in the following related context.

One says that a group Γ is *pro-p good* if for each $n \geq 0$, the homomorphism of cohomology groups

$$H^n(\widehat{\Gamma}_p;\mathbb{F}_p) \to H^n(\Gamma;\mathbb{F}_p)$$

induced by the natural map $\Gamma \to \widehat{\Gamma}_p$ is an isomorphism, where the group on the left is the continuous cohomology of $\widehat{\Gamma}_p$. One says that the group Γ is *cohomologically complete* if Γ is pro-p good for all primes p.

It is shown in [12] that many link groups are cohomologically complete, and indeed it was claimed by Hillman, Matei and Morishita [34] that all link groups are cohomologically complete. Counterexamples to the method of [34] were given in [11], but [11] left open the possibility that link groups might nevertheless be cohomologically complete. Here we note that homology boundary links provide counterexamples. In particular there are hyperbolic links that provide counterexamples.

Proposition 8.8. *Let L be a homology boundary link that is not the trivial link of m components. Then $\pi_1(S^3 \smallsetminus L)$ is not pro-p good for any prime p. In particular, $\pi_1(S^3 \smallsetminus L)$ is not cohomologically complete.*

Proof. Let $\Gamma = \pi_1(S^3 \smallsetminus L)$. Since Γ has the same nilpotent quotients as the free group of rank m, it follows that $\widehat{\Gamma}_p$ is isomorphic to a free pro-p group. Hence $H^2(\widehat{\Gamma}_p; \mathbb{F}_p) = 0$. On the other hand, since L is a non-trivial link with $m \geq 2$ components, $H^2(\Gamma; \mathbb{F}_p)$ has dimension $m - 1 > 0$ as an $\mathbb{F}_p$-vector space. □

By way of contrast, we also note that the lattice Γ described in §8.2 is *good* in the sense of Serre; i.e. for each $n \geq 0$ and for every finite Γ-module M, the homomorphism of cohomology groups

$$H^n(\widehat{\Gamma}; M) \to H^n(\Gamma; M)$$

induced by the natural map $\Gamma \to \widehat{\Gamma}$ is an isomorphism between the cohomology of Γ and the continuous cohomology of $\widehat{\Gamma}$. (Note that goodness deals with the profinite completion and not the pro-p completions.) The goodness of Γ follows from [31] since Γ is a subgroup of finite index in a Bianchi group.

8.6. Theorem 7.7 can be usefully applied to cases where F is not free; for example, F might be a non-abelian surface group or, more generally, a non-abelian limit group. That these satisfy the condition on the first L^2-Betti number can be seen in Example 7.2 for surface groups and [20] for limit groups. This leads to an analogue of Theorem 7.1 for *paralimit* groups.

Note also that, for both surface groups and limit groups, condition ($\star$) of [40] holds (since these groups are residually torsion-free nilpotent). Hence, groups that are paralimit or parasurface groups, in the above sense, also satisfy a version of Freiheitssatz as in Corollary 8.3.

The notion of a "parasurface group" was considered in [13].

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