# Arithmetic Hyperbolic Manifolds

**Alan W. Reid**

University of Texas at Austin

*Cortona, June 2017*

## Plan for the lectures

A basic example and some preliminary material on bilinear and quadratic forms, $\mathbb{H}^n$ and $\mathrm{Isom}(\mathbb{H}^{\mathrm{n}})$.

Arithmetic hyperbolic manifolds of simplest type.

Why you might care.

Geometric bounding

Dimensions 2 and 3 versus higher dimensions.

## A basic example $\mathrm{PSL}(2,\mathbb{Z})$

$\mathrm{SL}_2(\mathbb{R})$ acts on the set $\mathcal{S}$ of $2 \times 2$ real symmetric matrices.

Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, and $S = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in \mathcal{S}$ we have:

$$g \cdot S \mapsto gSg^t.$$

Note that since $g \in \mathrm{SL}_2(\mathbb{R})$,

$$\det(gSg^t) = \det(S) = xz - y^2.$$

$\mathcal{S}$ is a 3-dimensional vector space, and using a basis for $\mathcal{S}$ we get a representation $\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}_3(\mathbb{R})$:

$$\rho(g) = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & bc + ad & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

$\det(\rho(g)) = 1$ so $\rho(g) \in \mathrm{SL}_3(\mathbb{R})$.

Using $\det(gSg^t) = \det(S) = xz - y^2$, it follows that $\rho(g)$ preserves the quadratic form $xz - y^2$; i.e. Set

$$J = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$\rho(g).J.\rho(g)^t = J.$$

$\ker(\rho) = \pm I$ and so this gives an isomorphism of $PSL_2(\mathbb{R})$ onto a subgroup of:

$$SO(xz - y^2, \mathbb{R}) = \{X \in SL_3(\mathbb{R}) : XJX^t = J\}.$$

In fact $PSL_2(\mathbb{R}) \cong$ a subgroup of index 2.

Moreover this maps $PSL_2(\mathbb{Z})$ onto a subgroup of

$$SO(xz - y^2, \mathbb{Z}) = \{X \in SL_3(\mathbb{Z}) : XJX^t = J\}.$$

Make a change of basis: $u = (x+z)/2$ and $v = (x-z)/2$.

$$xz - y^2 = u^2 - v^2 - y^2.$$

$\mathrm{PSL}_2(\mathbb{R})$ still maps isomorphically onto a subgroup of $\mathrm{SO}(\mathrm{u}^2 - \mathrm{v}^2 - \mathrm{y}^2, \mathbb{R})$

$$= \{X \in \mathrm{SL}_3(\mathbb{R}) : \mathrm{X}\mathrm{diag}\{1,-1,-1\}\mathrm{X}^{\mathrm{t}} = \mathrm{diag}\{1,-1,-1\}\}$$

but:

$\mathrm{PSL}_2(\mathbb{Z})$ does not map into $\mathrm{SO}(\mathrm{u}^2 - \mathrm{v}^2 - \mathrm{y}^2, \mathbb{Z})$.

Another comment on this representation of $\mathrm{PSL}(2,\mathbb{R})$:

Suppose $n > 1$ and let $\Gamma_0(n) < \mathrm{PSL}(2,\mathbb{Z})$ denote the subgroup consisting of those elements congruent to $\pm \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \pmod{n}$.

Note that $\tau_n = \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}$ normalizes $\Gamma_0(n)$.

Hence $\langle \Gamma_0(n), \tau_n \rangle \subset N_{\mathrm{PSL}(2,\mathbb{R})}(\Gamma_0(n))$ is

commensurable with $\mathrm{PSL}(2,\mathbb{Z})$,

not a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ or even $\mathrm{PSL}(2,\mathbb{Q})$ if $n$ is square-free.

But under the representation $\rho$ described above:

$$\rho(\tau_n) = \begin{pmatrix} 0 & 0 & \frac{1}{n} \\ 0 & -1 & 0 \\ n & 0 & 0 \end{pmatrix}$$

it is rational!

## Whats so special about $xz - y^2$ or $u^2 - v^2 - y^2$

Take $ax^2 + by^2 - cy^2$, $a, b, c$ integers and $> 0$

Consider

$$= \{X \in \mathrm{SL}_3(\mathbb{R}) : \mathrm{Xdiag}\{\mathrm{a}, \mathrm{b}, -\mathrm{c}\}\mathrm{X}^{\mathrm{t}} = \mathrm{diag}\{\mathrm{a}, \mathrm{b}, -\mathrm{c}\}\}$$

and the discrete subgroup:

$$= \{X \in \mathrm{SL}_3(\mathbb{Z}) : \mathrm{Xdiag}\{\mathrm{a}, \mathrm{b}, -\mathrm{c}\}\mathrm{X}^{\mathrm{t}} = \mathrm{diag}\{\mathrm{a}, \mathrm{b}, -\mathrm{c}\}\}$$

What can we say about this discrete group?

They are infinite.

$\{X \in \mathrm{SL}_2(\mathbb{Z}) : \mathrm{Xdiag}\{\mathrm{b}, -\mathrm{c}\}\mathrm{X}^{\mathrm{t}} = \mathrm{diag}\{\mathrm{b}, -\mathrm{c}\}\}$ gives an infinite cyclic subgroup.

e.g Take $y^2 - 3z^2$, and $X = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, then
$X\mathrm{diag}\{1, -3\}\mathrm{X}^{\mathrm{t}} = \mathrm{diag}\{1, -3\}$.

If $ax_0^2 + by_0^2 - cz_0^2 = 0$ then can build a unipotent element ($x_0, y_0, z_0$ not all 0).

## Bilinear and Quadratic Forms

Let $V$ be a finite dimensional vector space over $k$, with characteristic of $k \neq 2$.

By a (symmetric) bilinear form $B$ on $V$, we mean a map

$$B : V \times V \longrightarrow k$$

such that

(i) $B(u, v) = B(v, u)$, for all $u, v \in V$.

(ii) $B(u + u', v) = B(u, v) + B(u', v)$. for all $u, u', v \in V$.

(iii) $B(\alpha u, v) = \alpha B(u, v)$, for all $\alpha \in k$ and $u, v \in V$.

Definition *With $V$ and $B$ as above, we call $(V, B)$ a* ***bilinear space****.*

Associated to $B$ is a **quadratic map**

$$q : V \longrightarrow k$$

defined by

$$q(v) = B(v, v).$$

We see that $q$ satisfies

(i)
$$q(\alpha v) = \alpha^2 q(v),$$
for all $\alpha \in k$ and $v \in V$.

(ii)
$$q(u + v) - q(u) - q(v) = 2B(u, v),$$
for all $u, v \in V$.

By specifying a basis for $V$, $\mathcal{B} = \{e_i\}$, one can write $B$ and $q$ as follows:

Associated to $B$ is the symmetric matrix

$$\Big(B(e_i, e_j)\Big)$$

and

$$q = q_{\mathcal{B}}(x) = x^T \Big(B(e_i, e_j)\Big) x.$$

is the associated quadratic form for the basis $\mathcal{B}$.

All bilinear forms (or quadratic forms) will be non-degenerate (i.e. $B(x, y) = 0$ for all $y \in V$ implies $x = 0$)

## Example:

Let $V = \mathbb{R}^{n+1}$ with the standard basis $\mathcal{B} = \{e_i\}$.

Define $B =< \cdot, \cdot >$ by

$$< x, y >= x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1},$$

where

$$x = (x_1, \ldots, x_{n+1}) \quad y = (y_1, \ldots, y_{n+1}).$$

and the quadratic form

$$q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2.$$

Let $V_1$ and $V_2$ be $n$-dimensional vector spaces over $k$ equipped with quadratic forms $q_1$ and $q_2$ (call the associated symmetric matrices $Q_1$ and $Q_2$).

Say $(V_1, q_1)$ is equivalent over $k$ to $(V_2, q_2)$ if there exists $T \in \mathrm{GL_n(k)}$ so that:

$$T^t Q_1 T = Q_2$$

Write $q_1 \simeq_k q_2$.

Example Take $V_1 = V_2 = \mathbb{R}^3$ and

$$q_1 = x_1^2 + x_2^2 - x_3^2, \ \ q_2 = x_1^2 + x_2^2 - 3x_3^2$$

$$q_3 = x_1^2 + x_2^2 - 4x_3^2, \ \ q_4 = x_1 x_2 + x_3^2$$

$q_1 \simeq_{\mathbb{R}} q_2$

$q_1 \simeq_{\mathbb{Q}} q_3$

Is $q_1 \simeq_{\mathbb{Q}} q_2$?

$q_1 \simeq_{\mathbb{Q}} q_4$.

## Equivalence over $\mathbb{R}$

Let $V = \mathbb{R}^n$, $B$ and $q$ be bilinear and quadratic forms.

Sylvester's Law: There exists a basis $\{v_1, \ldots, v_n\}$ of $V$ such that $q$ has the description

$$Q = \Big(B(v_i, v_j)\Big) = \begin{cases} 0, & i \neq j \\ 1, & 1 \leq i \leq p \\ -1, & p < i \leq n \end{cases},$$

for some $p$.

So

$$Q = \operatorname{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1),$$

with $p$ 1's and $s = (n-p)$ $-1$'s.

$(p, s)$ is called the signature of the form.

If $Q_1$ and $Q_2$ are symmetric, invertible matrices over $\mathbb{R}$, then

$$Q_1 \simeq_{\mathbb{R}} Q_2,$$

if and only if the signature of $Q_1$ and $Q_2$ are the same.

Example

The forms

$$q = x_1^2 + \ldots x_n^2 - x_{n+1}^2, \;\; q' = x_1^2 + \ldots x_n^2 - \sqrt{2}x_{n+1}^2$$

have signature $(n, 1)$ and so are equivalent over $\mathbb{R}$.

The forms

$$q_1 = x_1^2 + \ldots x_n^2 + x_{n+1}^2, \;\; q_1' = x_1^2 + \ldots x_n^2 + \sqrt{2}x_{n+1}^2$$

have signature $(n+1, 0)$ and so are equivalent over $\mathbb{R}$.

## Hyperboloid Model

$< \cdot, \cdot >$ will denote the bilinear form on $\mathbb{R}^{n+1}$ described earlier.

Let

$$\mathbb{H}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \ : \ < x, x >= -1, \ x_{n+1} > 0\}.$$

We shall define a metric $d$ on $\mathbb{H}^n$ and $(\mathbb{H}^n, d)$ will be the hyperboloid model of hyperbolic $n$-space.

**Proposition**

*Let*

$$d : \mathbb{H}^n \times \mathbb{H}^n \longrightarrow \mathbb{R}$$

*be the function that assigns to each pair* $(x, y) \in \mathbb{H}^n \times \mathbb{H}^n$ *the unique number* $d(x, y) \geq 0$ *such that*

$$\cosh d(x, y) = - < x, y > .$$

*Then* $d$ *is a metric on* $\mathbb{H}^n$*.*

Remarks: $\cosh d(x,y) = - < x, y >$ is well-defined since

$$< x, y > \leq -1 \text{ for all } x, y \in \mathbb{H}^n$$

(Cauchy Schwartz)

Equality holds iff $x = y$ since $< x, y >= -1$ iff $x = y$.

Symmetry follows from $< x, y >=< y, x >$.

Triangle inequality requires work —uses the Hyperbolic Law of Cosines:

Let $A$, $B$, and $C$ be distinct points in $\mathbb{H}^n$. Form the triangle with these points as the vertices: Let $\gamma$ be the angle at the vertex of $C$: With $a = d(B, C)$, $b = d(A, C)$, and $c = d(A, B)$,

Then

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.$$

## Isometries

$$O(n,1) = \{S \in GL_{n+1}(\mathbb{R}) \ : \ S^T J_n S = J_n\},$$

where

$$J_n = \mathrm{diag}(1, 1, \ldots, 1, -1)$$

is called the Orthogonal group of the quadratic form

$$x_1^2 + \cdots + x_n^2 - x_{n+1}^2.$$

This is equivalent to

$$O(n,1) = \{S \in GL_{n+1}(\mathbb{R}) \ : \ < Su, Sv > = < u, v >, \ u, v \in \mathbb{R}^{n+1}\},$$

where $< \cdot, \cdot >$ is as before.

Note that the matrix $\mathrm{diag}(-1, 1, \ldots, 1) \in \mathrm{O}(\mathrm{n}, 1)$ has determinant $-1$.

Define: $\mathrm{SO}(\mathrm{n}, 1) = \mathrm{O}(\mathrm{n}, 1) \cap \mathrm{SL}_{\mathrm{n}+1}(\mathbb{R})$, called the Special Orthogonal Group.

This has index 2 in $\mathrm{O}(\mathrm{n}, 1)$.

Define

$$\mathrm{O}_0(\mathrm{n}, 1) = \{\mathrm{S} \in \mathrm{O}(\mathrm{n}, 1) \ : \ \mathrm{S} \text{ preserves } \mathbb{H}^{\mathrm{n}}\},$$

and

$$\mathrm{SO}_0(\mathrm{n}, 1) = \mathrm{SO}(\mathrm{n}, 1) \cap \mathrm{O}_0(\mathrm{n}, 1).$$

$\mathrm{diag}(1, 1, \ldots, 1, -1) \in \mathrm{O}(\mathrm{n}, 1)$ flips $x_{n+1}$ to $-x_{n+1}$ and so

$$[\mathrm{O}(\mathrm{n}, 1) : \mathrm{O}_0(\mathrm{n}, 1)] = 2.$$

By construction of the metric $d$, $\mathrm{O}_0(\mathrm{n}, 1)$ preserves $d$. Thus

$$\mathrm{O}_0(\mathrm{n}, 1) \subset \mathrm{Isom}(\mathbb{H}^{\mathrm{n}}).$$

**Theorem 1**

$\mathrm{O}_0(\mathrm{n}, 1) = \mathrm{Isom}(\mathbb{H}^{\mathrm{n}})$ *and* $\mathrm{SO}_0(\mathrm{n}, 1) = \mathrm{Isom}^+(\mathbb{H}^{\mathrm{n}})$.

**Definition**

By a hyperbolic n-manifold we mean a manifold (resp. orbifold) $M^n = \mathbb{H}^n/\Gamma$ where $\Gamma < \mathrm{O}_0(\mathrm{n}, 1)$ is torsion-free (otherwise).

If $\Gamma < \mathrm{SO}_0(\mathrm{n}, 1)$, $M^n$ is orientable.

$M^n = \mathbb{H}^n/\Gamma$ has finite volume if $\Gamma$ admits a fundamental polyhedron of finite volume (say $\Gamma$ has finite co-volume).

Say $\Gamma$ is cocompact if $M^n$ is closed.

How do we construct examples closed or finite volume hyperbolic *n*-manifolds?

Note: $\mathrm{O(n, 1, \mathbb{Z}) = O(n, 1) \cap GL_{n+1}(\mathbb{Z})}$ is a discrete subgroup of $\mathrm{O(n, 1)}$

Example: $\mathrm{O_0(2, 1, \mathbb{Z}) = (2, 4, \infty)}$ reflection triangle group.

In particular $\mathrm{O_0(2, 1, \mathbb{Z})}$ has finite co-volume and is non-cocompact.

## Reflection generators:

$$\tau_{e_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(reflection in the plane $< z, e_2 >= 0$)

$$\tau_v = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(reflection in the plane $< z, (-1/\sqrt{2}, 1/\sqrt{2}, 0) >= 0$)

$$\tau_u = \begin{pmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix}$$

(reflection in the plane $< z, (1, 1, 1) >= 0$)

## Arithmetic groups of simplest type

Let $k$ be a totally real number field of degree $d$ over $\mathbb{Q}$ equipped with a fixed embedding into $\mathbb{R}$ which we refer to as the identity embedding, and denote the ring of integers of $k$ by $R_k$.

Let $V$ be an $(n+1)$-dimensional vector space over $k$ equipped with a quadratic form f (with associated symmetric matrix $F$) defined over $k$ which has signature $(n, 1)$ at the identity embedding, and signature $(n+1, 0)$ at the remaining $d-1$ embeddings.

Call such quadratic forms admissible.

Define the linear algebraic groups defined over $k$:

$$\mathrm{O(f)} = \{\mathrm{X} \in \mathrm{GL_{n+1}}(\mathbb{C}) : \mathrm{X^tFX} = \mathrm{F}\} \text{ and}$$

$$\mathrm{SO(f)} = \{\mathrm{X} \in \mathrm{SL_{n+1}}(\mathbb{C}) : \mathrm{X^tFX} = \mathrm{F}\}.$$

For a subring $L \subset \mathbb{C}$, we denote the $L$-points of $\mathrm{O(f)}$ (resp. $\mathrm{SO(f)}$) by $\mathrm{O(f, L)}$ (resp. $\mathrm{SO(f, L)}$).

An *arithmetic lattice* in $\mathrm{O(f)}$ (resp. $\mathrm{SO(f)}$) is a subgroup $\Gamma < \mathrm{O(f)}$ commensurable with $\mathrm{O(f, R_k)}$ (resp. $\mathrm{SO(f, R_k)}$).

Note that an arithmetic subgroup of $\mathrm{SO(f)}$ is an arithmetic subgroup of $\mathrm{O(f)}$, and an arithmetic subgroup $\Gamma < \mathrm{O(f)}$ determines an arithmetic subgroup $\Gamma \cap \mathrm{SO(f)}$ in $\mathrm{SO(f)}$.

## Examples:

1. Let $q = a_1x_1^2 + \cdots + a_nx_n^2 - a_{n+1}x_{n+1}^2$, where $a_i > 0$ and $a_i \in \mathbb{Z}$. This has signature $(n, 1)$.

2. $f = x_1^2 + \cdots + x_n^2 - \sqrt{2}x_{n+1}^2$.
Then for the two Galois embeddings

$$\sigma_1 : \quad \sqrt{2} \mapsto \sqrt{2}$$
$$\sigma_2 : \quad \sqrt{2} \mapsto -\sqrt{2}.$$

we see that $f$ has signature $(n, 1)$ and $f_2^{\sigma} = x_1^2 + \cdots + x_n^2 + \sqrt{2}x_{n+1}^2$, so has signature $(n + 1, 0)$.

With $q$ as in Example 1, $\mathrm{SO}(\mathrm{q}, \mathbb{Z})$ is clearly discrete.

What about Example 2?

What about co-compact or finite covolume?

Will discuss this below.

The form $q$ may or may not represent 0 over $\mathbb{Q}$ (equivalently $\mathbb{Z}$).

e.g. $x_1^2 + x_2^2 - x_3^2$ does, $x_1^2 + x_2^2 - 3x_3^2$ does not.
$x_1^2 + x_2^2 + x_3^2 - 3x_3^2$ does, $x_1^2 + x_2^2 + x_3^2 - 7x_3^2$ does not.

Let $d$ be any positive integer then $x_1^2 + x_2^2 + x_3^2 + x_4^2 - dx_5^2$ represents 0 ($d$ is a sum of 4 squares).

More generally Meyer's Theorem shows that whenever $n \geq 4$, $q$ as in Example 1 always represent 0 non-trivially.

In the second example there is no solution over $\mathbb{Q}(\sqrt{2})$ to the equation $f(x) = 0$.

## Recap from Lecture 1

### Arithmetic groups of simplest type

Let $k$ be a totally real number field of degree $d$ over $\mathbb{Q}$ equipped with a fixed embedding into $\mathbb{R}$ which we refer to as the identity embedding, and denote the ring of integers of $k$ by $R_k$.

Let $V$ be an $(n+1)$-dimensional vector space over $k$ equipped with a quadratic form f (with associated symmetric matrix $F$) defined over $k$ which has signature $(n, 1)$ at the identity embedding, and signature $(n+1, 0)$ at the remaining $d-1$ embeddings.

Call such quadratic forms admissible.

Define the linear algebraic groups defined over $k$:

$$\mathrm{O(f)} = \{\mathrm{X} \in \mathrm{GL_{n+1}}(\mathbb{C}) : \mathrm{X^tFX} = \mathrm{F}\}$$

$$\mathrm{SO(f)} = \{\mathrm{X} \in \mathrm{SL_{n+1}}(\mathbb{C}) : \mathrm{X^tFX} = \mathrm{F}\}.$$

For a subring $L \subset \mathbb{C}$, we denote the $L$-points of $\mathrm{O(f)}$ (resp. $\mathrm{SO(f)}$) by:

$$\mathrm{O(f,L)} = \mathrm{O(f)} \cap \mathrm{GL_{n+1}(L)}, \quad \mathrm{SO(f,L)} = \mathrm{SO(f)} \cap \mathrm{O(f,L)}$$

An arithmetic lattice in $\mathrm{O(f)}$ (resp. $\mathrm{SO(f)}$) is a subgroup $\Gamma < \mathrm{O(f)}$ commensurable with $\mathrm{O(f, R_k)}$ (resp. $\mathrm{SO(f, R_k)}$).

Two examples to keep in mind:

1. Let $q = a_1x_1^2 + \cdots + a_nx_n^2 - a_{n+1}x_{n+1}^2$, where $a_i > 0$ and $a_i \in \mathbb{Z}$. This has signature $(n, 1)$.

Meyer's Theorem says this represents 0 non-trivially over $\mathbb{Z}$ whenever $n \geq 4$.

2. $f = x_1^2 + \cdots + x_n^2 - \sqrt{2}x_{n+1}^2$.

$f$ has signature $(n, 1)$ and $f_2^\sigma = x_1^2 + \cdots + x_n^2 + \sqrt{2}x_{n+1}^2$, so has signature $(n+1, 0)$.

## Theorem 2

*Let $q$ be an admissible quadratic form over the totally real field $k$. Then $\mathrm{SO}(q, R_k)$ is a discrete subgroup of finite covolume in $\mathrm{SO}(q, \mathbb{R})$. Moreover it is cocompact if and only if $q$ is anisotropic.*

Suppose that $q$ is a quadratic form of signature $(n, 1)$.

By Sylvester's Theorem, there exists $T \in \mathrm{GL_{n+1}(R)}$ such that $T^t Q T = J_n$.

This effects a conjugation:

$$T^{-1}\mathrm{O(Q, \mathbb{R})T} = \mathrm{O(J_n, \mathbb{R})} = \mathrm{O(n, 1)}.$$

A subgroup $\Gamma < \mathrm{O}_0(\mathrm{n}, 1)$ is called *arithmetic of simplest type* if $\Gamma$ is commensurable with the image in $\mathrm{O}_0(\mathrm{n}, 1)$ of an arithmetic subgroup of $\mathrm{O}(\mathrm{f})$ (under the conjugation map described above).

An arithmetic hyperbolic $n$-manifold $M = \mathbb{H}^n/\Gamma$ is called *arithmetic of simplest type* if $\Gamma$ is.

The same set-up using special orthogonal groups constructs orientation-preserving arithmetic groups of simplest type (and orientable arithmetic hyperbolic $n$-manifolds of simplest type).

## Discreteness

**1.** Assume that there is a sequence $\{A_m = (a_{ij}^m)\} \subset \mathrm{O}(\mathrm{q}, \mathrm{R_k})$ such that $A_m \to I$.

For sufficiently large $m$, we have $|a_{ij}^m| < 2$.

**2.** If $\sigma$ is a non-identity embedding $q^\sigma$ is equivalent over $\mathbb{R}$ to $x_1^2 + x_2^2 + \ldots + x_{n+1}^2$.

Hence $\mathrm{O}(\mathrm{q}^\sigma, \mathbb{R})$ is conjugate to $\mathrm{O}(\mathrm{n}+1)$ and so is a compact group.

This implies that $|\sigma(a_{ij}^m)| < K_\sigma$ for some $K_\sigma \in \mathbb{R}$.

**3.** There are only finitely many algebraic integers $x$ of bounded degree, such that $x$ and all of its Galois conjugates are bounded.

Finite co-volume of these arithmetic groups follows from general results of Borel and Harish-Chandra.

## Cocompactness

$f$ be a diagonal, anisotropic (over $\mathbb{Q}$) quadratic form of signature $(n, 1)$ and $\mathbb{Z}$-coefficients. Then $\Lambda = \mathrm{SO}(\mathrm{f}, \mathbb{Z})$ is cocompact in $G = \mathrm{SO}(\mathrm{f}, \mathbb{R})$.

Caution Meyer's theorem implies that this is only possible for $n \leq 3$.

We have a map

$$\pi : \mathrm{SL_{n+1}}(\mathbb{R}) \to \mathrm{SL_{n+1}}(\mathbb{R})/\mathrm{SL_{n+1}}(\mathbb{Z})$$

Key Claim: Using $\pi$ , we can define a map

$$\phi : G/\Lambda \to \mathrm{SL_{n+1}}(\mathbb{R})/\mathrm{SL_{n+1}}(\mathbb{Z}).$$

The image of $\phi$ is compact.

This can be established using the Mahler Compactness Criterion

**Theorem 3 (Mahler's Compactness Criterion)**

*Let $C \subset \mathrm{SL_m}(\mathbb{R})$, then the image of $C$ in $\mathrm{SL_m}(\mathbb{R})/\mathrm{SL_m}(\mathbb{Z})$ is precompact (i.e. compact closure) if and only if $0$ is not an accumulation point of*

$$C\mathbb{Z}^m = \{c \cdot v \;:\; c \in C,\; v \in \mathbb{Z}^m\}.$$

Use this to prove: The image of $G$ in $\mathrm{SL_{n+1}}(\mathbb{R})/\mathrm{SL_{n+1}}(\mathbb{Z})$ is precompact

Uses anisotropic and defined over $\mathbb{Z}$ to ensure you stay away from 0.

Associated to $f$ is the bilinear form $B$, with

$$B(x,y) = \frac{1}{2}(f(x+y) - f(x) - f(y)).$$

This has $\mathbb{Z}$-coefficients ($f$ is diagonal with $\mathbb{Z}$-coefficients). Note that

(i) $B(\mathbb{Z}^{n+1}, \mathbb{Z}^{n+1}) \in \mathbb{Z}$.

(ii) $|B(v_m, v_m)| \geq 1$. (anisotropic)

You obtain a contradiction from a sequence of $g_m \in G$ and $v_m \in \mathbb{Z}^{n+1}$ such that

$$g_m v_m \to 0.$$

One can show the image is actually closed and this finishes the proof.

## Comments on non-compact case

### Theorem 4

*TFAE*

- $\mathrm{SO}(\mathrm{f}, \mathrm{R}_\mathrm{k})$ *is non-compact and finite volume.*
- $f$ *is defined over* $\mathbb{Q}$ *and is isotropic.*
- $\mathrm{SO}(\mathrm{f}, \mathrm{R}_\mathrm{k})$ *contains a unipotent element.*

# Finding unipotent elements

## Main idea

Let $a, b, c \in \mathbb{Z}$ and $f = ax^2 + by^2 + cz^2$ represent 0 non-trivially over $\mathbb{Z}$; e.g. $ax_0^2 + by_0^2 + cz_0^2 = 0$. Assume $x_0 \neq 0$.

This form is equivalent over $\mathbb{Q}$ to the form $t(xz - y^2)$ for some $t \in \mathbb{Q}$. Note $\mathrm{SO}(\mathrm{t}(\mathrm{xz} - \mathrm{y}^2), \mathbb{Q}) = \mathrm{SO}(\mathrm{xz} - \mathrm{y}^2, \mathbb{Q})$.

If $(x_0, y_0, z_0)$ is as above, consider the matrix:

$$T = \begin{pmatrix} bcx_0 & 0 & 4x_0 \\ bcy_0 & -4cz_0 & -4y_0 \\ bcz_0 & 4by_0 & -4z_0 \end{pmatrix}.$$

Then $T^t f T = \begin{pmatrix} 0 & 0 & t/2 \\ 0 & -t & -0 \\ t/2 & 0 & -0 \end{pmatrix}$ where $t = 16abcx_0^2$.

Already seen $\mathrm{SO}(xz - y^2, \mathbb{Z})$ contains unipotent elements.
This group up to finite index is the image of $\mathrm{PSL}(2, \mathbb{Z})$.
As we now discuss: Use the equivalence above to construct
unipotents in $\mathrm{SO}(f, \mathbb{Z})$ via commensurability.

## Commensurability

Suppose $q_1$ and $q_2$ are admissable quadratic forms over $k$ and $q_1 \simeq_k q_2$.

So there exists $T \in \mathrm{GL_{n+1}(k)}$ such that $T^t Q_1 T = Q_2$.

Claim:

$$T^{-1}\mathrm{O(Q_1, k)T = O(Q_2, k)}$$

The converse also holds.

However

$$T^{-1}\mathrm{O(Q_1, R_k)T \subset O(Q_2, k)}.$$

But need not preserve $R_k$-points

However, $T^{-1}\mathrm{O}(\mathrm{Q}_1,\mathrm{R_k})\mathrm{T}$ is commensurable with $\mathrm{O}(\mathrm{Q}_2,\mathrm{R_k})$.

Idea By considering the entries of $T$ and $T^{-1}$ can choose a congruence subgroup $\Gamma < \mathrm{O}(\mathrm{Q}_1,\mathrm{R_k})$ such that $T^{-1}\Gamma T < \mathrm{O}(\mathrm{Q}_2,\mathrm{R_k})$.

Example $f = xz - y^2$ and $f_n = nxz - y^2$.

These are equivalent over $\mathbb{Q}$: Let $Y = \begin{pmatrix} n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$Y^t \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix} Y = \begin{pmatrix} 0 & 0 & n/2 \\ 0 & -1 & 0 \\ n/2 & 0 & 0 \end{pmatrix}$$

Then if $X = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in \mathrm{O}(\mathrm{f}, \mathbb{Z})$,

$$Y^{-1}XY = \begin{pmatrix} a_1 & a_2/n & a_3/n \\ nb_1 & b_2 & b_3 \\ nc_1 & c_2 & c_3 \end{pmatrix}$$

which will lie in $\mathrm{O}(\mathrm{f_n}, \mathbb{Z})$ if we choose $a_2$ and $a_3$ divisible by $n$; so take $\Gamma$ to be the principal congruence subgroup of level $n$ in $\mathrm{O}(\mathrm{f}, \mathbb{Z})$.

## Examples

**1.** The $(2,4,6)$ triangle group arises from the form $x^2 + y^2 - 3z^2$.

**2.** The $(2,3,7)$ triangle group arises from a form defined over $\mathbb{Q}(\cos \pi/7)$.

Of course most Fuchsian groups are not arithmetic.

**3.** The Bianchi groups $\mathrm{PSL}(2, \mathrm{O_d})$ represent the totality of commensurability classes of non-cocompact arithmetic Kleinian groups.

These arise from the quadratic forms $dx_1^2 + x_2^2 + x_3^2 - x_4^2$.

**4.** The minimal volume hyperbolic 3-orbifold arises from the quadratic form $x_1^2 + x_2^2 + x_3^2 + (3 - 2\sqrt{5})x_4^2$.

Most arithmetic hyperbolic 3-manifolds are not of simplest type.

**5.** The groups generated by reflections in the compact 120 cell in $\mathbb{H}^4$ and the ideal 24-cell in $\mathbb{H}^4$ are arithmetic of simplest type. The quadratic forms are:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - (\frac{1+\sqrt{5}}{2})x_5^2$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2$$

## Some other remarks

There are volume formula for certain arithmetic groups of simplest type; e.g. maximal groups.

If $n$ is even and $f$ is an admissible quadratic form defined over $k$, then every arithmetic subgroup in $\mathrm{SO(f)}$ is contained in $\mathrm{SO(f,k)}$ (Borel). This is not true when $n$ is odd.

## Totally Geodesic Submanifolds

Consider the form $f = x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}$ and write as $f = x_1^2 + g$ where

$$g = x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2$$

Hence the group $\mathrm{SO}(\mathrm{f}, \mathbb{Z}[\sqrt{2}])$ is cocompact and contains a subgroup $\mathrm{SO}(\mathrm{g}, \mathrm{R_k})$ which is a cocompact subgroup of $\mathrm{SO}(\mathrm{g}, \mathbb{R})$.

This allows us to construct arithmetic hyperbolic $(n+1)$-manifolds that contains a hyperbolic $n$-manifold.
Indeed to construct arithmetic hyperbolic $(n+1)$-manifolds that contains an immersed hyperbolic $m$-submanifold for all $1 \leq m \leq n$.

## But we can get many many more....

The reason is this: If $T \in \mathrm{O(f,k)}$ then $T\mathrm{O(f,R_k)T^{-1}}$ is commensurable with $\mathrm{O(f,R_k)}$.

Given this: we have a subgroup $U = T\mathrm{O(f,R_k)T^{-1}} \cap \mathrm{O(f,R_k)}$ of finite index in both $T\mathrm{O(f,R_k)T^{-1}}$ and $\mathrm{O(f,R_k)}$.

We can therefore build another co-dimension 1 totally geodesic submanifold by considering $H = T\mathrm{SO(g,R_k)T^{-1}} \cap \mathrm{U} < \mathrm{O(f,R_k)}$.

Another way to say this is: Let $W \subset V$ be the subspace spanned by $\{e_2, \dots e_{n+1}\}$ so that $W$ equipped with $g$ gives a copy of $\mathbb{H}^{n-1}$.

Let $T \in \mathrm{O(f,k)}$ as above, then $T(W)$ is invariant by a cocompact subgroup $H$.

**Theorem 5**

*Let $M$ be an arithmetic hyperbolic n-manifold of simplest type. Then $M$ contains infinitely many immersed totally geodesic co-dimension $1$ submanifolds.*

## And more....

Again take $f = x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}$.

If we can find an admissible $n$-dimensional quadratic form $q$ defined over $\mathbb{Q}(\sqrt{2})$ and an element $a > 0 \in \mathbb{Q}(\sqrt{2})$ such that:

$$f \simeq_{\mathbb{Q}(\sqrt{2})} ax_1^2 + q = q'$$

then commensurability of $\mathrm{O(f}, \mathbb{Z}[\sqrt{2}])$ and $T^{-1}\mathrm{O(q'}, \mathbb{Z}[\sqrt{2}])\mathrm{T}$ can be used to built a subgroup of finite index in $T^{-1}\mathrm{O(q}, \mathbb{Z}[\sqrt{2}])\mathrm{T}$ in $\mathrm{O(f}, \mathbb{Z}[\sqrt{2}])$

## Recap from Lectures 1 and 2

Given an admissible quadratic form $f$ over a totally real field $k$ of signature $(n, 1)$ we have an algebraic group $\mathrm{O(f)}$ and an equivalence over $\mathbb{R}$ of $f$ with $J_n$ so that.

$$T^t F T = J_n \text{ implies a conjugation } T^{-1}\mathrm{O(f, \mathbb{R})T} = \mathrm{O(n, 1)}.$$

Then $T^{-1}\mathrm{O(f, R_k)T} \cap \mathrm{O_0(n, 1)}$ determines a commensurability class of arithmetic hyperbolic n-manifolds of simplest type.

Note if $f \simeq_k q$ then this provides further commensurabilities.

One striking fact about these arithmetic hyperbolic manifolds of simplest type is:

**Theorem 6**

*Let $M$ be an arithmetic hyperbolic $n$-manifold of simplest type. Then $M$ contains infinitely many immersed totally geodesic co-dimension $1$ submanifolds.*

Does this characterize arithmeticity?

## Embedding and Bounding

**Theorem 7 (Millson, Bergeron-Haglund-Wise)**

*Let $M$ be an arithmetic hyperbolic $n$-manifold of simplest type and $N$ an immersed co-dimension $1$ totally geodesic submanifold. Then $N$ embeds in a finite sheeted cover of $M$.*

Need to promote immersed to embedded.

# LERF

## Definition ($H$-separable)

Let $G$ be a group, $H < G$. We say $G$ is $H$-separable if for each $g \in G \setminus H$, there exists a subgroup $K < G$ of finite index such that $H \subset K$ and $g \notin K$.

Example Let $H = 1$. Then $G$ is $H$-separable is equivalent to saying $G$ is Residually Finite.

## Definition

Let $G$ be a group. We say that $G$ is subgroup separable or LERF if $G$ is $H$-separable for all finitely generated subgroups $H$.

Thanks to work of Peter Scott this is what is needed to promote immersed to embedded!

## Example

Take $f = x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}$, and write $f = x_1^2 + g$ with $g = x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2$.

We can inject $\mathrm{O(g)} \hookrightarrow \mathrm{O(f)}$ as follows:

$$A \in \mathrm{O(g)} \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right) \in \mathrm{O(f)}$$

Let the image of $\mathrm{O}(\mathrm{g}, \mathbb{Z}[\sqrt{2}])$ under this map be denoted by $H$.

Claim: $\mathrm{O}(\mathrm{f}, \mathbb{Z}[\sqrt{2}])$ is $H$-separable.

Let $\gamma \in \mathrm{O}(\mathrm{f}, \mathbb{Z}[\sqrt{2}]) \setminus \mathrm{H}$.

Two cases to focus on: (1) $\gamma = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & A \end{array}\right)$, $A \in \mathrm{O}(\mathrm{g}, \mathbb{Z}[\sqrt{2}])$ and

(2) $\gamma$ has a non-zero entry $x$ in the first column or row (and not in the $(1,1)$-entry).

In either case choose a prime ideal $P \subset \mathbb{Z}[\sqrt{2}]$ so that $P \neq < \sqrt{2} >$ and $x \notin P$.

Now let $\Gamma$ be a torsion free subgroup of finite index in $\mathrm{O}(\mathrm{f}, \mathbb{Z}[\sqrt{2}])$, then $\Gamma$ is $\Gamma \cap H$-separable.
This gives hyperbolic n-manifolds containing embedded totally geodesic submanifolds.
they may be non-orientable.

However, can always arrange (perhaps on a passage to a further finite sheeted cover) that everything is orientable.

So we can construct an orientable hyperbolic $n$-manifold $M$ containing an embedded co-dimension 1 totally geodesic submanifold.
It either is separating or non-separating.
Latter case gives a map $\pi_1(M) \to \mathbb{Z}$, the former $N$ bounds a compact hyperbolic $n$-manifold.

### Theorem 8 (Long, Bergeron)

*$M$ an arithmetic hyperbolic n-manifold of simplest type, and $N$ an immersed co-dimension one totally geodesic submanifold for which $N$ does not factor through a cover. Then $\pi_1(M)$ is $\pi_1(N)$-separable.*

### Embedded vs Bounding

## Geometrically bounding

### Definition

A closed connected orientable, hyperbolic $n$-manifold $M$ bounds geometrically if $M$ is realized as the totally geodesic boundary of a compact orientable, hyperbolic $(n+1)$-manifold $W$.

### Definition

A closed connected flat $n$-manifold $M$ bounds geometrically if $M$ is realized as the cusp cross-section of a finite volume 1-cusped hyperbolic $(n+1)$-manifold.

Can also make sense of just saying that complete orientable finite volume hyperbolic $n$-manifold bounds geometrically.

### Theorem 9 (Long-R)

1. *There are closed hyperbolic 3-manifolds that do not bound geometrically.*
2. *There are flat 3-manifolds that do not bound geometrically.*
3. *For every $n$ there are closed orientable hyperbolic $n$-manifolds which are arithmetic of simplest type that bound geometrically.*

Other results:

**Theorem 10 (Kolpakov-Martelli-Tschantz)**

*There exist infinitely many closed hyperbolic 3-manifolds (whose volumes are known) which bound geometrically a compact hyperbolic 4-manifold (whose volumes are known).*

This uses the tessellation of $\mathbb{H}^4$ coming from the 120-cell.

**Theorem 11 (Slavich)**

*The figure-eight knot complement bounds geometrically.*

Recent work:

**Theorem 12 (Kolpakov-R-Slavich)**

*Let $M = \mathbb{H}^n/\Gamma$ ($n \geq 2$ and even) be an orientable arithmetic hyperbolic n-manifold of simplest type.*

*Then $M$ embeds as a totally geodesic submanifold of an orientable arithmetic hyperbolic $(n+1)$-manifold $W$.*

*If $M$ double covers a non-orientable hyperbolic manifold, then $M$ bounds geometrically.*

Moreover, when $M$ is not defined over $\mathbb{Q}$ (and is therefore closed), the manifold $W$ can be taken to be closed.

Weaker statement can be made in odd dimensions.

## Obstructions to bounding

### Theorem 13 (Long-R)

*If $M$ is a closed hyperbolic 3-manifold or a flat 3-manifold that bounds geometrically then $\eta(M) \in \mathbb{Z}$.*

Meyerhoff-Neumann: *For surgeries on a hyperbolic knot in $S^3$ the $\eta$ invariant takes on a dense set of values in $\mathbb{R}$.*
This allows us to build hyperbolic examples that don't bound geometrically.

$\eta(M) \in \mathbb{Z}$ is rare: from the SnapPy census of approx. $11,000$ closed hyperbolic 3-manifolds 41 have this property.

For a flat 3-manifold $M$, it is known that $\eta(M)$ depends only on the topology of $M$ and is independent of the flat metric.

Take the unique orientable flat 3-manifold with base for the Seifert fibration $S^2$ and Seifert invariants $(2, 1)$, $(3, -1)$, $(6, -1)$. Then $\eta(M) = -4/3$

**Theorem 14 (Long-R, McReynolds)**

*Every flat n-manifold arises as the cusp cross-section of some cusp of a multi-cusped non-compact arithmetic hyperbolic $(n+1)$-manifold.*

WARNING At the time when we proved these results it was unknown as to whether there were any 1-cusped finite volume hyperbolic 4-manifolds.

### Theorem 15 (Kolpakov-Martelli)

*There exist* 1*-cusped arithmetic hyperbolic 4-manifolds of simplest type.*

These are built using the 24-cell, and are plentiful.

Still none known in any other dimension $> 4$.

### Theorem 16 (Stover)

*There are no 1-cusped arithmetic hyperbolic n-orbifolds whenever* $n \geq 30$.

## Non-arithmetic hyperbolic n-manifolds—after Gromov and Piatetski-Shapiro

Cut-and-paste arithmetic hyperbolic manifolds of simplest type along a common co-dimension 1 totally geodesic submanifold (not necessarily connected)

How to do this:

Take $f = x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}$, and write $f = x_1^2 + g$ with $g = x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2$.

Plan: Find a form $q = ax_1^2 + g$ such that $q$ is not $\simeq_{\mathbb{Q}(\sqrt{2})}$ to $f$.

If we can do this its a win: we cut and paste and glue. This produces a group $\Lambda$ built from the pieces that can't be arithmetic.

## Lemma 17

*Let* $\Gamma_1$ *and* $\Gamma_2$ *be arithmetic lattices in* $\mathrm{O}_0(\mathrm{n}, 1)$ *such that* $\Gamma_1 \cap \Gamma_2$ *is Zariski dense in* $\mathrm{O}_0(\mathrm{n}, 1)$. *Then* $\Gamma_1$ *and* $\Gamma_2$ *are commensurable.*

Take $\Gamma_1 = \mathrm{O}(\mathrm{f}, \mathbb{Z}[\sqrt{2}])$ and $\Gamma_2 = \Lambda$, and apply the lemma.

Do the same with $\Gamma_1 = \mathrm{O}(\mathrm{q}, \mathbb{Z}[\sqrt{2}])$ and $\Gamma_2 = \Lambda$ and apply the lemma.

The orthogonal groups are not commensurable.

*a* can be constructed using the theory of quadratic forms

Are there non-arithmetic hyperbolic n-manifolds $n \geq 4$ that are ”not built from arithmetic pieces”?

## GFERF

**Theorem 18 (Bergeron-Haglund-Wise)**

*Let $M$ be an arithmetic hyperbolic n-manifold of simplest type. Then $\pi_1(M)$ is virtually C-special; i.e. $\pi_1(M)$ contains a finite index subgroup contained in an abstract right angled Coxeter group. In particular $\pi_1(M)$ is separable on geometrically finite subgroups —it has the virtual retract property over geometrically finite subgroups.*

Remark: The theorem with Kolpakov-Slavich uses this as a key ingredient.

Starting point: Co-dimension 1 totally geodesic submanifolds are abundant in the following sense:

Given $u, v \in \partial\mathbb{H}^n$ there exists codimension 1 geodesic submanifold $H$ whose boundary $\partial H \subset \partial\mathbb{H}^n$ separates $u$ and $v$, and there exists $\Gamma < \pi_1(M)$ leaving $H$ invariant.

Wont say any more about this, but discuss an earlier special case of the Bergeron-Haglund-Wise result.

**Theorem 19 (Agol-Long-R)**

*The Bianchi groups are virtually C-special.*

Recent work of M. Chu improves the argument and controls the "virtual part":

### Theorem 20 (Chu)

*Let $R_d$ be the subgroup of the Bianchi group* $\mathrm{PSL_2(O_d)}$ *given by*

$$R_d = \left\{ \gamma \in \mathrm{PSL_2(\mathbb{Z}[\sqrt{-d}])} : \gamma \equiv \mathrm{Id} \mod 2 \right\}.$$

*Then $R_d$ embeds in a RACG and has index* $[\mathrm{PSL_2(O_d) : R_d}] =$

1. $48$ *if* $d \equiv 1, 2 \mod (4)$
2. $288$ *if* $d \equiv 7 \mod (8)$
3. $480$ *if* $d \equiv 3 \mod (8)$

## Ideas in the proof of ALR

As mentioned in Lecture 2 $\mathrm{PSL}(2, \mathrm{O_d})$ can be realized up to commensurability as $\mathrm{SO}(\mathrm{p_d}, \mathbb{Z})$ where $p_d = dx_1^2 + x_2^2 + x_3^2 - x_4^2$.

The form $du^2 + dv^2 + dw^2 + p_d$ is equivalent over $\mathbb{Q}$ to the form $J_6$ (uses the 4-squares theorem).

$\mathrm{O}(6, 1, \mathbb{Z})$ contains an all right Coxeter group of finite index.

Hence $\mathrm{PSL}(2, \mathrm{O_d})$ is virtually $C$-special.

## Contrast with dimensions 2 and 3

$M$ a hyperbolic 2,3-manifold. $\pi_1(M)$ is LERF.

**Theorem 21 (Hongbin Sun)**

*Let $M$ be an arithmetic hyperbolic n-manifold of simplest type with $n \geq 4$. Then $\pi_1(M)$ is not LERF.*

## Idea in the proof when $M$ is non-compact

## Exploit non-LERF non-geometric closed 3-manifolds

Comment: LERF for $\pi_1$(closed 3-manifold) completely understood now.

**Theorem 22 (Hongbin Sun)**

*$M$ a closed 3-manifold. $\pi_1(M)$ is LERF if and only $M$ is geometric.*

In fact following work of Yi Liu, Hongbin proves:

### Theorem 23

*Let $M$ be a mixed non-geometric closed 3-manifold. Then $\pi_1(M)$ contains a non-separable surface group.*

### How to exploit this?

Here is the basic idea now given Hongbin's previous result.

### Theorem 24 (Long-R)

*Let $M$ denote the exterior of the figure-eight knot complement, and $DM$ its double. Then $\pi_1(DM)$ admits a faithful representation (with geometrically finite image) into an arithmetic group of simplest type commensurable with* $\mathrm{SO}_0(4, 1, \mathbb{Z})$.

Putting these together:

## Corollary 25

$\mathrm{SO}_0(4, 1, \mathbb{Z})$ *is not LERF.*

Indeed one gets more:

## Corollary 26

$\mathrm{SO}_0(\mathrm{n}, 1, \mathbb{Z})$ *is not LERF.*

The general non-co-compact case is a generalization of this.

The closed case uses amalgams of closed hyperbolic 3-manifold groups along infinite cyclic groups ($n = 4$ needs a different argument).

THE END