THE DIRAC OPERATOR ON CUSPED HYPERBOLIC MANIFOLDS

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Abstract. We study how the spin structures on finite-volume hyperbolic n-manifolds restrict to cusps. When a cusp cross-section is a $(n-1)$-torus, there are essentially two possible behaviours: the spin structure is either *bounding* or *Lie*. We show that in every dimension n there are examples where at least one cusp is Lie, and in every dimension $n \leq 8$ there are examples where all the cusps are bounding.

By work of C. Bär, this implies that the spectrum of the Dirac operator is $\mathbb{R}$ in the first case, and discrete in the second. We therefore deduce that there are cusped hyperbolic manifolds whose spectrum of the Dirac operator is $\mathbb{R}$ in all dimensions, and whose spectrum is discrete in all dimensions $n \leq 8$.

Introduction

Let M be a finite-volume hyperbolic n-manifold all of whose cusp cross-sections are Euclidean $(n-1)$-tori. If M has a spin structure, it induces one on every orientable codimension-1 submanifold, and hence in particular on every $(n-1)$-torus cusp cross-section. Recall that there are two spin structures on a $(n-1)$-torus T up to automorphisms of T (see [2] for example), namely:

- the *bounding* spin structure, induced by the representation of T as the boundary of $D^2 \times S^1 \times \cdots \times S^1$ equipped with any spin structure,
- the *Lie* spin structure, induced by the representation of T as a Lie group.

These spin structures are sometimes called *non-trivial* and *trivial*, respectively. Our interest in the possible restrictions of spin structures to cusp cross-sections is motivated by the following theorem of Bär [2]:

Theorem 1 (Bär). *Let M be a finite volume hyperbolic n-manifold equipped with a spin structure all of whose cusp cross-sections are Euclidean $(n-1)$-tori. Then the spectrum of the Dirac operator of M is:*

(1) $\mathbb{R}$ *if at least one cusp cross-section inherits the Lie spin structure;*
(2) *discrete if all the cusp cross-sections inherit the bounding spin structure.*

We will not define the Dirac operator here and refer the reader to [2]. By explicit constructions, it is shown in [2] that both cases of Theorem 1 occur in dimensions $n = 2, 3$. Although it is known in dimensions $n \geq 4$, that there are many cusped hyperbolic n-manifolds admitting a spin structure (indeed every finite-volume hyperbolic manifold is virtually spinnable [10, 15]), we are not aware of any example where the spin structure is constructed explicitly and its restriction to the cusp cross-sections determined. In particular it seems unknown whether cases (1) or (2) of Theorem 1 occur in dimension $n \geq 4$. The main purpose of this article is to provide a partial answer.

Theorem 2. *The following hold:*

(1) *For every integer $n \geq 2$ there is a finite volume cusped orientable hyperbolic n-manifold equipped with a spin structure, all of whose cusp cross-sections are Euclidean $(n-1)$-tori, and such that the spin structure restricts to the Lie spin structure on at least one cusp cross-section;*
(2) *for every integer $n \leq 8$ there is a finite volume cusped orientable hyperbolic n-manifold equipped with a spin structure, all of whose cusp cross-sections are Euclidean $(n-1)$-tori, and such that the spin structure restricts to the bounding structure on all cusp cross-sections.*

Combining this result with Bär's Theorem we get:

Corollary 3. *The following hold:*

(1) *For every n there is a finite volume cusped orientable hyperbolic n-manifold with a spin structure for which the spectrum of the Dirac operator is $\mathbb{R}$;*
(2) *for every $n \leq 8$ there is a finite volume cusped orientable hyperbolic n-manifold with a spin structure for which the spectrum of the Dirac operator is discrete.*

The constructions of examples in (1) and (2) of Theorem 2 are very different in nature, and neither is completely straightforward, relying on results that have been proved recently.

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1. The Lie group case

1.1. **Preliminaries.** It will be convenient for this part of the discussion to identify hyperbolic n-space $\mathbb{H}^n$, with the hyperboloid model, defined using the quadratic form $j_n := x_0^2 + x_1^2 + \ldots x_{n-1}^2 - x_n^2$; i.e.

$$\mathbb{H}^n = \{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : j_n(x) = -1, x_n > 0\}$$

equipped with the Riemannian metric induced from the Lorentzian inner product associated to j_n. The full group of isometries of $\mathbb{H}^n$ is then identified with $\mathrm{O}^+(n,1)$, the subgroup of

$$\mathrm{O}(n,1) = \{A \in \mathrm{GL}(n+1,\mathbb{R}) : A^t J_n A = J_n\},$$

preserving the upper sheet of the hyperboloid $j_n(x) = -1$, and where J_n is the symmetric matrix associated to the quadratic form j_n. The full group of orientation-preserving isometries is given by $\mathrm{SO}^+(n,1) = \{A \in \mathrm{O}^+(n,1) : \det(A) = 1\}$.

If M is a finite volume hyperbolic manifold admitting a spin structure (i.e. the first and second Stiefel Whitney classes are both zero), then the set of spin structures on M can be identified with $H^1(M,\mathbb{Z}/2\mathbb{Z})$. This identification is not canonical, although it can be made canonical after choosing a "base" spin structure corresponding to $0 \in H^1(M,\mathbb{Z}/2\mathbb{Z})$. Moreover, $H^1(M,\mathbb{Z}/2\mathbb{Z})$ acts freely and transitively on the set of spin structures.

We will make use of the following lemma (c.f. [2, Lemma 3]).

Lemma 4. *Let M be a finite volume hyperbolic n-manifold admitting a spin structure, all of whose cusp cross-sections are Euclidean $(n-1)$-tori, and assume that for some cusp cross-section T, the inclusion map $T \hookrightarrow M$ induces a direct sum decomposition, $H_1(M,\mathbb{Z}) = H_1(T,\mathbb{Z}) \oplus A$ for some finitely generated Abelian group A. Then given any spin structure σ on T, there exists a spin structure s_σ on M such that s_σ restricts to σ on T. In particular this holds for the Lie spin structure on T.*

Proof. Since $H_1(T,\mathbb{Z})$ is a direct summand of $H_1(M,\mathbb{Z})$, it follows that $H^1(M,\mathbb{Z}/2\mathbb{Z})$ surjects on $H^1(T,\mathbb{Z}/2\mathbb{Z})$. So given any spin structure σ on T we can find a spin structure s_σ on M such that s_σ restricts to σ on T. □

1.2. **Cusped arithmetic hyperbolic manifolds.** The proof of Theorem 2(1) will make use of cusped arithmetic, hyperbolic n-manifolds whose definition we now recall (see [16] for further details). Suppose that $X = \mathbb{H}^n/\Gamma$ is a finite volume cusped hyperbolic n-manifold. Then X is arithmetic if Γ is commensurable with a group $\Lambda < \mathrm{SO}^+(n,1)$ as described below.

Let f be a non-degenerate quadratic form defined over $\mathbb{Q}$ of signature $(n,1)$, which we can assume is diagonal and has integer coefficients. Then f is equivalent over $\mathbb{R}$ to the form j_n defined above; i.e. there exists $T \in \mathrm{GL}(n+1,\mathbb{R})$ such that $T^t F T = J_n$, where F and J_n denote the symmetric matrices associated to f and j_n respectively. Then $T^{-1}\mathrm{SO}(f,\mathbb{Z})T \cap \mathrm{SO}^+(n,1)$ defines the arithmetic subgroup $\Lambda < \mathrm{SO}^+(n,1)$.

Note that the form f is anisotropic (ie does not represent 0 non-trivially over $\mathbb{Q}$) if and only if the group Γ is cocompact, otherwise the group Γ is non-cocompact (see [4]). By Meyer's Theorem [14, §IV.3.2, Corollary 2], the case that f is anisotropic can only occur when $n = 2, 3$.

The main result we will need from the theory of cusped arithmetic hyperbolic manifolds is that their fundamental groups are virtually special [3]. We will not define this here, but we will prove a result which is a consequence of being virtually special.

Theorem 5. *Let $\mathbb{H}^n/\Lambda$ be a cusped arithmetic hyperbolic n-manifold. Then Λ has a finite index subgroup Γ for which $\mathbb{H}^n/\Gamma$ has all cusp cross-sections being Euclidean $(n-1)$-tori and for one such cusp cross-section T, there exists a retraction $\Gamma \to \pi_1(T)$.*

Proof. We indicate how this follows from [3]. By [12] for example, we can first pass to a finite cover M_1 of $\mathbb{H}^n/\Lambda$ so that all cusp cross-sections are Euclidean $(n-1)$-tori. Next, we use [3, Theorem 1.4] together with the last sentence of [3] which states:

"*In fact, the proof of Theorem 1.4 extends to show that non-cocompact arithmetic lattices virtually retract onto their geometrically finite subgroups.*"

in the following way (note that [3, Theorem 1.4] deals with certain closed arithmetic hyperbolic manifolds).

Let $T_1 \subset M_1$ be a cusp cross-section, since $\pi_1(T_1)$ is geometrically finite, we can use the previous paragraph to arrange finite covers $M = \mathbb{H}^n/\Gamma \to M_1$ and $T \to T_1$ (and $\pi_1(T)$ geometrically finite) together with a retraction from $\Gamma \to \pi_1(T)$. □

1.3. **The proof of Theorem 2(1).** Let X be a cusped arithmetic hyperbolic manifold of dimension n (which we can assume is at least 4 by the results of [2]). Using Sullivan's result [15] we can pass to a finite cover that admits a spin structure and also (as noted in the proof of Theorem 5), so that all cusp cross-sections are Euclidean $(n-1)$-tori. From Theorem 5, we can pass to a further finite sheeted cover $M \to X$, that admits a spin structure and for which $\pi_1(M)$ retracts onto $\pi_1(T)$. This retraction determines a decomposition $H_1(M,\mathbb{Z}) = H_1(T,\mathbb{Z}) \oplus A$, for some finitely generated Abelian group A, and the result now follows from Lemma 4. □

We conclude §1 with some remarks on arranging the Lie spin structure on multiple cusp cross-sections. As above, M will be a cusped arithmetic hyperbolic manifold of dimension n (of dimension at least 4) admitting a spin structure with all cusp cross-sections $T_1, T_2, \ldots, T_s$ being Euclidean $(n-1)$-tori. For d a positive integer, let G_j^d be the characteristic subgroup of $\pi_1(T_j)$ arising as the kernel of the homomorphism $\pi_1(T_j) \to (\mathbb{Z}/d\mathbb{Z})^{n-1}$. Standard combination techniques can be used to show that for d sufficiently large, the group $G = G_1^d * G_2^d * \ldots * G_s^d$ is a geometrically finite subgroup of $\pi_1(M)$, and so we can apply [3] (see above), to obtain finite index subgroups $\Gamma < \pi_1(M)$ and $L < G$ with Γ admitting a retraction onto L.

Note that since $[G : L] < \infty$, L contains a finite index subgroup of each of G_j^d for $j = 1, \ldots s$, and each such will be isomorphic to $\mathbb{Z}^{n-1}$ and conjugate into a peripheral subgroup of Γ. Furthermore, by the Kurosh subgroup theorem (see [11, Chapter IV Theorem 1.10] for example), L has the form $F * (*_\alpha A_\alpha)$ where F is a free group and A_α is the intersection of L with a conjugate of some G_j^d. Since $[G : L] < \infty$ it follows that $A_\alpha \cong \mathbb{Z}^{n-1}$ for each α. Putting this together we deduce that there is a retraction $\Gamma \to (\mathbb{Z}^{n-1})^K$ where each $\mathbb{Z}^{n-1}$ is a peripheral subgroup of Γ and K some large positive integer.

We can now use an extension of Lemma 4 to get a direct sum decomposition, $H_1(\Gamma, \mathbb{Z}) = (\mathbb{Z}^{n-1})^K \oplus A$ for some finitely generated Abelian group A, and we can follow the proof of Lemma 4 to obtain the Lie spin structure on a large number of cusp cross-sections of $\mathbb{H}^n/\Gamma$.

2. The bounding case

In this section we prove Theorem 2(2), the method of proof being very different from that in §1. The core of the argument is a Dehn filling trick.

2.1. **The Dehn filling trick.** Let M be a finite-volume hyperbolic n-manifold, all of whose cross-sections are $(n-1)$-tori. We say that a closed smooth n-manifold N is a *Dehn filling* of M if N contains some disjoint $(n-2)$-tori with trivial normal bundles whose complement is diffeomorphic to M. Given M, it is possible to construct a Dehn filling N by attaching a copy of $D^2 \times S^1 \times \cdots \times S^1$ to each truncated cusp along some diffeomorphism. The resulting N of course depends on the chosen diffeomorphisms.

Here is a crucial observation: if a Dehn filling N has a spin structure, it induces one in M that is of bounding type on every cusp cross-section, because it extends by construction to the adjacent $D^2 \times S^1 \times \cdots \times S^1$. The existence of a spin structure on M that restricts to bounding spin structures on every cross-section is in fact equivalent to the existence of a spinnable Dehn filling N of M.

To prove Theorem 2(2) it therefore suffices to exhibit a hyperbolic n-manifold that can be Dehn-filled to a spinnable closed n-manifold. We are able to do this for every $n \leq 8$ using right-angled polytopes and a vanishing theorem for Stiefel-Whitney classes of moment-angled manifolds recently discovered in [8].

2.2. **Right-angled polytopes.** Let $P^3, \ldots, P^8$ be the notable sequence of finite-volume right-angled polytopes $P^n \subset \mathbb{H}^n$ already considered by various authors [1, 5, 9, 13]. These polytopes are combinatorially dual to the Euclidean *Gosset polytopes* [7] discovered by Gosset in 1900. The only information that we need here is that $P^n \subset \mathbb{H}^n$ is right-angled and has at least one ideal vertex, its combinatorics will not be important.

Let Γ be the reflection group of P^n. It follows easily from the standard Coxeter presentation of Γ that its abelianisation is isomorphic to the finite group $(\mathbb{Z}/2\mathbb{Z})^f$ where f is the number of facets of P^n. It is also a standard fact that the kernel $\Gamma' = [\Gamma, \Gamma]$ of the abelianization contains no torsion and hence $M^n = \mathbb{H}^n/\Gamma'$ is a cusped finite-volume hyperbolic manifold tessellated by 2^f copies of P^n. If we use

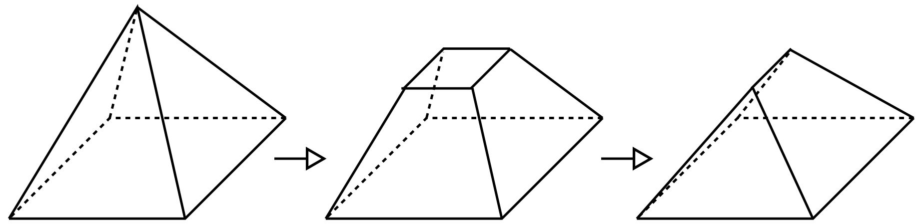

FIGURE 1. We Dehn fill P^n by replacing every ideal vertex with a $(n-2)$-cube. Here $n = 3$. The resulting (abstract) polytope $\bar{P}^n$ is simple.

the *colouring* language as in [9], this is the manifold that we get by colouring all the facets of P^n with distinct colours. It also follows from [9, Proposition 7] that all the cusps of M^n are $(n-1)$-dimensional tori. The number of cusps may be very big.[1] Note that the manifolds $M^3, \ldots, M^8$ defined here are much larger than the manifolds considered in [9], that are some quotients of these.

We will prove the following.

Theorem 6. *The hyperbolic cusped n-manifold M^n has a spinnable Dehn filling. Therefore it has a spin structure where every cusp cross-section inherits a bounding spin structure. This holds for every $3 \leq n \leq 8$.*

To prove this theorem we need to introduce some tools. We start by describing the Dehn fillings of M^n.

2.3. **Dehn fillings of polytopes with ideal vertices.** The following procedure works with every right-angled hyperbolic polytope containing some ideal vertex, but we focus on the poyhedra $P^3, \ldots, P^8$ for simplicity.

Consider P^n as a Euclidean polytope, using the Klein model for $\mathbb{H}^n$. We now *Dehn fill* P^n to produce a new abstract compact polytope $\bar{P}^n$, by substituting each ideal vertex of P^n with a $(n-2)$-cube.

This operation goes as follows. At each ideal vertex v of P^n, we first truncate it, thus producing a new small $(n-1)$-cubic facet C; then we choose two opposite facets F_1, F_2 of C, we foliate C with $(n-2)$-cubes parallel to F_1 and F_2, and we identify all the leaves to a single $(n-2)$-cube F. See Figure 1.

If we perform this operation at every ideal vertex v of P^n we get a topological disc $\bar{P}^n$ that has the structure of an *abstract simple polytope*, that is its boundary is stratified into faces which intersect minimally, *i.e.* exactly k of them at each $(n-k)$-stratum (this is a consequence of Remark 8 below). Very often $\bar{P}^n$ may itself be realized as a polytope in $\mathbb{R}^n$, but we will not need that. Note that P^n is not simple

[1]The polytope P^8 has 2160 ideal vertices and 240 facets. The discussion in [9, Section 1.2] implies that M^8 has $2160 \cdot 2^{240-14} \sim 10^{71}$ cusps. This is still below the number of atoms in the observable universe, that is around 10^{80}.

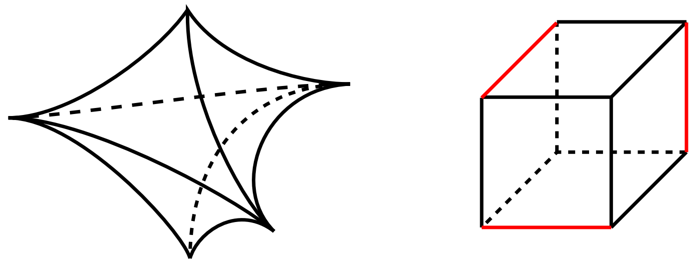

FIGURE 2. The polytope P^3 is a right-angled bipyramid with three ideal vertices along the horizontal plane and two real ones (top and bottom in the figure). By Dehn filling P^3 we may obtain a cube: the three ideal vertices have been substituted with three edges (red in the figure).

precisely at the ideal vertices: therefore $\bar{P}^n$ may be seen as a perturbation of P^n that transforms it into a simple polytope.

The strata of $\bar{P}^n$ are the same of P^n, except that every ideal vertex v is replaced with a $(n-2)$-cube F. The facets of $\bar{P}^n$ are those of P^n, with some additional adjacencies: at every ideal vertex v, the two facets of P^n that were opposite with respect to v and contained F_1 and F_2 are now adjacent in $\bar{P}^n$, since they intersect in the new $(n-2)$-cube F.

Note that at every ideal vertex v there are $n-1$ different Dehn fillings to choose from, one for every pair of opposite facets F_1, F_2 in the small $(n-1)$-cube C.

Example 7. The polytope P^3 is the bipyramid shown in Figure 2, with 3 ideal vertices and 2 real ones. The Dehn filling $\bar{P}^3$ is obtained by replacing every ideal vertex with an edge as in Figure 1. At every ideal vertex there are two possible choices, so there are $2^3 = 8$ choices overall. In all cases we get an abstract simple polytope with 6 faces. It is possible to get a $\bar{P}^3$ that is combinatorially a cube as in Figure 2.

Remark 8. We may also describe $\bar{P}^n$ using dual polytopes. The Gosset polytope dual to P^n is a Euclidean polytope $G^n \subset \mathbb{R}^n$ with two types of facets: some simplexes, dual to the real vertices of P^n, and some cross-polytopes (also called *hyper-octahedra*), dual to the ideal vertices of P^n. (Every ideal vertex of P^n has a cubical link, hence the dual facet is a cross-polytope, that is dual to that cube.)

Let K^{n-1} be the simplicial complex obtained from the boundary of the Gosset polytope G^n by subdividing each cross-polytope facet into 2^{n-1} simplexes as follows. A cross-polytope in $\mathbb{R}^{n-1}$ is the convex hull of $\pm e_1, \ldots \pm e_{n-1}$. Choose two opposite vertices, say $\pm e_1$ for simplicity, and subdivide the cross-polytope into the simplexes with vertices $e_1, -e_1, \pm e_2, \ldots, \pm e_n$. There are 2^{n-1} possible signs, hence 2^{n-1} simplexes. When $n = 2, 3$ this is the standard decomposition of a square into two triangles and of an octahedron into four simplexes. It depends on the choice of two opposite vertices, that is of a diagonal of the cross-polytope.

The stratification of $\partial\bar{P}^n$ is dual of the simplicial complex K^{n-1}. The choice of two opposite facets F_1, F_2 at each ideal vertex v for P^n corresponds to the choice of a diagonal in each cross-polytope of G^n, and the additional $(n-2)$-cube F in $\bar{P}^n$ corresponds to the additional diagonal in K^{n-1}. Perturbing P^n to the simple polyhedron $\bar{P}^n$ corresponds dually to subdividing the complex ∂G^n into the simplicial complex K^{n-1}.

We think of $\bar{P}^n$ as an abstract right-angled compact polytope, that is a topological disc with right-angled corners. In some fortunate cases as in Figure 2 the polyhedron $\bar{P}^n$ may indeed be interpreted as a right-angled polyhedron in some geometry ($\bar{P}^3$ is a Euclidean cube in the figure), but this does not hold in general, and we do not need it.

We may assign distinct colours to all the facets of $\bar{P}^n$ and apply the colouring construction to $\bar{P}^n$ as in [9]. The result is a closed topological manifold $\bar{M}^n$.

The manifold $\bar{M}^n$ is naturally a Dehn filling of M^n. Using the techniques of [9] we see that $\bar{M}^n$ decomposes into 2^f identical copies of $\bar{P}^n$, and that the pre-image of every additional $(n-2)$-cube F of $\bar{P}^n$ in $\bar{M}^n$ consists of many $(n-2)$-tori, each tessellated into 2^{2n-4} copies of F. Since $\bar{P}^n$ is obtained from P^n by substituting ideal vertices with $(n-2)$-cubes, the manifold M^n is naturally homeomorphic to $\bar{M}^n$ minus all the $(n-2)$-tori that are the pre-images of these $(n-2)$-cubes F. The closed manifold $\bar{M}^n$ also inherits a smooth structure from M^n.

Example 9. We Dehn fill the polyhedron P^3 to a cube $\bar{P}^3$. The filled manifold $\bar{M}^3$ is a 3-torus tessellated into 2^6 copies of the cube $\bar{P}^3$. Therefore M^3 is the complement of a 12-components link in the 3-torus $\bar{M}^3$. The link consists of the pre-images of the red edges in the cube shown in Figure 2. In fact the containment of M^3 in the 3-cube is a 8-fold covering of the usual description of the Borromean ring complement as a link complement in the 3-torus. So in particular M^3 is an 8-fold cover of the Borromean ring complement, and in fact P^3 is commensurable with the ideal regular right-angled octahedron.

2.4. **Dehn fillings are spinnable.** To conclude the proof of Theorem 6, and hence of Theorem 2(2), it remains to show that the Dehn filling $\bar{M}^n$ constructed above is spinnable. This is an instance of a more general theorem proved recently by Hasui – Kishimoto – Kizu in [8] in the context of *moment angle manifolds*. This theorem shows in fact that all the Stiefel-Whitney classes of $\bar{M}^n$ vanish.

We briefly recall this setting. Let K be a simplicial complex with vertices $\{1, \ldots, m\}$. The *real moment-angle complex* determined by K is

$$\mathbb{R}Z_K = \bigcup_{\sigma \in K} \mathbb{R}Z_\sigma \subset [-1, 1]^m$$

where

$$\mathbb{R}Z_\sigma = X_1 \times \cdots \times X_m$$

such that X_i equals $[-1, 1]$ if $i \in \sigma$ and $\{-1, 1\}$ if $i \notin \sigma$. The symbol $\mathbb{R}$ is used only to distinguish the real moment-angle complex from its complex version Z_K, that we will not use here.

By construction $\mathbb{R}Z_K$ is a cube subcomplex of $[-1, 1]^m$ that is symmetric with respect to the m reflections along the hyperplanes $x_i = 1/2$, $i = 1, \ldots, m$. These symmetries act transitively on the 2^m vertices of the cube complex. It is quite easy to check that the link of a vertex in $\mathbb{R}Z_K$ is isomorphic to the simplicial complex K itself. In particular, if K is homeomorphic to a $(n-1)$-sphere then $M = \mathbb{R}Z_k$ is a topological n-manifold. The manifold M is only topological, but it is noted in [6, 8] that Stiefel – Whitney classes need no smooth structure to be defined, and moreover the following holds.

Theorem 10 (Hasui – Kishimoto – Kizu)**.** *If K is a topological sphere, the Stiefel-Whitney classes of the real moment-angle manifold $\mathbb{R}Z_K$ all vanish.*

This theorem applies to our Dehn fillings $\bar{M}^n$ because of the following.

Proposition 11. *Let $K = K^{n-1}$ be the simplicial complex constructed by subdividing the Gosset polytope in Remark 8. The real moment-angle manifold $\mathbb{R}Z_K$ is homeomorphic to $\bar{M}^n$, for every $3 \leq n \leq 8$.*

Proof. Recall that K^{n-1} is dual to $\partial\bar{P}^n$ and that $\bar{M}^n$ is obtained from $\bar{P}^n$ by colouring its facets with all distinct colours. The decomposition of $\bar{M}^n$ into identical copies of $\bar{P}^n$ is dual to a cube complex C described in [9, Section 2]. The cube complexes $\mathbb{R}Z_K$ and C are in fact isomorphic by construction. □

By combining these two results we get that the Stiefel – Whitney classes of $\bar{M}^n$ vanish, so in particular $\bar{M}^n$ is spinnable and Theorem 2(2) is proved. □

3. Final remarks and questions

Motivated by Bär's result Theorem 1, the most interesting question about the nature of the spectrum of the Dirac operator that remains after out work is captured by the following.

Question 1: *For $n \geq 9$, does there exist examples of cusped orientable hyperbolic n-manifolds that have all cusp cross-sections being $(n-1)$-tori, which admits a spin structure that restricts to each cusp cross-section as the bounding spin structure?*

As noted in §2, this is equivalent to the existence of a spinnable Dehn filling N of the cusped hyperbolic manifold. Thus we pose:

Question 2: *For each $n \geq 9$, does there exist a cusped orientable hyperbolic n-manifold M having all cusp cross-sections being $(n-1)$-tori which admits a Dehn filling N (in the sense of §2) a closed n-manifold that is spinnable?*

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