

PROFINITE RIGIDITY, KLEINIAN GROUPS, AND THE COFINITE HOPF PROPERTY

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ABSTRACT. Let Γ be a non-elementary Kleinian group and $H < \Gamma$ a finitely generated, proper subgroup. We prove that if Γ has finite co-volume, then the profinite completions of H and Γ are not isomorphic. If H has finite index in Γ , then there is a finite group onto which H maps but Γ does not. These results streamline the existing proofs that there exist full-sized groups that are profinitely rigid in the absolute sense. They build on a circle of ideas that can be used to distinguish among the profinite completions of subgroups of finite index in other contexts, e.g. limit groups. We construct new examples of profinitely rigid groups, including the fundamental group of the hyperbolic 3-manifold $\text{Vol}(3)$ and of the 4-fold cyclic branched cover of the figure-eight knot. We also prove that if a lattice in $\text{PSL}(2, \mathbb{C})$ is profinitely rigid, then so is its normalizer in $\text{PSL}(2, \mathbb{C})$.

Dedicated to Gopal Prasad on the occasion of his 75th birthday

1. INTRODUCTION

Rigidity theorems are propositions that tell us objects of a certain kind are equivalent in a weak sense only if they are equivalent in an apparently-stronger sense. A celebrated prototype for such theorems comes from the work of Mostow [36] and Prasad [39]: beyond dimension 2, if complete hyperbolic manifolds of finite volume have isomorphic fundamental groups, then the manifolds must be isometric. In our previous work [13, 9, 14, 11, 12] we showed that hyperbolic orbifolds also play an important role in the study of profinite rigidity.

Profinite rigidity theorems are propositions that tell us groups of a certain kind will have the same set of finite quotients only if the groups are isomorphic. A finitely generated, residually finite group Γ is *profinutely rigid* in the absolute sense if its set of finite quotients distinguishes it from all other finitely generated, residually finite groups: in the language of profinite completions (recalled in Section 2), $\widehat{\Lambda} \cong \widehat{\Gamma} \implies \Lambda \cong \Gamma$.

In our work with McReynolds and Spitler [11, 12], we constructed arithmetic lattices in $\text{PSL}(2, \mathbb{R})$ and $\text{PSL}(2, \mathbb{C})$ that are profinitely rigid in the absolute sense. It remains unknown whether all such lattices are profinitely rigid. The proofs in [11, 12] have two stages: first, given the lattice Γ and a finitely generated, residually finite group Λ with $\widehat{\Lambda} \cong \widehat{\Gamma}$, one tries to construct a Zariski-dense representation $\Lambda \rightarrow \text{PSL}(2, \mathbb{C})$ with image in Γ ; one then has to argue that if $\Lambda \rightarrow \Gamma$ were not surjective, a contradiction would ensue. Our main objective in the current article is to develop a general framework that, in particular, provides a more uniform and comprehensive treatment of the second stage in these proofs.

Theorem A. *Let Γ be a Kleinian group of finite co-volume. If $H < \Gamma$ is a finitely generated, proper subgroup, then $\widehat{\Gamma}$ and \widehat{H} are not isomorphic.*

The most interesting part of the proof of this theorem concerns subgroups of finite index, where our arguments apply to finitely generated, non-elementary Kleinian groups in general.

Key words and phrases. Profinite rigidity, Kleinian group.

Theorem B. *Let Γ be a finitely generated, non-elementary Kleinian group. For each proper subgroup of finite index $H < \Gamma$, there exists a finite group Q such that H maps onto Q but Γ does not.*

Our proof of Theorem B relies on specific properties of Kleinian groups, including Liu's recent result [29] that if M is a complete, orientable hyperbolic 3-manifold of finite volume, then there are only finitely many other such manifolds N with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$. But the general outline of our proof is built around general facts concerning (virtual) epimorphisms of finitely generated profinite groups that have wider implications for the study of profinite rigidity; see in particular Propositions 3.1 and 3.3. The applications in this article are based on the following consequence of these results.

Proposition C. *Let G be a finitely generated, residually finite group and $H < G$ a subgroup of finite index. If every finite quotient of H is also a quotient of G , then there is a continuous surjection $\widehat{G} \twoheadrightarrow \widehat{H}$ with finite kernel.*

This proposition is most useful in geometric situations where one can control finite normal subgroups and one has non-zero invariants (for example ℓ_2 -beti numbers) that increase on passage to finite-index subgroups. In such circumstances, one can often prove that the groups concerned are *cofinitely Hopfian*, meaning that every endomorphism $\phi : G \rightarrow G$ with $[G : \phi(G)] < \infty$ must be an automorphism [8]. We illustrate the potential of this circle of ideas by proving that every proper, finite-index subgroup of a non-abelian limit group has a finite quotient that the ambient group does not have (Proposition 3.7). We also explain, in Section 4, how these general observations, combined with Mostow-Prasad rigidity, allow one to extend profinite rigidity results for lattices to over-lattices. For example:

Theorem D. *If a lattice $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is profinitely rigid, then so is its normaliser in $\mathrm{PSL}(2, \mathbb{C})$.*

We have discussed how Theorems A and B simplify the latter stages of the proofs in [11]. This simplification provides us with encouragement to search for further examples of profinitely rigid lattices $\Gamma < \mathrm{PSL}(2, \mathbb{C})$. We take up this challenge in the final sections of this paper, where we study cyclic branched covers of the figure-eight knot and related families of groups. In particular we prove the following theorem. Here, $\mathrm{Vol}(3)$ is a particular closed orientable hyperbolic 3-manifold whose volume is that of the regular ideal tetrahedron in \mathbb{H}^3 ; it has recently been established [19] that $\mathrm{Vol}(3)$ is the unique closed orientable hyperbolic 3-manifold with this volume, which is the third-smallest possible volume for closed manifolds.

Theorem E. *The fundamental group of the 4-fold cyclic branched cover of the figure-eight knot is profinitely rigid, and so is the fundamental group of $\mathrm{Vol}(3)$.*

In order to illustrate the practical nature of the results in Section 4, we shall explain how profinite rigidity extends to lattices containing those in Theorem E; see Corollaries 6.3 and 6.4. Our results in Section 7 also establish *Galois rigidity* for a class of groups that includes these. Galois rigidity (whose definition we recall in Section 3) provides tight control on the $(\mathrm{P})\mathrm{SL}_2$ -character variety of the groups concerned and plays a crucial role in the first stage of the proofs in [11, 12].

A further notion of rigidity relevant to Theorem A is *Grothendieck rigidity*. In [20], Grothendieck initiated the study of isomorphisms $\widehat{u} : \widehat{H} \xrightarrow{\sim} \widehat{\Gamma}$ induced by inclusions of discrete groups $u : H \hookrightarrow \Gamma$. One calls $(\Gamma, H)_u$ a *Grothendieck pair* if Γ is finitely generated and residually finite and \widehat{u} is an isomorphism but u is not. If there is no Grothendieck pair $(\Gamma, H)_u$ with H finitely generated, then Γ is said to be *Grothendieck rigid*. Bridson and Grunewald [7] proved that there exist Grothendieck pairs of finitely presented groups. The analogous problem for finitely generated groups was settled earlier by Platonov and Tavgen [38]. Long and Reid [31] established Grothendieck rigidity for the fundamental groups of all closed geometric 3-manifolds and finite volume hyperbolic 3-manifolds. Theorem A proves something stronger in the hyperbolic case, ruling out abstract isomorphisms

$\widehat{H} \cong \widehat{\Gamma}$, not just those induced by inclusions $H \hookrightarrow \Gamma$. The analogous stronger statement is false in the non-hyperbolic setting [24, 21].

An alternative approach to Grothendieck rigidity proceeds via subgroup separability. If $(\Gamma, H)_u$ is a Grothendieck pair, then H is dense in the profinite topology on Γ . Thus any residually finite group that is LERF (all finitely generated subgroups are closed in the profinite topology) will be Grothendieck rigid. This observation was applied in [38] to free groups and Fuchsian groups. Following the work of [1] and [45] one knows that all finitely generated Kleinian groups are LERF, so they too are Grothendieck rigid. Recently, Sun proved that the fundamental group of every compact 3-manifold is Grothendieck rigid [44].

This paper is organised as follows. In Section 2 we gather the definitions and background material needed in the sequel. In Section 3 we present two basic results about (virtual) surjections of profinite groups, Propositions 3.1 and 3.3, and illustrate their utility with some immediate applications. In Section 4 we delve further into the significance of finite normal subgroups in profinite completions and discuss results about extending profinite rigidity to over-lattices, including Theorem D. Section 5 contains the remainder of the proof of Theorems A and B. In Section 6 we construct a new (finite) family of profinitely rigid lattices in $\mathrm{PSL}(2, \mathbb{C})$, starting with Theorem E; Section 7 contains an explanation of the Mathematica calculations used to compute the character variety of the fundamental group of the 4-fold cyclic branched cover of the figure-eight knot.

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2. PRELIMINARIES

We shall assume that the reader is familiar with basic facts about profinite groups; the book [42] is a standard reference. By definition, the *profinite completion* $\widehat{\Gamma}$ of a group Γ is the limit of the inverse system consisting of the finite quotients Γ/N and the natural maps $\Gamma/N \rightarrow \Gamma/M$ for $N < M$. Beyond the basic theory, we shall need the Nikolov-Segal Theorem [37] that every subgroup of finite index in a finitely generated profinite group is open. We shall make frequent use of the following basic result describing the correspondence between the finite-index subgroups of a discrete group and those of its profinite completion (see [42, Proposition 3.2.2], and note that we have used [37] to replace “open” by “finite index”).

Notation. Given a subset X of a profinite group G , we write \overline{X} to denote the closure of X in G .

Proposition 2.1. *If Γ is a finitely generated, residually finite group, then there is a one-to-one correspondence between the set \mathcal{X} of subgroups of finite index in Γ and the set \mathcal{Y} of subgroups of finite index in $\widehat{\Gamma}$. Identifying Γ with its image in $\widehat{\Gamma}$, this correspondence is given by:*

- For $H \in \mathcal{X}$, $H \mapsto \overline{H}$.
- For $Y \in \mathcal{Y}$, $Y \mapsto Y \cap \Gamma$.

If $H, K \in \mathcal{X}$ and $K < H$ then $[H : K] = [\overline{H} : \overline{K}]$. Moreover, $K \triangleleft H$ if and only if $\overline{K} \triangleleft \overline{H}$, and in this case $\overline{H}/\overline{K} \cong H/K$.

The last clause implies that Γ and $\widehat{\Gamma}$ have the same finite quotients. A useful refinement of this fact is the observation that the set of finite quotients of Γ determines $\widehat{\Gamma}$ up to isomorphism – see [42, Theorem 3.2.7]), for example, which we repeat here using [37] to replace *topological isomorphism*

with *isomorphism* (as abstract groups). We write $\mathcal{C}(\Gamma)$ to denote the set of isomorphism classes of finite groups onto which Γ surjects.

Theorem 2.2. *Suppose that Γ_1 and Γ_2 are finitely generated abstract groups. Then $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic if and only if $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$.*

2.1. Kleinian groups and Mostow-Prasad rigidity. A *Kleinian group* is, by definition, a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$. We shall be concerned only with finitely generated Kleinian groups. A finitely generated Kleinian group is *elementary* if it contains an abelian subgroup of finite index; otherwise it is *non-elementary*. The *co-volume* of a Kleinian group Γ is the Riemannian volume of the quotient orbifold \mathbb{H}^3/Γ . Thus Kleinian groups of finite co-volume are precisely the lattices in $\mathrm{PSL}(2, \mathbb{C})$.

Mostow [36] (for uniform lattices) and Prasad [39] (covering the non-uniform case) proved that if G is a semisimple Lie group with trivial center and no compact factors, and G is not locally isomorphic to $\mathrm{PSL}(2, \mathbb{R})$, then any pair of lattices $\Gamma_1, \Gamma_2 < G$ that are isomorphic as abstract groups must be conjugate in G . We shall make frequent use of this fact in the case $G = \mathrm{PSL}(2, \mathbb{C})$.

2.2. First ℓ_2 -beti numbers. The first ℓ_2 -beti number of a finitely presented group Γ can be calculated (or defined) using Lück approximation: $b_1^{(2)}(\Gamma) = \lim_n b_1(H_n)/[\Gamma : H_n]$, where $b_1(H)$ is the standard first betti number and (H_n) is a nested sequence of finite-index normal subgroups of Γ intersecting in the identity. The first assertion in the following lemma is immediate from this description and the second is [9, Corollary 2.11].

Lemma 2.3. *Let Γ_1 and Γ_2 be finitely presented groups.*

- (1) *If $H < \Gamma_1$ is a subgroup of finite index d , then $b_1^{(2)}(H) = d b_1^{(2)}(\Gamma_1)$.*
- (2) *If $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ then $b_1^{(2)}(\Gamma_1) = b_1^{(2)}(\Gamma_2)$.*

2.3. Goodness and torsion. Following Serre [43], one says that a group Γ is *good* if for every finite Γ -module M , the homomorphism of cohomology groups $H^n(\widehat{\Gamma}; M) \rightarrow H^n(\Gamma; M)$ induced by the natural map $\Gamma \rightarrow \widehat{\Gamma}$ is an isomorphism between the cohomology of Γ and the continuous cohomology of $\widehat{\Gamma}$. (See [43] and [42, Chapter 6] for details about the cohomology of profinite groups.)

Fuchsian groups are good [22]. Serre [43, Chapter 1, Section 2.6] gives a criterion for when extensions of good groups by good groups are good; by applying this, one sees that the fundamental groups of surface bundles over the circle are good. Goodness is an invariant of commensurability, so it follows from Agol's Virtual Fibration Theorem [1] that all lattices in $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{PSL}(2, \mathbb{C})$ are good. More generally, all Kleinian groups are good; see [2, p.102]. The most important consequence of this for us in the present paper comes from the following well-known fact (which is [9, Corollary 2.16]).

Lemma 2.4. *If a finitely generated, residually finite group G is good and has finite cohomological dimension, then \widehat{G} is torsion-free.*

Corollary 2.5. *If Γ is a finitely generated, discrete torsion-free subgroup of $\mathrm{PSL}(2, \mathbb{C})$, then $\widehat{\Gamma}$ is torsion-free.*

We shall also need a variant of Lemma 2.4 that allows for limited torsion. The following is a weak form of a result of Minasyan and Zalesskii [35, Corollary 3.5].

Proposition 2.6. *Let Γ be a finitely generated, residually finite group that has a subgroup of finite index with finite cohomological dimension. If Γ is good, then every element $g \in \widehat{\Gamma}$ of prime order is conjugate into Γ .*

Corollary 2.7. *If $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is a finitely generated non-elementary Kleinian group, then $\widehat{\Gamma}$ does not contain a non-trivial, finite normal subgroup.*

Proof. A non-trivial finite normal subgroup $N < \widehat{\Gamma}$ would contain an element g of prime order. Proposition 2.6 tells us that the conjugacy class of g , which is contained in N , must intersect Γ . Thus $H = N \cap \Gamma$ is a non-trivial finite normal subgroup of Γ . But Γ does not contain such subgroups, because if it did then the non-empty, totally-geodesic submanifold $\mathrm{Fix}(H) \subset \mathbb{H}^3$ would be invariant under Γ ; this submanifold cannot be 2-dimensional, because Γ (hence H) acts effectively and does not contain orientation-reversing isometries; and if $\mathrm{Fix}(H)$ were a line or a point, then the discrete action of Γ on it would imply that Γ was virtually cyclic, hence elementary. \square

Remark 2.8. There is an alternative proof of Corollary 2.7 wherein one argues as in the proof of Theorem 1.2 of [30] that for infinitely many finite fields \mathbb{F} , Γ maps onto the finite simple group $\mathrm{PSL}(2, \mathbb{F})$ with torsion-free kernel. As these groups are simple and unbounded in size, for infinitely many of the induced maps $\widehat{\Gamma} \rightarrow \mathrm{PSL}(2, \mathbb{F})$, any finite normal subgroup $N < \widehat{\Gamma}$ would be in the kernel. But this would contradict Corollary 2.5, because the kernel of $\widehat{\Gamma} \rightarrow \mathrm{PSL}(2, \mathbb{F})$ is the completion of the kernel of $\Gamma \rightarrow \mathrm{PSL}(2, \mathbb{F})$, which is torsion-free.

2.4. Trace fields, character varieties, and Galois rigidity. In order to fully understand the construction of profinitely rigid groups in the penultimate section of this paper, the reader will need to be familiar with the basic arithmetic theory of Kleinian groups, as described in [34], and to have some familiarity with the main ideas in our papers with McReynolds and Spitler [11, 12], where we proved that certain Kleinian and Fuchsian groups are profinitely rigid. For the convenience of the reader, we recall the basic terminology here.

Let Γ be a finitely generated subgroup of $\mathrm{PSL}(2, \mathbb{C})$ and let Γ_1 be its pre-image in $\mathrm{SL}(2, \mathbb{C})$. It will be convenient to say Γ is *Zariski dense* in $\mathrm{PSL}(2, \mathbb{C})$ when what we actually mean is that Γ_1 is Zariski dense in $\mathrm{SL}(2, \mathbb{C})$. The *trace-field* of Γ is defined to be the field

$$K_\Gamma = \mathbb{Q}(\mathrm{tr}(\gamma) : \gamma \in \Gamma_1).$$

The algebra consisting of finite K_Γ -linear combinations of the matrices $\gamma \in \Gamma_1$ is a quaternion algebra, denoted $A_0\Gamma$; see [34, Chap 3]. If Γ is a lattice, then K_Γ is a number field, of degree n_K say. If $K_\Gamma = \mathbb{Q}(\theta)$ for some algebraic number θ , then the Galois conjugates of θ , say $\theta = \theta_1, \dots, \theta_{n_K}$ provide embeddings $\sigma_i : K_\Gamma \rightarrow \mathbb{C}$ defined by $\theta \mapsto \theta_i$. These in turn can be used to build n_K Zariski dense non-conjugate representations $\rho_{\sigma_i} : \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{C})$ with the property that $\mathrm{tr}(\rho_{\sigma_i}(\gamma)) = \sigma_i(\mathrm{tr}(\rho(\gamma)))$ for all $\gamma \in \Gamma$. These are the *Galois conjugates* of the inclusion map $\Gamma \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$.

As in [11] we will be most interested in groups Γ with the fewest possible Zariski dense representations. We say that Γ is *Galois rigid* if, up to conjugacy, the only Zariski dense representations of Γ in $\mathrm{PSL}(2, \mathbb{C})$ are the Galois conjugates of the inclusion. (In [11] we defined Galois rigidity for more general finitely generated groups, but the interests here are narrower.) The crucial results relating Galois rigidity to profinite rigidity are Theorem 4.8 and Corollaries 4.10 and 4.11 of [11].

3. TWO BASIC RESULTS

In this section we present two general results concerning (virtual) epimorphisms of profinite groups. The proofs that we shall present for the theorems described in the introduction will illustrate the utility of these results. We expect these results to have many further applications in the context of profinite rigidity.

3.1. Profinite surjections. We begin with a one-sided version of Theorem 2.2.

Proposition 3.1. *If G and H are finitely generated profinite groups and every finite quotient of H is also a quotient of G , then there is a continuous surjection $\widehat{G} \twoheadrightarrow \widehat{H}$.*

Proof. The proposition is trivial if H is finite, so suppose H is infinite. For each positive integer n we denote by I_n the intersection of all subgroups in H that have index at most n , and we consider the quotient

$$H(n) := H/I_n.$$

There is an obvious surjection $H(n+1) \twoheadrightarrow H(n)$ and H is the inverse limit of the system $\{H(n+1) \twoheadrightarrow H(n)\}_n$.

Let $T(n)$ be the set of (necessarily open) normal subgroups $L \leq G$ such that $G/L \cong H(n)$. Let $T = \bigcup_n T(n)$. Note that $T(1) = G$ and $T(n)$ is non-empty since $H(n)$ is a quotient of G . Each $L \in T(n)$ is contained in some $L' \in T(n-1)$, namely the kernel of the composition $G \rightarrow G/L \cong H(n) \rightarrow H(n-1)$. We say L is a descendant of L' and draw an arrow from L' to L . This defines a connected, locally-finite directed graph with root G and vertex set T . Let $G = L_1 > L_2 > L_3 > \dots$ be an infinite directed path in this graph with $G/L_n \cong H(n)$ for all n . The quotient of G by the closed subgroup $N = \bigcap_n L_n$ is isomorphic to the inverse limit of the $H(n)$, which is H . \square

Remark 3.2. Theorem 2.2 follows easily from Proposition 3.1 and the fact that finitely generated profinite groups are Hopfian, i.e. every continuous surjection $G \rightarrow G$ is an isomorphism (as explained in the last paragraph of the proof of Proposition 3.3 below).

Proposition 3.1 is useful in much the same way as Theorem 2.2, particularly in combination with Proposition 2.1.

3.2. The cofinite Hopf property. The second general tool that we gather comes from the observation that Hirshon's proof [26] of the discrete analogue of the following proposition carries over to the profinite setting.

Proposition 3.3. *If G is a finitely generated profinite group and $H < G$ is a subgroup of finite index, then every epimorphism $\alpha : G \rightarrow H$ has finite kernel; in particular, if G has no finite normal subgroup then α is injective.*

Proof. Let $\alpha : G \twoheadrightarrow H$ be an epimorphism, define $H_n := \alpha^n(G)$, let $\theta_n : H_n \twoheadrightarrow H_{n+1}$ be the restriction of α and let $K_n = \ker \theta_n$. Note that $H_{n+1} < H_n$ and that H_n has finite index in G . Also, $K_{n+1} < K_n = H_n \cap \ker \alpha$.

Let N_n be the core of H_{n+1} in H_n , that is, the intersection of the conjugates $g^{-1}H_{n+1}g$ with $g \in H_n$. Then $\theta_n(N_n) \subseteq N_{n+1}$, so θ_n induces an epimorphism $\bar{\theta}_n : H_n/N_n \twoheadrightarrow H_{n+1}/N_{n+1}$. As the groups H_i/N_i are finite, for sufficiently large n , say $n \geq n_0$, the map $\bar{\theta}_n$ must be an isomorphism. This forces $K_n < N_n$. But $N_n < H_{n+1}$, and by definition $K_i = H_i \cap \ker \alpha$, so $K_n = K_{n+1}$ for all $n \geq n_0$.

In order to obtain a contradiction, we assume that $\ker \alpha$ is infinite. In this case, $K_{n_0} = H_{n_0} \cap \ker \alpha$ is infinite, in particular non-trivial. Let k a non-trivial element of K_{n_0} . As $K_{n_0} = K_n < H_n$ for all $n \geq n_0$, there exist elements $k_n \in G$ such that $\alpha^n(k_n) = k$. Note that $\alpha^{n+1}(k_n) = \alpha(k) = 1$.

As G is residually finite, there is a finite quotient $\pi : G \rightarrow Q$ such that $\pi(k) \neq 1$. The contradiction that we seek is obtained by noting that the maps $p_n = \pi \circ \alpha^n : G \rightarrow Q$ are all distinct, because $p_n(k_n) \neq 1$ but $p_m(k_n) = 1$ for all $m > n$; this is nonsense because there are only finitely many maps from a finitely generated profinite group to a finite group. \square

Remark 3.4. The above argument shows that θ_n is an isomorphism for sufficiently large n , regardless of whether G contains finite normal subgroups.

The following result is an immediate consequence of Propositions 3.1 and 3.3. Recall that, given a class of groups \mathcal{G} , a group $\Gamma \in \mathcal{G}$ is said to be *profinutely rigid in \mathcal{G}* if $\hat{\Lambda} \cong \hat{\Gamma}$ implies $\Lambda \cong \Gamma$ provided $\Lambda \in \mathcal{G}$.

Corollary 3.5. *Let \mathcal{G} be a class of finitely generated groups that is closed under passage to subgroups of finite index. Suppose that $\Gamma \in \mathcal{G}$ is profinitely rigid in \mathcal{G} and that $\widehat{\Gamma}$ does not contain a non-trivial finite normal subgroup. If $H < \Gamma$ is a proper subgroup of finite index, then either $H \cong \Gamma$ or else H has a finite quotient that Γ does not have.*

The results of [9] imply that Corollary 3.5 applies to any non-elementary Fuchsian group Γ (cf. [12, Corollary 3.7]). It also applies when \mathcal{G} is the class of 3-manifold groups and Γ is the fundamental group of any once-punctured torus bundle over the circle [14].

3.3. Fuchsian groups and limit groups: $b_1^{(2)} > 0$ and goodness. In order to apply Corollary 3.5 to non-elementary Fuchsian groups we had to quote one of the main results of [9] – non-isomorphic Fuchsian groups can be distinguished from each other by their finite quotients. A more useful observation is that the proofs of profinite rigidity in [9] and [12] can be shortened considerably by invoking the following consequence of Propositions 3.1 and 3.3.

Proposition 3.6. *If H is a proper subgroup of finite index in a finitely generated non-elementary Fuchsian group Γ , then H has a finite quotient that Γ does not have.*

Proof. If the proposition were to fail for a finite-index subgroup $H < \Gamma$, then Proposition 3.1 would provide a continuous epimorphism $\widehat{\Gamma} \twoheadrightarrow \widehat{H}$. As Γ is a non-elementary Fuchsian group, $\widehat{\Gamma}$ does not contain a non-trivial finite normal subgroup [9, Corollary 5.2], so Proposition 3.3 implies that $\widehat{\Gamma} \twoheadrightarrow \widehat{H}$ is an isomorphism. On the other hand, since $b_1^{(2)}(\Gamma) > 0$ for non-elementary Fuchsian groups (see for example [9, Proposition 3.5]) and since Fuchsian groups are finitely presented, Lemma 2.3 tells us that $\widehat{\Gamma} \not\cong \widehat{H}$ if $H < \Gamma$ is a proper subgroup of finite index – a contradiction. \square

The proof of Proposition 3.6 serves as a template for other classes of groups in which (1) one can control finite normal subgroups in $\widehat{\Gamma}$ and (2) one has a profinite invariant (for example $b_1^{(2)}(\Gamma)$) that is non-zero and increases when one passes to finite-index subgroups (cf. [8]). Goodness provides a means of controlling finite subgroups (Lemma 2.4 and Proposition 2.6), so it is natural to look for further applications in settings where one knows the groups are good. Recall that a finitely generated group Γ is called a *limit group* if it is fully residually free, i.e. for every finite subset $X \subset \Gamma$, there is a homomorphism from Γ to a free group that restricts to an injection on X .

Proposition 3.7. *Let Γ be a non-abelian limit group. If H is a proper subgroup of finite index in Γ , then H has a finite quotient that Γ does not have.*

Proof. We follow the proof of Proposition 3.6. Limit groups have finite classifying spaces and are good, by [22], so $\widehat{\Gamma}$ is torsion-free by Lemma 2.4. Thus if the proposition were to fail for $H < \Gamma$, then by combining Propositions 3.1 and 3.3 we would get an isomorphism $\widehat{\Gamma} \cong \widehat{H}$. But $b_1^{(2)}(\Gamma) > 0$ for non-abelian limit groups, by [10]. Lemma 2.3 provides us with the desired contradiction, as before. \square

A further application of the same argument yields a special case of Theorem B, for which we need the following lemma.

Lemma 3.8. *Let Γ be a finitely generated non-elementary Kleinian group. Then $b_1^{(2)}(\Gamma) = 0$ if and only if Γ has finite co-volume.*

Proof. When Γ is torsion-free, this is proved in [32]. The general case follows from Lemma 2.3, since finitely generated Kleinian groups are finitely presented. \square

Proposition 3.9. *Let Γ be a finitely generated non-elementary Kleinian group that has infinite co-volume. If $H < \Gamma$ is a proper subgroup of finite index, then H has a finite quotient that Γ does not have.*

Proof. Corollary 2.7 assures us Γ has no non-trivial finite normal subgroup, and Lemma 3.8 tells us $b_1^{(2)}(\Gamma) > 0$. \square

4. PROFINITE RIGIDITY OF OVER-LATTICES

Before turning to the proof of Theorems A and B, we elaborate further on the significance of finite normal subgroups in profinite completions, which emerged in the previous section. By focusing on this, we shall see in particular that the normaliser of any profinitely rigid lattice $\Gamma < (\mathrm{P})\mathrm{SL}(2, \mathbb{C})$ is itself profinitely rigid. Mostow-Prasad rigidity plays an important role in these arguments.

Proposition 4.1. *Let Γ be a lattice in $\mathrm{PSL}(2, \mathbb{C})$ that is profinitely rigid (in the absolute sense). Let Λ be a finitely generated, residually finite group that has a subgroup of finite index Δ with $\widehat{\Delta} \cong \widehat{\Gamma}$. Then Λ is isomorphic to a lattice in $\mathrm{PSL}(2, \mathbb{C})$ if and only if $\widehat{\Lambda}$ does not contain a non-trivial finite normal subgroup.*

Proof. If Λ is Kleinian, then Corollary 2.7 assures us there is no non-trivial finite normal subgroup in $\widehat{\Lambda}$. The profinite rigidity of Γ implies $\Delta \cong \Gamma$. Thus Λ is commensurable to a lattice in $\mathrm{PSL}(2, \mathbb{C})$, and by Mostow-Prasad Rigidity there is a short exact sequence

$$1 \rightarrow N \rightarrow \Lambda \rightarrow \Omega \rightarrow 1$$

with N finite and Ω a lattice in $\mathrm{PSL}(2, \mathbb{C})$ that contains Γ . If Λ is not itself a lattice, then $N \neq 1$ is the non-trivial finite normal subgroup in $\widehat{\Lambda}$ we seek. \square

Corollary 4.2. *Let Λ_1 and Λ_2 be finitely generated, residually finite groups with $\widehat{\Lambda}_1 \cong \widehat{\Lambda}_2$. Let $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ be a lattice that is profinitely rigid. If Λ_1 has a finite-index subgroup isomorphic to Γ , then so does Λ_2 . In this case, either Λ_1 and Λ_2 are (isomorphic to) commensurable lattices in $\mathrm{PSL}(2, \mathbb{C})$ with the same covolume, or else neither is a lattice.*

Proof. Suppose Λ_1 contains Γ as a subgroup of index d . Proposition 2.1 implies that Λ_2 contains a subgroup Δ of index d with $\widehat{\Delta} \cong \widehat{\Gamma}$. As Γ is profinitely rigid, $\Delta \cong \Gamma$. So if Λ_1 and Λ_2 are lattices, then they are commensurable and the covolume of each is $1/d$ that of Γ . And Proposition 4.1 tells us they are lattices if and only if $\widehat{\Lambda}_1 \cong \widehat{\Lambda}_2$ does not contain a non-trivial finite normal subgroup. \square

Recall that the *profinite genus* of a finitely generated group G consists of the set of isomorphism classes of finitely generated, residually finite groups H such that $\widehat{H} \cong \widehat{G}$.

In the following statement, we use Mostow-Prasad rigidity to conjugate the representatives of the isomorphism classes in the genus of Δ so that they contain Γ itself rather than an isomorphic copy. The statement about normality is immediate from Proposition 2.1.

Corollary 4.3. *Let $\Gamma < \Delta < \mathrm{PSL}(2, \mathbb{C})$ be lattices. If Γ is profinitely rigid, then the profinite genus of Δ is a subset of the lattices in $\mathrm{PSL}(2, \mathbb{C})$ that contain Γ with index $[\Delta : \Gamma]$. If Γ is normal in Δ , then it is normal in all of the lattices in the profinite genus of Δ .*

It follows from Corollary 4.3 that if Δ is the only lattice of a given covolume containing Γ , up to conjugacy, then Δ is profinitely rigid. Further rigid lattices can be obtained by using Corollary 4.3 in combination with other profinite invariants: for example, by goodness, one can distinguish lattices that contain torsion from those that are torsion-free. The fact that normality is preserved in the standard correspondence of Proposition 2.1 can also be used in this context.

Theorem 4.4. *If a lattice in $\mathrm{PSL}(2, \mathbb{C})$ is profinitely rigid (in the absolute sense), then so is its normaliser in $\mathrm{PSL}(2, \mathbb{C})$.*

Proof. By Mostow-Prasad rigidity, the normaliser of a lattice $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ can be characterised as the unique lattice $\Omega < \mathrm{PSL}(2, \mathbb{C})$ of minimal co-volume containing Γ as a normal subgroup (necessarily of finite index).

Let Λ be a finitely generated, residually finite group with $\widehat{\Lambda} \cong \widehat{\Omega}$. Then, as in Corollary 4.3, Λ is isomorphic to a lattice with the same covolume as Ω that contains a normal subgroup Γ_0 isomorphic to Γ . By conjugating the image of Λ in $\mathrm{PSL}(2, \mathbb{C})$ if necessary, we may assume $\Gamma_0 = \Gamma$. Then, by the characterisation of Ω in the first sentence of the proof, $\Lambda = \Omega$. \square

4.1. Arithmetic lattices of simplest type. For the most part, our focus in this article is on hyperbolic orbifolds of dimension 3. This reflects the special role that arithmetic plays in dimension 3, in particular the existence and importance of the invariant trace field, and the fact that lattices in $\mathrm{PSL}(2, \mathbb{C})$ are good. But there are also lattices in higher dimensional hyperbolic lattices that fit well with the circle of ideas developed here, namely *standard* arithmetic lattices, which are also known as arithmetic lattices of *simplest type* – their definition is recalled below.

Let k be a totally real number field of degree d over \mathbb{Q} equipped with a fixed embedding into \mathbb{R} which we refer to as the identity embedding, and denote the ring of integers of k by R_k . Let V be an $(n+1)$ -dimensional vector space over k equipped with a non-degenerate quadratic form f defined over k which has signature $(n, 1)$ at the identity embedding, and signature $(n+1, 0)$ at the remaining $d-1$ embeddings.

Given this, the quadratic form f is equivalent over \mathbb{R} to the quadratic form $J_n = x_0^2 + x_1^2 + \dots + x_{n-1}^2 - x_n^2$, and for any non-identity Galois embedding $\sigma : k \rightarrow \mathbb{R}$, the quadratic form f^σ (obtained by applying σ to each entry of f) is equivalent over \mathbb{R} to $x_0^2 + x_1^2 + \dots + x_{n-1}^2 + x_n^2$. We call such a quadratic form *admissible*.

Let F be the symmetric matrix associated to the quadratic form f and let $O(f)$ and $SO(f)$ denote the linear algebraic groups defined over k described as:

$$O(f) = \{X \in \mathrm{GL}(n+1, \mathbb{C}) : X^t F X = F\} \text{ and } SO(f) = \{X \in \mathrm{SL}(n+1, \mathbb{C}) : X^t F X = F\}.$$

For a subring $L \subset \mathbb{C}$, we denote the L -points of $O(f)$ (resp. $SO(f)$) by $O(f, L)$ (resp. $SO(f, L)$).

Note that, given an admissible quadratic form defined over k of signature $(n, 1)$, there exists $T \in \mathrm{GL}(n+1, \mathbb{R})$ such that $T^{-1} SO(f, \mathbb{R}) T = SO(n, 1)$. Let $\mathrm{Isom}^+(\mathbb{H}^n)$ denote the full group of orientation-preserving isometries of \mathbb{H}^n . This can be identified with the group $SO^+(J_n, \mathbb{R}) = SO^+(n, 1)$, which is the subgroup of $SO(n, 1)$ preserving the upper-half sheet of the hyperboloid $J_n = -1$.

A subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ is called *arithmetic of simplest type* (alternatively, a *standard arithmetic lattice*) if Γ is commensurable with the image in $\mathrm{Isom}^+(\mathbb{H}^n)$ of the subgroup of $SO(f, R_k)$ (under the conjugation map described above). An arithmetic hyperbolic n -manifold $M = \mathbb{H}^n / \Gamma$ is called arithmetic of simplest type if Γ is.

The following analogue of Corollary 4.3 holds for arithmetic lattices of simplest type.

Theorem 4.5. *Let $n \geq 2$, and let $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ be an arithmetic lattice of simplest type which is profinitely rigid. Suppose that $\Delta < \mathrm{Isom}^+(\mathbb{H}^n)$ contains Γ as a subgroup of finite index. Then the profinite genus of Δ is a subset of the lattices in $\mathrm{Isom}^+(\mathbb{H}^n)$ that contain Γ with index $[\Delta : \Gamma]$. If Γ is normal in Δ , then it is normal in all of the lattices in the profinite genus of Δ .*

Proof. The proof of Corollary 4.3 will apply once we have established that the lattices we are studying do not contain non-trivial finite, normal subgroups. To see that this is the case, we appeal to Proposition 2.6. Subgroups of $\mathrm{Isom}^+(\mathbb{H}^n)$ have torsion-free subgroups of finite index, and hence discrete subgroups have finite virtual cohomological dimension, so the only non-trivial point to address is the goodness of arithmetic lattices of simplest type. This is a consequence of [4], as explained in [28, Theorem 6.5].

Proposition 2.6 implies that if $\widehat{\Delta}$ had a non-trivial finite normal subgroup, then Δ would too. But Δ has no such subgroup, because if it did then, as in the proof of Corollary 2.7 there would be a Δ -invariant totally geodesic submanifold of positive codimension in \mathbb{H}^n (the fixed point set of the finite normal subgroup) and this is impossible since Δ is a lattice. \square

We should offset the preceding discussion by noting that it is unknown whether there exist profinitely rigid lattices in $\text{Isom}^+(\mathbb{H}^n)$ for $n > 3$.

4.2. Flexibility in rigid sandwiches. Theorem 4.4 and the examples in Section 6 illustrate how the foregoing results can be used to generate pairs of lattices $\Gamma < \Omega$ both of which are profinitely rigid. Ultimately, one expects all intermediate lattices $\Gamma < \Delta < \Omega$ to be profinitely rigid as well, but the following construction indicates that this is less obvious than one might naively imagine.

It is easy to see that if N is finite then $N \times \mathbb{Z}$ is profinitely rigid. Gilbert Baumslag [3] pointed out that, strikingly, this rigidity fails if one replaces $N \times \mathbb{Z}$ with a semidirect product. We shall modify his construction to exhibit pairs of non-isomorphic groups H_1 and H_2 that have the same finite quotients and have the same finite index in a group $N \times \mathbb{Z}$ with N finite. Thus H_1 is sandwiched between the profinitely rigid groups \mathbb{Z} and $N \times \mathbb{Z}$ but is not itself profinitely rigid.

Our construction applies (mutatis mutandis) to each of the pairs of metacyclic groups considered by Baumslag in [3], but for the sake of clarity we concentrate on

$$B_1 = (\mathbb{Z}/25) \rtimes_{\alpha} \mathbb{Z} \quad B_2 = (\mathbb{Z}/25) \rtimes_{\beta} \mathbb{Z}$$

where, in multiplicative notation, $\alpha \in \text{Aut}(\mathbb{Z}/25)$ is $\alpha(x) = x^6$ and $\beta(x) = x^{11}$. Note that $\alpha = \beta^3$ and $\beta = \alpha^2$ generate the same cyclic subgroup of order 5 in $\text{Aut}(\mathbb{Z}/25)$. One can prove by direct argument that $B_1 \not\cong B_2$ but $\hat{B}_1 \cong \hat{B}_2$.

Let $N = (\mathbb{Z}/25) \rtimes_{\alpha} (\mathbb{Z}/5) = \langle x, y \rangle$, where x generates the first factor and the generator y in the second factor acts as α . Let t be a generator for the second factor of $N \times \mathbb{Z}$. Then $H_1 = \langle x, yt \rangle$ and $H_2 = \langle x, y^2t \rangle$ both have index 5 in $N \times \mathbb{Z}$, and $H_1 \cong B_1$ while $H_2 \cong B_2$.

By replacing B_1 and B_2 in the above construction by the surface-bundle groups of Hempel [24], one can arrange for all the groups in the sandwich to be fundamental groups of closed 3-orbifolds.

5. KLEINIAN GROUPS: THEOREMS A AND B

In this section, we complete the proofs of Theorems A and B and then reflect on how these results can be used simplify arguments in [11].

Proof of Theorem B: Let $\Gamma < \text{PSL}(2, \mathbb{C})$ be a finitely generated, non-elementary Kleinian group and let $H < \Gamma$ be a subgroup of finite index $d > 1$. If every finite quotient of H is also a quotient of Γ , then Propositions 3.1 and 3.3 yield an epimorphism $\hat{\Gamma} \rightarrow \hat{H}$ with finite kernel. Corollary 2.7 tells us that Γ does not contain a non-trivial, finite normal subgroup, so in fact $\hat{\Gamma} \cong \hat{H}$.

Proposition 2.1 then provides an index- d subgroup $H_1 < H$, corresponding to $H < \Gamma$, with $\hat{H}_1 \cong \hat{H} \cong \hat{\Gamma}$. Repeating this argument, we obtain an infinite sequence of subgroups $H_n < \Gamma$ with $[\Gamma : H_n] = d^n$ such that $\hat{H}_n \cong \hat{\Gamma}$.

If Γ is torsion-free and has finite co-volume, then this contradicts Liu's result [29] that there can be only finitely many finite-volume hyperbolic 3-manifolds whose fundamental groups have the same profinite completion. (The covolume of H_n increases with n , so no two of the manifolds \mathbb{H}^3/H_n are homeomorphic, by Mostow-Prasad rigidity.) This completes the proof in the torsion-free case.

If Γ is a lattice with torsion, then we pass to a torsion-free subgroup of finite index $\Delta < \Gamma$ and consider the subgroups $\Delta_n < H_n$ obtained from the isomorphism $\hat{\Gamma} \cong \hat{H}_n$ via the correspondence of Proposition 2.1. Then $\hat{\Delta}_n \cong \hat{\Delta}$ for all n . As $[H_n : \Delta_n] = [\Gamma : \Delta]$, the covolume of the lattices Δ_n increases with n . Lemma 2.4 assures us that $\Delta_n \hookrightarrow \hat{\Delta}$ is torsion-free. Thus we once again obtain a contradiction to Liu's Theorem.

Proposition 3.9 deals with the case where Γ has infinite co-volume. \square

Proof of Theorem A: We must show that if $\Gamma < \text{PSL}(2, \mathbb{C})$ is a lattice and $H < \Gamma$ is a proper subgroup, then $\hat{\Gamma}$ is not isomorphic to \hat{H} . In the light of Theorem B we may assume that H is of infinite index. If a finitely generated, residually finite group has the same profinite completion as

a finitely generated virtually abelian group, then it is itself virtually abelian; this distinguishes Γ from its elementary subgroups. When H is non-elementary of infinite index, Lemma 3.8 tells us that $b_1^{(2)}(H) \neq 0$ while $b_1^{(2)}(\Gamma) = 0$. And $b_1^{(2)}$ is an invariant of $\widehat{\Gamma}$ (Lemma 2.3). \square

5.1. Reworking the end game in [11]. The proofs used in [11] and [12] to establish the profinite rigidity of arithmetic Fuchsian and Kleinian groups such as the triangle group $\Delta(3, 3, 4)$, the Bianchi group $\mathrm{PSL}(2, \mathbb{Z}[\omega])$, and the fundamental group of the Weeks manifold, had two main steps. The first and deepest part of the proof uses Galois rigidity of these examples to deduce that if Γ is one of the above groups and Λ is any finitely generated, residually finite group with $\widehat{\Lambda} \cong \widehat{\Gamma}$, then there exists a homomorphism $\Lambda \rightarrow \Gamma$ with non-elementary image, L say. A further suite of arguments of a more *ad hoc* nature was then used in [11] and [12], exploiting specific characteristics of the finite index subgroups of Γ to show that $L = \Gamma$. We then had an epimorphism $\widehat{\Gamma} = \widehat{\Lambda} \rightarrow \widehat{L} = \widehat{\Gamma}$, and the Hopfian property of profinite groups (explained in the last paragraph of the proof of Proposition 3.3) assures us that this is injective, hence $\Lambda \cong \Gamma$. The Galois rigidity of these examples is a consequence of a fine understanding of the arithmetic of the invariant trace-field and quaternion algebra of the given lattice.

Theorem A provides a more uniform and conceptual alternative to this second suite of arguments, removing the need to analyse the case where L has finite index in Γ . (The case where L has infinite index is easy to deal with using topological arguments.)

Proposition 5.1. *Let Γ be a non-elementary Kleinian group and let Λ be a finitely generated, residually finite group such that there is a homomorphism $\Lambda \rightarrow \Gamma$ whose image is a proper subgroup of finite index in Γ . Then $\widehat{\Lambda}$ is not isomorphic to $\widehat{\Gamma}$.*

6. NEW EXAMPLES OF PROFINITELY RIGID KLEINIAN GROUPS

We closed the previous section by discussing how Theorems A and B allow one to avoid significant difficulties in the second stage of the proofs of profinite rigidity in [11] and [12]. This raises the hope that in other cases where one can prove Galois rigidity, one might be able to use Proposition 5.1 to deduce profinite rigidity. In this section we shall illustrate the potential of this approach by exhibiting new examples of profinitely rigid Kleinian groups. We shall also illustrate the potential of the observations in Section 4 by identifying further profinitely rigid groups among over-lattices of our first example. Along the way, we shall describe an infinite family of Galois rigid Kleinian groups.

Let $K \subset S^3$ denote the figure-eight knot, let Q_n be the orbifold obtained by $(n, 0)$ -Dehn filling on K and let M_n be the n -fold cyclic branched cover of K . Note that M_n can also be regarded as an n -fold cyclic (orbifold) cover of Q_n ; this is the maximal abelian cover. When $n = 2$, M_2 is the Lens Space $L(5, 3)$, and so Q_2 is finite; when $n = 3$, M_3 is the flat manifold known as the Hantzsche-Wendt manifold and so Q_3 is a Euclidean crystallographic group; when $n \geq 4$, M_n and Q_n are hyperbolic (see [17] for example). In addition, it was shown in [23] that $\Gamma_n = \pi_1(M_n) \cong F(2, 2n)$ where $F(2, 2n)$ is one of the Fibonacci groups of [16] (a presentation for $F(2, 8)$ is given below). Let Δ_n be the orbifold fundamental group of Q_n . Then $Q_n = \mathbb{H}^3/\Delta_n$ and $\Gamma_n = [\Delta_n, \Delta_n]$ is a subgroup of index n .

The following theorem will be proved in the next section.

Theorem 6.1. *In the notation established above,*

- (1) Δ_p is Galois rigid for all primes $p \geq 5$;
- (2) Δ_4, Δ_6 and Δ_9 are Galois rigid;
- (3) Γ_4 is Galois rigid.

Taking Theorem 6.1(3) as our starting point, we shall construct a new family of profinitely rigid groups.

Theorem 6.2. Γ_4 is profinitely rigid.

Our proof relies on the following additional information concerning Γ_4 . It is proved in [40, §4.1] that Γ_4 is an arithmetic Kleinian group with (invariant) trace-field $k = \mathbb{Q}(\sqrt{-3})$ and that its invariant quaternion algebra B over k is ramified at the places ν_2 and ν_3 associated to the prime ideals of norm 4 and 3, respectively, in k . (The arithmeticity of Δ_4 was also proved in [25].) Note that these are the unique primes of these norms in k . This implies that B is *locally uniform* in the sense of [11, Definition 4.10]. Using volume considerations, it is also shown in [40, pp. 172–173] that $\Delta_4^{(2)} = \Gamma_{\mathcal{O}}^1$, the image in $\mathrm{PSL}(2, \mathbb{C})$ of the group of elements of norm one in the unique (up to B^* -conjugacy) maximal order $\mathcal{O} \subset B$ (that is to say, B has type number 1). It is clear from the construction of Q_n that $\Delta_4^{\mathrm{ab}} \cong \mathbb{Z}/4\mathbb{Z}$, so $\Gamma_4 = [\Delta_4, \Delta_4] \subset \Delta_4^{(2)}$ has index 2. It was shown in [23] that Γ_n is isomorphic to the Fibonacci group $F(2, 2n)$ from [16], and the abelianization of Γ_4 can be calculated from the presentation of $F(2, 8)$ given below, $\Gamma_4^{\mathrm{ab}} = H_1(M_4, \mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$.

Proof. Suppose that Λ is a finitely generated residually finite group with $\widehat{\Gamma} \cong \widehat{\Lambda}$. Using Theorem 6.1(3), local uniformity and, the uniqueness of the maximal order \mathcal{O} (as noted above), together with [11, Corollary 4.10], we can build a representation $\phi : \Lambda \rightarrow \Gamma_{\mathcal{O}}^1$, with non-elementary image L say. We claim that $L \subset \Gamma_4$. This follows from the calculation of Γ_4^{ab} and the fact that $[\Gamma_{\mathcal{O}}^1 : \Gamma_4] = 2$: if L were not contained in Γ_4 , then $L \cap \Gamma_4$ would have index 2 in L , but since Γ_4^{ab} cannot map onto $\mathbb{Z}/2\mathbb{Z}$, neither can Γ_4 or L . As Γ_4 is torsion-free, if L were of infinite index in Γ_4 then it would be the fundamental group of a 3-manifold whose compact core had a boundary component of positive genus, and it would follow by duality (“half lives, half dies”) that L had infinite abelianization. But Γ_4 has finite abelianization, so Λ (hence L) does too. Thus we may assume that L has finite index in Γ_4 , in which case Proposition 5.1 applies to show $L = \Gamma_4$. The argument is completed, as in Section 5.1, by appealing to the Hopfian property for profinite groups. \square

6.1. Some lattices containing Γ_4 . The results in this section illustrate how the ideas presented in Section 4 can be used in practice. A discussion of the lattices that contain Γ_4 as a normal subgroup will suffice to make this point but a wider analysis covering all of the lattices containing Γ_4 would be feasible.

Borel [5] classified the maximal lattices in the commensurability classes of arithmetic lattices. In the case of Γ_4 , as noted above, the invariant quaternion algebra B has type number 1, and so there is a unique lattice of minimal co-volume in its commensurability class, up to conjugacy. Borel’s volume formula shows that this lattice $\Gamma_{\emptyset, \emptyset}$ has volume $v_0/8$, where v_0 is the volume of the regular ideal simplex in \mathbb{H}^3 . Jones and Reid [27] obtain the following presentation,

$$\Gamma_{\emptyset, \emptyset} = \langle x, y, z \mid x^2 = y^2 = z^2 = (xyz)^4 = (xyxyz)^2 = (yzxz)^2 = 1 \rangle.$$

The arguments on pages 172–176 of [40] identify $\Gamma_{\emptyset, \emptyset}$ as the normalizer in $\mathrm{PSL}(2, \mathbb{C})$ of $\Gamma_{\mathcal{O}}^1$ and we shall see in a moment that $\Gamma_{\emptyset, \emptyset}$ is also the normalizer of Γ_4 . Thus we have a chain of normal subgroups

$$\Gamma_4 < \Delta_4^{(2)} = \Gamma_{\mathcal{O}}^1 < \Delta_4 < \Gamma_{\emptyset, \emptyset}$$

where the first two are index-2 and the last is index-4. The other lattices Ω with $\Gamma_4 < \Omega < \Gamma_{\emptyset, \emptyset}$ correspond to subgroups in $\Gamma_{\emptyset, \emptyset}/\Gamma_4$, which is a group of order 16.

It is shown in [40, pp. 172–173] that M_4 double covers a manifold M which is now known as $\mathrm{Vol}(3)$. This is the only manifold that M_4 covers non-trivially. The name $\mathrm{Vol}(3)$ was given to reflect the fact that this manifold had the third smallest known volume of a closed hyperbolic 3-manifold (namely v_0). As noted in Section 1, it has recently been proved [19] that $\mathrm{Vol}(3)$ is the unique closed orientable hyperbolic 3-manifold of volume v_0 . (It was previously known that $\mathrm{Vol}(3)$ was

the only closed orientable arithmetic hyperbolic 3-manifold with volume is v_0 .) In particular, up to conjugacy, $\Pi_3 := \pi_1(\text{Vol}(3))$ is the only torsion-free lattice among the Ω described above. It is proved in [27] that Π_3 is normal in $\Gamma_{\emptyset, \emptyset}$ with quotient a dihedral group of order 8. Moreover $H_1(\text{Vol}(3), \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, so Π_3 has a unique subgroup of index 2, which is Γ_4 . As Γ_4 is characteristic in Π_3 , it too is normal in $\Gamma_{\emptyset, \emptyset}$. And since $\Gamma_{\emptyset, \emptyset}$ is a maximal subgroup $\text{PSL}(2, \mathbb{C})$, this is precisely the normalizer of Γ_4 in $\text{PSL}(2, \mathbb{C})$.

Corollary 6.3. *The groups $\Gamma_{\mathcal{O}}^1$, Δ_4 , Π_3 and $\Gamma_{\emptyset, \emptyset}$ are all profinitely rigid in the absolute sense.*

Proof. Non-elementary representations of $\Gamma_{\mathcal{O}}^1$ restrict to non-elementary representations of Γ_4 and non-conjugate ones remain non-conjugate. Therefore $\Gamma_{\mathcal{O}}^1 = \Delta_4^{(2)}$ is Galois rigid and the proof of Theorem 6.2 shows that it is profinitely rigid.

It follows from Corollary 4.3 that in order to prove that Π_3 is profinitely rigid, we need only distinguish it from lattices of the same co-volume that contain Γ_4 as a subgroup of index 2. By Corollary 2.5 we can also restrict to lattices that are torsion-free. And, as we noted above, Π_3 is the unique such lattice.

As $\Gamma_{\emptyset, \emptyset}$ is the unique lattice of minimal co-volume containing Γ_4 , Corollary 4.3 tells us that it too is profinitely rigid.

Using Magma [6] to enumerate the subgroups of index 4 in $\Gamma_{\emptyset, \emptyset}$, one finds that there are 11 conjugacy classes, only one of which has abelianization $\mathbb{Z}/4\mathbb{Z}$, so this must be Δ_4 . As $\Gamma_{\emptyset, \emptyset}$ is the normalizer of $\Gamma_{\mathcal{O}}^1 = \Delta_4^{(2)}$, it follows that, up to conjugacy, Δ_4 is the only lattice that contains $\Gamma_{\mathcal{O}}^1$ as a subgroup of index 2 and has abelianization $\mathbb{Z}/4\mathbb{Z}$. Thus Δ_4 is profinitely rigid, by Corollary 4.3. \square

With further computation one could extend the above analysis to the other Ω with $\Gamma_4 < \Omega < \Gamma_{\emptyset, \emptyset}$. We mention one other example in order to illustrate a further point concerning the ideas in Section 4, namely that one can use Δ/Λ as an invariant when $\Lambda < \Delta$ is normal and profinitely rigid. In the current setting, as $\Gamma_{\emptyset, \emptyset}/\Pi_3$ is dihedral of order 8, there is a unique lattice $\Omega_4 < \Gamma_{\emptyset, \emptyset}$ that contains Π_3 as a normal subgroup such that the quotient is cyclic of order 4. From Corollary 4.3 and the last item in Proposition 2.1 we deduce:

Corollary 6.4. *Ω_4 is profinitely rigid.*

7. GALOIS RIGIDITY FOR CYCLIC BRANCHED COVERS OF THE FIGURE-EIGHT KNOT

Our proof of Galois rigidity for Δ_n (Theorem 6.1) is based on the understanding of the $\text{PSL}(2, \mathbb{C})$ representations of the fundamental group of the figure-eight knot complement

$$\pi_1(S^3 \setminus K) = \langle \alpha, \beta \mid (\alpha\beta^{-1}\alpha^{-1}\beta)\alpha(\beta^{-1}\alpha\beta\alpha^{-1} = \beta) \rangle.$$

In this presentation, the (conjugate) generators α and β represent meridians of the knot, so $\Delta_n = \pi_1^{\text{orb}}(Q_n)$ is obtained by adding the relation $\alpha^n = 1$. It is easy to check that Δ_2 is the dihedral group of order 8 while Δ_3 is a virtually abelian group.

7.1. Δ_n is Galois rigid if $n \geq 5$ is prime. Although we wish to understand $\text{PSL}(2, \mathbb{C})$ representations, it will be convenient to work with $\text{SL}(2, \mathbb{C})$ representations. To that end, consider the canonical component X_0 of the $\text{SL}(2, \mathbb{C})$ character variety for $\pi_1(S^3 \setminus K)$; this is the component containing the characters of all irreducible $\text{SL}(2, \mathbb{C})$ representations of $\pi_1(S^3 \setminus K)$. It can be identified with the vanishing set of the polynomial

$$P(T, R) = 1 + R - R^2 - 2T^2 + RT^2,$$

where $T = \chi_\rho(\alpha) = \chi_\rho(\beta)$ and $R = \chi_\rho(\alpha\beta)$ (see [15, Section 7] for example). The character of each irreducible representation of Δ_n corresponds to a point on X_0 where the relation $\alpha^n = 1$ holds in $\text{PSL}(2, \mathbb{C})$, i.e. where T specializes to $T_{n,k} = 2\cos(k\pi/n)$ with $k \in \{1, \dots, n-1\}$. Recall that

when n is prime $\mathbb{Q}(\cos \pi/n)$ is a Galois extension of \mathbb{Q} of degree $(n-1)/2$, and moreover given $k \in \{1, \dots, n-1\}$ there is exactly one j such that $2\cos(j\pi/n) = -2\cos(k\pi/n)$.

Each of these specializations gives rise to a character determined by solving for R in

$$p_{n,k}(R) = 1 + R - R^2 - 2T_{n,k}^2 + RT_{n,k}^2.$$

Thus Δ_n has (at most) $2(n-1)$ irreducible $\mathrm{SL}(2, \mathbb{C})$ representations, up to conjugacy. Furthermore, the preceding comment about signs implies that the set of characters for these representations can be divided up so that each character in one half is the negative of one in the other half. Thus, on projecting to $\mathrm{PSL}(2, \mathbb{C})$, we have exactly $n-1$ representations, up to conjugacy. Hence we will be done if we can argue that the trace-field of Δ_n has degree $(n-1)$ (since $n \geq 5$ is prime the trace-field coincides with the invariant trace-field).

To see that this is the case, we consider the character of the discrete faithful representation: here, T specializes to $T_{1,n} = 2\cos(\pi/n)$ and we note that $p_{n,1}(R)$ must be irreducible over $\mathbb{Q}(\cos(\pi/n))$, for if not then the trace field $\mathbb{Q}(\mathrm{tr}(\Delta_n)) = \mathbb{Q}(T_{1,n}, R_n) = \mathbb{Q}(T_{1,n}, \bar{R}_n)$ would be totally real, which it is not since Q_n is a closed hyperbolic 3-orbifold. (Here, we have denoted the roots of $p_{1,n}(R)$ by R_n, \bar{R}_n .) Thus the trace field of Δ_n is a number field of degree $(n-1)$ and so Δ_n is Galois rigid. (The representations corresponding to characters arising from the specializations $T = T_{k,n}$ with $k > 1$ are appearing as Galois conjugates of the discrete faithful representation.) \square

7.2. Δ_n is Galois rigid if $n \in \{4, 6, 9\}$. We noted earlier that $\pi_1(S^3 \setminus K)$ becomes finite if one imposes the relation $\alpha^2 = 1$ and virtually abelian if one imposes $\alpha^3 = 1$. Thus, for $n \in \{4, 6, 9\}$, in any irreducible representation of Δ_n , the image of α must have order at least 4. With this additional information, the preceding proof applies upon specializing T in the following way: $\pm 2\sqrt{2}$ (when $n = 4$), $\pm 2\sqrt{3}$ (when $n = 6$) and $\pm 2\cos(\pi/9)$, $\pm 2\cos(2\pi/9)$, $\pm 2\cos(4\pi/9)$ ($n = 9$). \square

Remark 7.1. Note Galois rigidity fails for $n = 8$ since Δ_8 admits an epimorphism onto Δ_4 .

7.3. Γ_4 is Galois rigid. This proof is of a different nature to the one above. Instead of locating representations within the well-understood character variety of a group mapping onto Γ_4 , we take a more brute force approach, analysing directly what the irreducible representations of Γ_4 can be, starting from a specific finite presentation and making explicit computations in Mathematica [46], as described below. The Mathematica notebook is available from the authors upon request. The presentation that we work with comes from the isomorphism $\Gamma_4 \cong F(2, 8)$ with a Fibonacci group,

$$\langle x_1, \dots, x_8 \mid x_1x_2x_3^{-1}, x_2x_3x_4^{-1}, \dots, x_6x_7x_8^{-1}, x_7x_8x_1^{-1}, x_8x_1x_2^{-1} \rangle,$$

which reduces to the following two generator presentation:

$$(1) \quad \Gamma_4 = \langle a, b \mid ba^{-2}ba^{-1}b^2ab^2a^{-1}, a^2bab^2aba^2b^{-1} \rangle$$

where $a = x_3$ and $b = x_4$. One can check directly (by hand or computer) that imposing either of the additional relations $a^2 = 1$ or $a^3 = 1$ yields a finite quotient, and this will be useful in our analysis of the characters of irreducible $\mathrm{SL}(2, \mathbb{C})$ -representations of Γ_4 . Before we begin, we note that $H^2(\Gamma_4, \mathbb{Z}/2) = H_1(\Gamma_4, \mathbb{Z}/2) = 0$ by Poincaré duality, so the standard discrete faithful representation $\Gamma_4 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ and its complex conjugate lift to $\mathrm{SL}(2, \mathbb{C})$, and so as above, it is convenient to work in SL rather than PSL .

Since we are only interested in irreducible representations, we can assume that if $\rho : \Gamma_4 \rightarrow \mathrm{SL}(2, \mathbb{C})$ is such a representation then $\rho(\Gamma_4)$ is not conjugate into the subgroup consisting of upper triangular matrices in $\mathrm{SL}(2, \mathbb{C})$. Standard considerations ensure that we can conjugate ρ so that $\rho(a)$ fixes ∞ and $\rho(b)$ fixes 0; i.e.

$$\rho(a) = \begin{pmatrix} x & 1 \\ 0 & 1/x \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} y & 0 \\ r & 1/y \end{pmatrix}.$$

Referring to (1), to handle evaluation in Mathematica of the first relation on the matrices, we split it up as follows: `rel1=Factor[w1-Inverse[w2]]`, where $w1 = ba^{-2}ba^{-1}b^2$ and $w2 = ab^2a^{-1}$. This results in a matrix R whose entries are shown in §7.4 and which solves to zero. Using the (1,2)-entry of R we can solve for r in terms of x and y , namely

$$r = \frac{x^4 + x^3y^4 - x^3 + x^2 + y^2}{xy(x^2 - xy^2 - x + 1)}.$$

We then re-evaluate the first relation in (1) using this expression for r . This gives the matrix R_1 shown in §7.4 solving to zero. To evaluate the second relation of (1) on matrices, we again split this up: `rel2=Factor[w3-Inverse[w4]]`, where $w3 = a^2bab^2$ and $w4 = aba^2b^{-1}$. Using the value of r displayed above, results in a matrix S which solves to zero. The numerators s_i $i = 1, \dots, 4$ of the entries of S are displayed in §7.4.

As discussed in §7.4, an analysis using Mathematica shows that the only possible solution for x arises from roots of the polynomial $p(x) = x^4 - x^3 + 3x^2 - x + 1 = 0$. Note that $p(x)$ is a symmetric polynomial, so that setting x_0 to be a root of $p(x)$, the other roots are $1/x_0$, \bar{x}_0 , and $1/\bar{x}_0$. Converting to characters shows that $\chi_\rho(a)$ is a root of the equation $X^2 - X + 1 = 0$. A similar analysis shows that this polynomial also provides the only possibilities for y , and we see that $\chi_\rho(b) = \chi_\rho(a)$ or $\chi_\rho(b) = \bar{\chi}_\rho(a)$. It is also shown in §7.4 that r satisfies a degree 4 polynomial.

As discussed in §7.4, the character of the faithful discrete representation ρ_0 of Γ_4 satisfies $\chi_{\rho_0}(b) = \bar{\chi}_{\rho_0}(a)$, and $\chi_{\rho_0}(ab) = \chi_{\rho_0}(a)$, and that the only other distinct character of an irreducible representation of Γ_4 is that obtained by applying complex conjugation to ρ_0 . It follows that Γ_4 is Galois rigid as claimed. \square

7.4. Mathematica output. The matrix R :

$$\begin{aligned} (1, 1) - \text{entry} &= \frac{y^6 - rx^3y^5 - 2rxy^5 + r^2x^4y^4 + r^2x^2y^4 - rx^5y^3 - rx^3y^3 + rx^2y^3 - rxy^3 + r^2x^4y^2 + r^2x^2y^2 - rx^5y - rx^3y + rx^2y - x^3}{x^3y^2} \\ (1, 2) - \text{entry} &= -\frac{x^4 + y^4x^3 - rxy^3 - x^3 + ry^3x^2 + ryx^2 + x^2 - ryx + y^2}{x^2y^2} \\ (2, 1) - \text{entry} &= \frac{r(y^2x^6 + x^6 - 2ry^3x^5 - 2ryx^5 + r^2y^4x^4 + y^4x^4 + r^2y^2x^4 - ry^5x^3 - ry^3x^3 - ryx^3 + r^2y^4x^2 + r^2y^2x^2 - 2ry^5x + y^4x - ry^3x + y^2x + y^6)}{x^3y^3} \\ (2, 2) - \text{entry} &= -\frac{x^2y^6 + rxy^5 + ry^3 + rxy^3 - r^2x^3y^2 - r^2xy^2 + 2rx^4y + rx^2y - x^5}{x^2y^4} \end{aligned}$$

The matrix R_1 has (1,2)-entry 0 and

$$\begin{aligned} (1, 1) - \text{entry} &= \frac{(x^2+1)(x^2+x+1)(xy^4+x^2-x+1)(x^2y^4-xy^4+y^4+2x^2y^2-3xy^2+2y^2+x^2-x+1)}{x^2(x^2-y^2x-x+1)^2} \\ (2, 1) - \text{entry} &= \frac{(x^2+1)(x^2y^4-xy^4+y^4+2x^2y^2-3xy^2+2y^2+x^2-x+1)(x^4+y^4x^3-x^3+x^2+y^2)(x^3y^4+x^2y^4+xy^4+x^3y^2+x^2y^2+x^2-x+1)}{x^3y^2(x^2-y^2x-x+1)^3} \\ (2, 2) - \text{entry} &= \frac{(x^2+1)(x^2+x+1)(x^2y^4-xy^4+y^4+2x^2y^2-3xy^2+2y^2+x^2-x+1)}{x(x^2-y^2x-x+1)^2} \end{aligned}$$

We require R_1 to be the zero matrix in order to get a representation, and focussing on the (2,2) entry will be particularly informative: the factors $x^2 + 1$ and $x^2 + x + 1$ can be disregarded, because setting them equal to zero would imply that a had order 2 or 3, and we noted earlier that Γ_4 does not have an infinite quotient in which a has order 2 or 3. Thus the (2,2) entry tells us that $x^2y^4 - xy^4 + y^4 + 2x^2y^2 - 3xy^2 + 2y^2 + x^2 - x + 1 = 0$.

The matrix S .

$$\begin{aligned} s_1 &= x^{10}y^6 + 2x^{10}y^4 + 2x^{10}y^2 + x^{10} - x^9y^6 - 3x^9y^4 - 4x^9y^2 - 2x^9 + 2x^8y^6 + 6x^8y^4 - x^8y^3 + 8x^8y^2 + 5x^8 + \\ &+ x^7y^8 - x^7y^7 + x^7y^6 + x^7y^5 - 3x^7y^4 + 2x^7y^3 - 9x^7y^2 - 6x^7 + x^6y^9 - x^6y^8 + x^6y^7 - x^6y^5 + 6x^6y^4 - 4x^6y^3 + \\ &+ 12x^6y^2 + 8x^6 + x^5y^8 - 2x^5y^7 + 2x^5y^6 + 2x^5y^5 - 3x^5y^4 + 4x^5y^3 - 10x^5y^2 - 6x^5 + x^4y^9 - x^4y^8 + x^4y^7 + \end{aligned}$$

$$x^4y^6 - 2x^4y^5 + 7x^4y^4 - 5x^4y^3 + 11x^4y^2 + 5x^4 + x^3y^8 + x^3y^6 + 4x^3y^5 - 4x^3y^4 + 4x^3y^3 - 6x^3y^2 - 2x^3 - x^2y^7 + 2x^2y^6 - 4x^2y^5 + 5x^2y^4 - 4x^2y^3 + 5x^2y^2 + x^2 + xy^7 - xy^6 + 3xy^5 - 2xy^4 + 2xy^3 - xy^2 + y^6 - y^5 + 2y^4 - y^3 + y^2$$

$$s_2 = (y+1)(x^6y^4 - x^6y^3 + 2x^6y^2 - x^6y + x^6 - x^5y^4 + x^5y^3 - 2x^5y^2 + x^5y - x^5 + 2x^4y^4 - 2x^4y^3 + 4x^4y^2 - 3x^4y + 3x^4 - x^3y^4 + 2x^3y^3 - 3x^3y^2 + 2x^3y - 2x^3 + 2x^2y^4 - 2x^2y^3 + 4x^2y^2 - 3x^2y + 3x^2 - xy^4 + xy^3 - 2xy^2 + xy - x + y^4 - y^3 + 2y^2 - y + 1)$$

$$s_3 = -(x^4 + x^3y^4 - x^3 + x^2 + y^2)(x^5y^5 + x^5y^3 - x^4y^4 - x^4y^2 + x^4y - x^4 + x^3y^5 + x^3y^2 - x^3y + x^3 - x^2y^4 + x^2y^3 - 2x^2y^2 + 2x^2y - 2x^2 + xy^4 - xy^3 + 2xy^2 - xy + x - y^4 + y^3 - 2y^2 + y - 1)$$

$$s_4 = x^9y^7 + 2x^9y^5 + x^9y^3 - x^8y^7 - 2x^8y^5 + x^8y + 2x^7y^7 + 3x^7y^5 - x^7y^3 - 2x^7y - x^6y^7 + 4x^6y^3 + 4x^6y + x^6 + 2x^5y^7 + 2x^5y^5 + x^5y^4 - 4x^5y^3 - x^5y^2 - 4x^5y - 2x^5 - x^4y^7 - x^4y^6 + x^4y^5 - x^4y^4 + 5x^4y^3 + x^4y^2 + 4x^4y + 4x^4 + x^3y^7 + x^3y^4 - 3x^3y^3 - 3x^3y^2 - 2x^3y - 4x^3 + 2x^2y^5 + x^2y^4 + 3x^2y^3 + 3x^2y^2 + x^2y + 4x^2 - xy^5 - xy^4 - xy^3 - 3xy^2 - 2x + y^5 + y^3 + y^2 + 1$$

To determine the possibilities for x , y and r we use the **Resultant** feature in Mathematica to eliminate variables and determine solutions. We have already noted that from the $(2, 2)$ entry of R_1 we know that $x^2y^4 - xy^4 + y^4 + 2x^2y^2 - 3xy^2 + 2y^2 + x^2 - x + 1$ must vanish. On taking resultant with s_i , for $i = 1, \dots, 4$ to eliminate y we are left with the polynomial $p(x) = x^4 - x^3 + 3x^2 - x + 1$.

On taking the resultant of $p(x)$ and $(x^2y^4 - xy^4 + y^4 + 2x^2y^2 - 3xy^2 + 2y^2 + x^2 - x + 1)$ to eliminate x we obtain two possible polynomials: $(y^4 - y^3 + 3y^2 - y + 1)$ and $(y^4 + y^3 + 3y^2 + y + 1)$. Note the latter polynomial is the former evaluated at $-y$. This determines the same $\text{PSL}(2, \mathbb{C})$ representation, and so we deduce the polynomial that y satisfies is also $p(x)$.

Returning to the matrix R , we can use the polynomials that were obtained for x and y and use resultants with each of the numerators of the entries of R to determine r . The only possible common factor is the polynomial $r^4 - 9r^2 + 36$. Note this is not quadratic, but we know that for the faithful discrete representation ρ_0 , characters $\chi_{\rho_0}(a)$, $\chi_{\rho_0}(b)$ and $\chi_{\rho_0}(ab)$ lie in $\mathbb{Q}(\sqrt{-3})$. This occurs when $y = \bar{x}_0$, where x_0 is (approximately) the root $0.14840294359835 - 0.632502179219i$ of $p(x)$ and the corresponding value of r is (approximately) $r_0 = -2.29128784747792 + 0.8660254037844386467i$. Setting $x = 1/x_0$ and $y = 1/\bar{x}_0$ with $r = r_0$ provides a representation with the same character as ρ_0 and hence determines a conjugate representation (see [18, Proposition 1.5.2]). Finally we note that setting $x = y = x_0$ does not yield a representation (since no value of r solves the first relation of our presentation).

Thus we have shown that the only possible values of x, y, r are those that give the discrete faithful representation and its conjugate, and the proof that Γ_4 is Galois rigid is complete.

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