MOST HITCHIN REPRESENTATIONS ARE STRONGLY DENSE

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Abstract. We prove that generic Hitchin representations are strongly dense: every pair of non commuting elements in their image generate a Zariski-dense subgroup of $\mathrm{SL}_n(\mathbb{R})$. The proof uses a theorem of Rapinchuk, Benyash-Krivetz and Chernousov, to show that the set of Hitchin representations is Zariski-dense in the variety of representations of a surface group in $\mathrm{SL}_n(\mathbb{R})$.

1. Introduction

Following Breuillard, Green, Guralnick and Tao [2], we say that a subgroup $\Gamma \subset \mathrm{SL}_n(\mathbb{R})$ is *strongly dense* if any pair of non-commuting elements of Γ generate a Zariski-dense subgroup of $\mathrm{SL}_n(\mathbb{R})$. They proved that, among many other semisimple algebraic groups, the group $\mathrm{SL}_n(\mathbb{R})$ contains a strongly dense non abelian free subgroup [2, Theorem 4.5]. In this note, we extend the Breuillard, Green, Guralnick and Tao result to certain (discrete and) faithful representations of surface groups of genus at least two into $\mathrm{SL}_n(\mathbb{R})$.

To describe this more carefully, we introduce some background and terminology. For fixed $g \geqslant 2$, and base field k, the set of representations of the surface group $\pi_1(\Sigma_g)$ to $\mathrm{SL}_n(k)$ is denoted by $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(k))$ and is naturally an affine subvariety of k^{2gn^2} known as the *representation variety*. In the case of $k = \mathbb{R}$, those representations of interest to us, the Hitchin representations, are of particular geometric importance and can be defined as follows.

The *Teichmüller representations* in $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ are those obtained by composing any faithful and discrete representation $\pi_1(\Sigma_g) \to \mathrm{SL}_2(\mathbb{R})$ with an irreducible representation $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$. The Hitchin representations are those that lie in the same connected component (for the usual, Euclidean topology) of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ as a Teichmüller representation. Note that, depending on the parity of n, there may be more than one such component, but we simply choose one and denote it by HIT_n. [1]

We say that a representation is strongly dense if its image is a strongly dense subgroup of $\mathrm{SL}_n(\mathbb{R})$, and we say that a subset of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ is *generic* if its complement consists of a countable union of proper subvarieties. The main result of this note is:

Theorem 1.1. *Let $n \geqslant 3$. Then the set of strongly dense representations of $\pi_1\Sigma_g$ is generic in HIT_n.*

It is known that all the representations in HIT_n are faithful and discrete (see [8, Theorem 1.5]), so this provides the representations promised in the first paragraph. In fact, it is also known that generic Hitchin representations are Zariski-dense (see [7, 12]). We note that the result of Theorem 1.1 was obtained recently in [9] in the

[1] We note that a *Hitchin component* more usually refers to a connected component of the *character variety* $X(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R})))$ and the notation Hit_n is frequently used, but in this note it will be technically simpler to work at the level of representations.

case of $n = 3$ by direct geometric methods.

To prove Theorem 1.1 we prove the following result, which seems independently interesting, and uses a result of Rapinchuk, Benyash-Krivetz and Chernousov [11], that $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{C}))$ is an irreducible subvariety of $\mathbb{C}^{2gn^2}$; in fact, it is connected for the Zariski topology and for the classical (Euclidean) topology.

Theorem 1.2. *For all $n \geqslant 2$, the set HIT_n is Zariski-dense in the affine algebraic set $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{C}))$.*

The case $n = 2$ was already essentially observed in [5, Chapter 3].

As we describe below, Theorem 1.1 follows from Theorem 1.2 together with [2] and the fact that surface groups are residually free [1]. The idea of combining the irreducibility of representation spaces with residual properties of surface groups was already used, for example in [3, 4].

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2. Proofs.

Proof of Theorem 1.2. As noted in §1, $R(\mathbb{C}) = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{C}))$ is an affine subvariety of $\mathbb{C}^{2g \cdot n^2}$, and it was proved in [11, Theorem 3] to be irreducible of dimension $(2g-1)(n^2-1)$.

The set HIT_n is, by definition, a (topological) connected component of $R(\mathbb{R})$, which is a real algebraic variety, and hence HIT_n is open. We claim that it contains smooth points of $R(\mathbb{R})$, or equivalently, of $R(\mathbb{C})$: in fact, we will show that all its points are regular.

Indeed, by a result of Goldman [6, Propositon 1.2], at each point ρ of $R(\mathbb{R})$, the dimension of the Zariski tangent space at ρ equals $(2g-1)(n^2-1)+\dim(\zeta(\rho(\pi_1\Sigma_g)))$, where $\zeta(\rho(\pi_1\Sigma_g))$ is the centralizer of the image group $\rho(\pi_1\Sigma_g)$ in $\mathrm{SL}_n(\mathbb{R})$.

We will make use of the following facts proved by Labourie (see [8, Theorem 1.5 and Paragraph 10]). First, if $\rho \in \mathrm{HIT}_n$, then ρ is irreducible, and second, for all nonidentity elements $\gamma \in \pi_1(\Sigma_g)$, the matrix $\rho(\gamma)$ is diagonalizable with pairwise distinct real eigenvalues.

Fix such a γ_0; by conjugating the image of ρ in $\mathrm{SL}_n(\mathbb{R})$, we may suppose that $\rho(\gamma_0)$ is diagonal. Let ξ be an element of $\zeta(\rho(\pi_1(\Sigma_g)))$. Since ξ commutes with $\rho(\gamma_0)$, it is also diagonal, and if λ is an eigenvalue of ξ, the matrix $\xi - \lambda I$ also commutes with $\rho(\pi_1(\Sigma_g))$. Hence $\ker(\xi - \lambda I)$ is invariant by $\rho(\pi_1(\Sigma_g))$. However, ρ is irreducible, and so this implies that ξ is a scalar matrix, that is to say, $\xi = \pm I$.

Thus, the Zariski tangent space at any representation $\rho \in \mathrm{HIT}_n$ has minimal dimension, $(2g-1)(n^2-1)$, in other words, these are regular points of the varieties $R(\mathbb{R})$ and $R(\mathbb{C})$.

Now, the result follows from the following general fact from real algebraic geometry: suppose V is an irreducible complex affine variety defined by real polynomials, and suppose H is a connected component of $V(\mathbb{R})$ which has a smooth real point.

Then H is Zariski-dense in V. This is a slight variation of the statement [10, Theorem 2.2.9] (with the same proof). □

Proof of Theorem 1.1. For every pair of non commuting elements $a, b \in \pi_1(\Sigma_g)$, let $\mathrm{Bad}(a, b)$ denote the subset of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ consisting of representations ρ such that $\rho(a)$ and $\rho(b)$ do not generate a Zariski-dense subgroup of $\mathrm{SL}_n(\mathbb{R})$, and let $\mathrm{Good}(a, b)$ denote its complement.

The proof will be complete once we know that for every pair of non commuting elements $a, b \in \pi_1(\Sigma_g)$, the set $\mathrm{Bad}(a, b) \cap \mathrm{HIT}_n$ is Zariski-closed, and that it is a proper subset of HIT_n.

The fact that the sets $\mathrm{Bad}(a, b)$ are Zariski-closed follows from [2, Theorem 4.1].

Now let us check that $\mathrm{Bad}(a, b) \cap \mathrm{HIT}_n$ is a proper subset of HIT_n, or equivalently, that $\mathrm{Good}(a, b) \cap \mathrm{HIT}_n$ is nonempty. Since $\mathrm{Good}(a, b)$ is Zariski-open, and since HIT_n is Zariski-dense in $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ by Theorem 1.2, it suffices to check that $\mathrm{Good}(a, b)$ is nonempty.

By [2, Theorem 4.5], there exists a strongly dense representation $\rho_0 \colon F_2 \to \mathrm{SL}_n(\mathbb{R})$. Let $a, b \in \pi_1(\Sigma_g)$ be a pair of non commuting elements. Since $\pi_1(\Sigma_g)$ is residually free (see Baumslag [1]) and $[a, b] \neq 1$, there exists a surjective morphism ψ from $\pi_1\Sigma_g$ onto a free group F, such that $\phi([a, b]) \neq 1$. By composing ψ with an injective morphism $F \to F_2$, this yields a morphism $\varphi \colon \pi_1(\Sigma_g) \to F_2$ such that $\varphi([a, b]) \neq 1$. Thus, $\varphi(a)$ and $\varphi(b)$ do not commute, hence $\rho_0(\varphi(a))$ and $\rho_0(\varphi(b))$ generate a Zariski dense subgroup of $\mathrm{SL}_n(\mathbb{R})$. In other words, $\rho_0 \circ \varphi$ lies in $\mathrm{Good}(a, b)$, so this set is non empty. □

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