APPENDIX: FILLING LINKS IN RANK 2

Abstract. We show that a closed orientable 3-manifold whose fundamental group has rank 2 contains a filling hyperbolic link.

1. Rank 2

In this appendix we prove the following result which provides the first examples of closed orientable 3-manifolds with Heegaard genus > 1 that contain a filling link. To state this, recall that the *rank* of a finitely generated group Γ is the minimal cardinality of a generating set for Γ. When $\Gamma = \pi_1(M)$ and M is a compact 3-manifold, we define the *rank of M* to be the rank of Γ.

Theorem 1.1. *Let M be a closed orientable 3-manifold of rank 2. Then M contains a filling hyperbolic link.*

For convenience, we will discuss what lies behind the crucial feature that we exploit to exhibit filling links using the assumption of rank 2; namely a classical result from 3-manifold topology (see [4, Theorem VI.4.1]) which affords a classification of 2-generator subgroups of the fundamental group of a compact atoroidal, irreducible 3-manifold (see also §2 below). Very briefly, let X be such a 3-manifold, $H \subset \pi_1(X)$ a 2-generator subgroup, and $Y_H \to X$ the cover corresponding to H with compact core C_H. As an illustration of the main part of the argument, assume that $\pi_1(X)$ is freely indecomposable and ∂C_H is non-empty and contains no 2-sphere components. Then the 2-generator assumption, together with the fact that the first Betti number of a 3-manifold is at least half the first Betti number of its boundary, implies that ∂C_H (which may be disconnected) has genus 1 or 2. In the case when ∂C_H is connected of genus 2, it follows that $b_1(C_H) = 2$, and some simple homological algebra then shows:

$$-1 = \chi(\partial C_H)/2 = \chi(C_H) = b_2(C_H) - b_1(C_H) + b_0(C_H) = b_2(C_H) - 1.$$

It follows that $b_2(C_H) = 0$, and one can now deduce in this case that H is free of rank 2, contradicting the assumption that H is indecomposable. The remainder of the argument is completed by analyzing the case when ∂C_H consists of tori; either one or two incompressible tori, or when ∂C_H contains a compressible torus. The upshot is a limited set of possibilities for what C_H, and hence H can be, and these are listed in §2.

Before commencing with the proof of Theorem 1.1 we list some corollaries. Closed orientable 3-manifolds of Heegaard genus 2 are examples of manifolds covered by Theorem 1.1, so an immediate corollary is:

Corollary 1.2. *Let M be a closed orientable 3-manifold such that M has Heegaard genus 2. Then M contains a filling hyperbolic link.*

Note that there there are closed orientable 3-manifolds M for which $\pi_1(M)$ has rank 2, but the Heegaard genus is 3 (see [1]). However, an interesting case of Corollary 1.2 is the following.

Using Perelman's resolution of the Geometrization Conjecture [5]) it is known that all closed orientable 3-manifolds with non-trivial finite fundamental have Heegaard genus 1 or 2. To see this, the resolution of the Geometrization Conjecture proves that a closed 3-manifold M with finite fundamental is covered by S^3, in which case M is a Seifert fibered space. Assuming that M is not

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S^3, or a Lens Space, then M is a Seifert fibered space over S^2 with three exceptional fibers (see [3, VI.11]), from which a genus 2 Heegaard splitting may be constructed directly (see [1, Proposition 1.3] for example). The existence of filling links in manifolds of genus 1 was noted after the proof of Proposition 2, so Theorem 1.1 now shows.

Corollary 1.3. *Let M be a closed orientable 3-manifold such that $\pi_1(M)$ is finite and non-trivial. Then M contains a filling link.*

Remark 1.4. An alternative proof of Corollary 1.3 bypassing the use of Heegaard genus is the following. Using [6, Section 3], one obtains a classification of finite groups that act freely on S^3. Perelman's resolution of the Geometrization Conjecture eliminates the one class of finite groups from [6] that are not subgroups of SO(4). The five families of non-cyclic subgroups of SO(4) can all be seen to have rank 2.

2. Proof of Theorem 1.1:

We begin with a preliminary remark. The hypothesis that M is closed, orientable and $\pi_1(M)$ has rank 2 implies that $\pi_1(M)$ is non-Abelian. The reason is this: From [2, Theorem 9.13] the only abelian groups occurring as the fundamental group of a closed orientable 3-manifold are $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, and $\mathbb{Z}^3$, and these are all excluded by the rank hypothesis.

Let $L \subset M$ be a hyperbolic link with at least 3 components. To find such a link, one may start with any link L' in M with at least 2 components, then appealing to [7, Corollary 6.3] we may choose a knot K in $M \setminus L'$ with hyperbolic complement, and set $L = L' \cup K$.

As in the introduction G will be a spine of M, with $G \cap L = \emptyset$. By definition, $\pi_1(G)$ surjects $\pi_1(M)$. We need to prove that the induced homomorphism $\pi_1(G) \to \pi_1(M \setminus L)$ is injective. Let H be be the image group of this homomorphism. The result will follow from the Hopfian property for free groups once we establish that H is free of rank 2. To that end we argue as follows.

Appealing to the classical 3–manifold result sketched in the previous section, namely [4, Theorem VI.4.1], observe that since $M \setminus L$ is hyperbolic the possibilities for H are:

(1) H is free of rank 2, or
(2) H is free abelian of rank ≤ 2, or
(3) H finite index in $\pi_1(M \setminus L)$.

If H is free abelian of rank 1 or 2, then the image in $\pi_1(M)$ via the homomorphism induced by inclusion $M \setminus L \to M$ is the quotient of a free abelian group of rank at most 2, and is in particular abelian. Since G is a spine, H surjects $\pi_1(M)$, which contradicts the fact noted above that $\pi_1(M)$ is non-abelian. Therefore, case (2) is impossible. Case (3) can also be eliminated as follows. As noted above, the first Betti number of a 3-manifold is at least half the first Betti number of its boundary, and so we deduce from this that $\mathbb{H}^3/H$ has at most two cusps. On the other hand, by construction, $M \setminus L$ has at least 3 cusps, and $\mathbb{H}^3/H \to M \setminus L$ is a finite sheeted cover, a contradiction. We therefore conclude that the only possibility for H is that it is free of rank 2 and the proof is complete. □

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