Generalized Dedekind sums.

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Dedicated to Andrew Casson on the occasion of his 60th birthday

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1 Introduction

A classical and important construction which arises in many contexts is that of the *Dedekind sum* which is defined for coprime integers a and c by

$$s(a,b) = \sum_{k=1}^{|b|-1} ((\frac{k}{b}))((\frac{ka}{b}))$$

where $((x)) = x - [x] - 1/2$. Dedekind sums arise naturally in various topological settings, one of the most famous being Hirzebuch's description of $4s(b,a)$ as the signature defect of the Lens space $L(a,b)$ coming from Rademacher's cotangent formula

$$s(a,b) = \frac{1}{4|b|} \sum_{k=1}^{|b|-1} \cot(\frac{k\pi}{b})((\frac{ka\pi}{b})),$$

as well as in Walker's formula for the generalized Casson invariant.

From the point of view of this note, it is the beautiful construction in [1] of Dedekind sums based upon the classical modular group $\mathrm{PSL}(2,\mathbf{Z})$ that is of interest. We describe some this briefly, as it is useful in the development of what follows. It is shown in [1] that there exists a 2-cocycle $\epsilon : \mathrm{PSL}(2,\mathbf{Z}) \times \mathrm{PSL}(2,\mathbf{Z}) \to \mathbf{Z}$ and a function $\phi : \mathrm{PSL}(2,\mathbf{Z}) \to \mathbf{Z}$ (the Rademacher ϕ function) which satisfy $\delta\phi = 3\epsilon$ (where δ is the coboundary operator). Furthermore, it is shown in [1] that the function ϕ is closely related to the Dedekind sums mentioned above. Namely, in [1] the authors define a Dedekind symbol S on $\mathbf{Q} \cup \infty$ which maps ∞ to ∞ and otherwise, $S(\frac{a}{c}) = \phi(M) + \chi(M)$ where $M \in \mathrm{PSL}(2,\mathbf{Z})$ satisfies $M(\infty) = \frac{a}{c}$ and χ is

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a function depending on the entries of M (see §2.2). As pointed out in (0.8) of [1], the relationship between S and the Dedekind sum s above is $S(\frac{a}{c}) = 12\mathrm{sign}(c)s(a,c)$.

For us, since $\mathbf{Q} \cup \infty$ coincides with the cusp set (that is the set of all parabolic fixed points) of $\mathrm{PSL}(2,\mathbf{Z})$, S can be viewed as a function defined on the cusp set of $\mathrm{PSL}(2,\mathbf{Z})$. In [2] it was shown that there exist finite coarea Fuchsian groups not commensurable with the modular group but whose cusp set is precisely $\mathbf{Q} \cup \infty$. The purpose of this note is to show that these groups give rise to very natural generalizations of Dedekind sums.

We begin by recalling briefly the construction of [2]. The starting point of that paper was to take the two generator group $\Delta(u^2, 2t)$ generated by elements g_1 and g_2 as below

$$g_1 = \begin{pmatrix} (-1+t)/\sqrt{-1+t-u^2} & u^2/\sqrt{-1+t-u^2} \\ \\ 1/\sqrt{-1+t-u^2} & 1/\sqrt{-1+t-u^2} \end{pmatrix}$$

and

$$g_2 = \begin{pmatrix} u/\sqrt{-1+t-u^2} & u/\sqrt{-1+t-u^2} \\ \\ 1/(u\sqrt{-1+t-u^2}) & (t-u^2)/u\sqrt{-1+t-u^2} \end{pmatrix}$$

where the parameters u^2 and t are real and satisfy $t > u^2+1$.

One sees easily that in the hyperbolic plane, g_1 maps the directed edge $\{-1, 0\}$ to the directed edge $\{\infty, u^2\}$ and g_2 mapping $\{\infty, -1\}$ to $\{u^2, 0\}$, and moreover the commutator

$$g_1.g_2^{-1}.g_1^{-1}.g_2 = \begin{pmatrix} -1 & -2t \\ 0 & -1 \end{pmatrix}$$

is parabolic and generates the stabiliser of infinity. It follows that $\mathbf{H}^2/\Delta(u^2, 2t)$ is a complete finite area once punctured torus. This family includes a modular torus as $\Delta(1, 6)$, as well as other arithmetic once punctured tori, and if u^2 and t are chosen to be rational the set of cusps of these groups must be a subset of $\mathbf{Q} \cup \infty$. In the arithmetic cases, the cusp set is precisely $\mathbf{Q} \cup \infty$, although this is not always the case for rational pairs $(u^2, 2t)$. (See [2]).

Despite the apparently complicated nature of the entries in these matrices because of the presence of square roots, an easy computation shows that if one considers $G = \ker\{\Delta \to \mathbf{Z}/2 \oplus \mathbf{Z}/2\}$, then the trace-field of G, and hence the invariant trace-field of $\Delta(u^2, 2t)$ is the field $\mathbf{Q}(u^2, t)$. In fact all the entries of the matrix representatives for G lie in the field $\mathbf{Q}(u^2, t)$. This real field will be called the *invariant field of definition* of $\Delta(u^2, 2t)$ as it is the most germane field for our considerations. In particular, the cusp set of $\Delta(u^2, 2t)$ can clearly be no larger than the field $\mathbf{Q}(u^2, t) \cup \infty$

The main result of [2] is that there are rational choices of parameters $(u^2, 2t)$ which give rise to nonarithmetic groups whose cusp sets are precisely the rationals. Such groups we call *pseudomodular.* There is a good deal of evidence that such groups exist for fields more general than the rationals, that is to say, their cusp sets are equal to their invariant field of definition - such groups we will describe as *maximally cusped.* It is these groups which we will use to construct Dedekind sums; since our family includes the modular group, it will include a construction of the classical Dedekind sum. In this note we will show

Theorem 1.1 *Suppose that* Δ *as above has invariant field of definition* K *and is maximally cusped. Then associated to* Δ *is a function*

$$S_\Delta : K \cup \infty \to K \cup \infty$$

Such functions we say are **generalized Dedekind sums**.

2 The Construction.

Following [1], we first construct an analogoue of the Rademacher ϕ function. Fix one of the groups $\Delta(u^2, 2t)$ of [2]; (at this stage it is not necessary that the group be pseudomodular) and suppose that its invariant field of definition is K.

All once punctured tori are hyperelliptic so we can adjoin to this group the orientation preserving involution τ which conjugates the generators to their inverses, to form a new discrete group Γ. The surface $F = \mathbf{H}^2/\Gamma$ is a sphere with three cone points of angle π and a cusp. Note that as an element of $\mathrm{GL}(2,\mathbf{R})$, τ is represented by the matrix $\begin{pmatrix} 0 & 2u \\ -2/u & 0 \end{pmatrix}$, so $\tau(\infty) = 0$.

Following [1], we define an area 2-cocycle

$$\epsilon : \Gamma \times \Gamma \to \mathbf{Z}$$

by setting $\epsilon(A, B) = \mathrm{area}(\infty, A\infty, AB\infty)/\pi$ where this area is to be regarded as oriented, it follows that ϵ takes on the values $0, \pm 1$.

Equivalently, one can usefully think of $\epsilon(A, B)$ as the sign of $AB\infty - A\infty$, where this is to be interpreted as zero if either term of the difference is infinite.

Notice that ϵ is a cocycle, because the coboundary

$$\delta\epsilon(A, B, C) = \epsilon(B, C) - \epsilon(AB, C) + \epsilon(A, BC) - \epsilon(A, B)$$

involves four triangular areas and the first has vertices $(\infty, B\infty, BC\infty)$ which has same oriented area as $(A\infty, AB\infty, ABC\infty)$, so that taken together with other three this forms a tetrahedron, and hence the total area is 0.

Lemma 2.1 *There is a unique* K*-valued* 1*-cochain* $\Gamma \to K$ *with coboundary* ϵ*.*

Proof. Note that $\Gamma \cong \mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$ so that

$$H^1(\Gamma; \mathbf{Z}) \cong 0$$

and

$$H^2(\Gamma; \mathbf{Z}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

since the integral homology of $\mathbf{Z}/2$ is zero in odd dimensions and $\mathbf{Z}/2$ in even dimensions. For our purposes, we need only use that $H^2(\Gamma; K) = H^1(\Gamma; K) = 0$. The fact that $H^2(\Gamma; K) = 0$ implies immediately the existence of a K-valued 1-cochain with coboundary ϵ.

We prove uniqueness as follows. If $\delta(\phi_1) = \epsilon = \delta(\phi_2)$, then ϕ_i are both cocycles and hence since $H^1(\Gamma; K) = 0$, both are coboundaries. It follows that there is a 0-cochain β with $\delta(\beta) = \phi_1 - \phi_2$. We are computing group cohomology with trivial coefficients, so that this coboundary map is zero and $\phi_1 = \phi_2$ as required. $\square$

Definition. We shall denote this K-valued 1-cochain by ϕ.

2.1 Computation of ϕ

It will be useful to have a computation of the cochain ϕ. A consequence of Lemma 2.1 is that there is a function $\phi : \Gamma \to K$ which satisfies

$$\phi(AB) - \phi(A) - \phi(B) = -\lambda \mathrm{sign}(AB\infty - A\infty) \qquad (*)$$

for some $\lambda \in K$ which will be determined.

Taking $A = B = I$ we see that $\phi(I) = 0$. Taking $A = B = -I$, we also get $\phi(-I) = 0$. Taking $A = -I$ and $B = g$, we deduce from $(*)$ that $\phi(g) = \phi(-g)$ for every $g \in \Gamma$.

More generally, if A and B both stabilise ∞, then the relation says

$$\phi(AB) - \phi(A) - \phi(B) = 0$$

that is to say, ϕ is a homomorphism on stab(∞).

Note that in the group Γ, we have that $g_1 g_2^{-1} \tau$ stabilises infinity and one checks easily that this is the generating matrix for the parabolic subgroup and is given by $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

By scaling by an appropriate element of K, we may assume that λ is chosen so that ϕ maps this generating parabolic matrix to t, so that ϕ is now determined on the parabolic subgroup.

It also follows from $(*)$ that

$$\phi(\alpha^{-1}) = -\phi(\alpha) = -\phi(-\alpha)$$

for any element α, in particular, if ξ is any projective involution in Γ, (that is to say $\xi^2 = \pm I$) we deduce that $\phi(\xi) = 0$.

Now in the notation introduced above we have

$$\phi(g_1\tau) - \phi(g_1) - \phi(\tau) = -\lambda \mathrm{sign}(u^2 - t + 1)$$

Since $g_1\tau$ and τ are both projective involutions and recalling that the groups in question are required to have $0 > 1 + u^2 - t$ we get

$$\phi(g_1) = -\lambda$$

By considering τg_2, a similar computation also shows $\phi(g_2) = -\lambda$.

Now for any $k \in K$, for which the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ lies in Γ, we have that

$$\phi(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.\tau) - \phi(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}) - 0 = 0$$

In the special case that $k = t$, the leftmost term is the product $(g_1 g_2^{-1}\tau) \cdot \tau = -g_1 g_2^{-1}$, so we deduce from the properties described above that $\phi(g_1 g_2^{-1}) = t$.

Since $\begin{pmatrix} t/u & -u \\ 1/u & 0 \end{pmatrix} = g_1 g_2^{-1}$, (or from purely geometric considerations) we see that $g_2 g_1^{-1}\infty = 0$. Finally, noting that $g_2\infty = u^2 > 0$ together with the relation

$$\phi(g_2 g_1^{-1}) - \phi(g_2) + \phi(g_1) = -\lambda \mathrm{sign}(0 - u^2)$$

it follows that $\lambda = -t$, since the leftmost term is $-t$ by the previous calculation and the inverse rule.

To sum up, we now have a complete inductive description of ϕ on the group Γ, namely it satisfies

$$\phi(AB) - \phi(A) - \phi(B) = t \cdot \mathrm{sign}(AB\infty - A\infty)$$

and

$$\phi(g_1) = \phi(g_2) = \phi(g_2 g_1^{-1}) = t$$

Remark This is in keeping with the computations of [1] which are for the modular group and have $\lambda = -3$.

2.2 Generalized Dedekind sums.

Now fix some maximally cusped $\Delta = \Delta(u^2, 2t)$ defined over the field K.

For any $M \in \Delta$, by applying the cocycle condition we have

$$\phi(M.\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}) - \phi(M) - k = t \cdot \operatorname{sign}(M\infty - M\infty) = 0$$

from which it follows that

$$\phi(M.\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}) = \phi(M) + k \qquad (**)$$

For $M \in \Delta \setminus \operatorname{stab}(\infty)$, set

$$\chi(M) = (M_{1,1} + M_{2,2})/M_{2,1}.$$

Since $M_{2,1} \neq 0$, the value $\chi(M)$ is an element of the field K, since the groups Δ consist of matrices of the shape $\sqrt{r}X$ for a matrix $X \in \mathrm{GL}(2, K)$ and $r \in K$. Now a matrix computation shows that

$$\chi(M.\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}) = \chi(M) + k$$

so that by taking the difference between this and $(**)$ we get

$$\phi(M.\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}) - \chi(M.\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}) = \phi(M) - \chi(M)$$

which is to say the function

$$S(M) = \phi(M) - \chi(M)$$

is invariant under right muliplication by the parabolic subgroup.

These observations are independent of whether Δ is maximally cusped or not. If we now assume that it is, we can define a **generalized Dedekind sum** as follows.

Given any element $\kappa \in K$, since Δ is maximally cusped, there is an element $M \in \Delta$ with $M(\infty) = \kappa$ and we may set

$$S_\Delta(\kappa) = S(M)$$

The ambiguity in such $M \in \Delta$ is accounted for by right multiplication by elements of the parabolic subgroup $\operatorname{stab}(\infty)$ so that this function depends only on κ. We will define $S_\Delta(\infty) = \infty$, and this defines the advertized function in Theorem 1.1.

Remark. This construction gives is a scalar multiple of the classical Dedekind sum when $(u^2, 2t) = (1, 6)$. (See [1], (0.8))

Examples. It is proved in [2] that the group $\Delta(3/5, 4)$ is pseudomodular so provides an example of a generalized Dedekind sum of this type. It is not difficult to write a computer program which computes its values based upon the iterative procedure outlined above. A table of the groups currently proven to be pseudomodular (and some conjectural examples) is provided in [2].

In subsequent work, the authors have extended this table of conjectural examples to groups which are maximally cusped for real quadratic number fields, for example $\Delta(1, 2(1+\sqrt{13})/2)$ appears to be maximally cusped. Questions about whether there are analogues of, for example, Dedekind reciprocity and formulae of the classical type seem interesting and appear worthy of further investigation.

References

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