PROPERTY FA IS NOT A PROFINITE PROPERTY

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Abstract. We exhibit infinitely many pairs of non-isomorphic finitely presented, residually finite groups Δ and Γ with Δ having Property FA, Γ having a non-trivial action on a tree and Δ and Γ having isomorphic profinite completions.

Dedicated to Pavel Zalesskii on the occasion of his 60*th birthday*

1. Introduction

Let Γ be a finitely generated residually finite group, the *profinite completion* $\widehat{\Gamma}$ of Γ is the inverse limit of the inverse system consisting of the finite quotients Γ/N and the natural epimorphisms $\Gamma/N \to \Gamma/M$. A property $\mathcal{P}$ for Γ is said to be a *profinite property* if whenever Δ is a finitely generated residually finite group with $\widehat{\Delta} \cong \widehat{\Gamma}$, if Γ has $\mathcal{P}$ then Δ also has $\mathcal{P}$. The abelianization is a profinite property (in particular the first Betti number is), a more subtle profinite property is the first ℓ^2-Betti number [3, Corollary 3.3]. On the other hand in [1], it is shown that Property T is not a profinite property, and it remains an open question of Remeslennikov [12, Question 5.48] as to whether being a free group is a profinite property. As is well-known (see [18]), if a group Γ has Property T then it also has Property FA of Serre; i.e. every action of Γ on a tree has a global fixed point. The main result of this note strengthens that of [1].

Theorem 1.1. *There are infinitely many pairs of non-isomorphic, finitely presented, residually finite groups Δ and Γ satisfying the following:*

(1) $\widehat{\Delta} \cong \widehat{\Gamma}$*;*
(2) Δ *has Property T and hence Property FA;*
(3) Γ *has neither Property T nor Property FA.*

Recall that a finitely generated group G *does not* have Property FA if and only if G is an HNN extension or G admits a decomposition as a non-trivial free product with amalgamation. For a finitely generated group, being an HNN extension is equivalent to having infinite abelianization, which, as noted above, is a profinite property. Hence a corollary of Theorem 1.1 is the following.

Corollary 1.2. *Being a free product with amalgamation is not a profinite property.*

To put Theorem 1.1 and Corollary 1.2 in context we make two remarks:

(1) The aforementioned examples of Aka [1] provide pairs of groups G and H with the same profinite completion with G having Property T, H does not, but we emphasize that both of G and H have Property FA. This is clear for G since as remarked upon above, having Property T implies Property FA [18]. That H has Property FA can be seen as follows: H is an arithmetic lattice in the Lie group $\mathrm{Spin}(n,1) \times \mathrm{Spin}(n,1)$ which has rank 2, and so H and all subgroups of finite index have finite abelianization. Hence H cannot be an HNN extension. That H does not admit a decomposition as a free product with amalgamation

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follows from work of Margulis [8] exploiting the fact that H is an arithmetic lattice in a product of rank one groups none of which is p-adic. Alternative proofs of this are now known; see for example [10], [11] and [17]. In particular the examples of [1] cannot be used to prove Theorem 1.1 and Corollary 1.2.

(2) It was previously known that there exist non-isomorphic groups that are free products with amalgamation having isomorphic profinite completions. For example, in [7, Example 4.11], Grunewald and Zalesskii provide examples of non-isomorphic free products with amalgamation where the groups in question are finite, and in [19, Section 10], examples of non-isomorphic graph manifold groups are given that are free products with amalgamation associated to non-trivial JSJ decompositions of the 3-manifolds. For emphasis, we note that it remains an open question (see [6, Question 3]) as to whether being a free product is or is not a profinite property (as remarked upon above, even for the free group). In connection with this, and in the spirit of this work, in [6], it was shown that a subgroup of a finitely generated virtually free group G is a free factor if and only if its closure in the profinite completion of G is a profinite free factor

The infinitude of examples stated in Theorem 1.1 are S-arithmetic groups that arise by choice of a rational prime. Fixing a prime p, the group Δ of Theorem 1.1 is the S-arithmetic group $\mathrm{SL}(4,\mathbb{Z}[\frac{1}{p}])$. Hence a corollary of independent interest is the following. Recall that a finitely generated residually finite group Γ is *profinitely rigid* if whenever Λ is a finitely generated residually finite group with $\widehat{\Lambda} \cong \widehat{\Gamma}$ then $\Lambda \cong \Gamma$.

Corollary 1.3. $\mathrm{SL}(4,\mathbb{Z}[\frac{1}{p}])$ *is not profinitely rigid.*

It remains open as to whether $\mathrm{SL}(4,\mathbb{Z})$ (or indeed any $\mathrm{SL}(n,\mathbb{Z})$, $n \geq 2$) and also $\mathrm{SL}(n,\mathbb{Z}[\frac{1}{p}])$ for $n = 2, 3$ are profinitely rigid.

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2. The groups

For the remainder of this note, we fix a rational prime p. As noted in §1, the group Δ will be taken to be $\mathrm{SL}(4,\mathbb{Z}[\frac{1}{p}])$, an S-arithmetic lattice in $\mathrm{SL}(4,\mathbb{R}) \times \mathrm{SL}(4,\mathbb{Q}_p)$. We record some facts about Δ.

Lemma 2.1. *The group Δ is finitely presented, has Property T, and hence has Property FA.*

Proof. Since Δ is an S-arithmetic lattice it is finitely presented by [13, Theorem 5.11]. The groups $\mathrm{SL}(4,\mathbb{R})$ and $\mathrm{SL}(4,\mathbb{Q}_p)$ have Property T (see [9, Theorem III.5.6]), and so the product $\mathrm{SL}(4,\mathbb{R}) \times \mathrm{SL}(4,\mathbb{Q}_p)$ also has Property T by [9, Corollary III.2.10]. Since Δ is a lattice in $\mathrm{SL}(4,\mathbb{R}) \times \mathrm{SL}(4,\mathbb{Q}_p)$, it follows that Δ has Property T by [9, Theorem III.2.12]. A theorem of Watatani [18] (see also [9, Theorem III.3.9]), then implies that Δ has Property FA. □

2.1. Congruence completion and the Congruence Subgroup Property.

For convenience we recall the construction of *congruence subgroups* of the group Δ.

Let $n \in \mathbb{Z}$ be a positive integer co-prime to p, then by reducing entries modulo n we have an epimorphism $\phi_n : \Delta = \mathrm{SL}(4,\mathbb{Z}[\frac{1}{p}]) \to \mathrm{SL}(4,\mathbb{Z}/n\mathbb{Z})$ whose kernel $\Delta(n)$ is the *principal congruence subgroup* of level n. A subgroup $H \leq \mathrm{SL}(4,\mathbb{Z}[\frac{1}{p}])$ is called a *congruence subgroup* if there exists an n such that $\Delta(n) \leq H$. It is a result of [2] that Δ has the Congruence Subgroup Property in the sense that *every* subgroup of finite index in Δ contains some principal congruence subgroup.

An alternative way to describe this is as follows. For any prime l different from p, there is an embedding of $\mathbb{Q}$ into $\mathbb{Q}_l$, the l-adic numbers. Using this we obtain an embedding of $\mathrm{SL}(4,\mathbb{Q})$ into the S-adelic group $\prod'_{l\neq p}\mathrm{SL}(4,\mathbb{Q}_l)$, which is the restricted product of the $\mathrm{SL}(4,\mathbb{Q}_l)$ and is given the restricted product topology. The closure of Δ within $\prod'_{l\neq p}\mathrm{SL}(4,\mathbb{Q}_l)$ is the so-called *congruence completion* $\overline{\Delta}$. This is a profinite group which is exactly the inverse limit of the congruence quotients $\mathrm{SL}(4,\mathbb{Z}/n\mathbb{Z})$ of Δ, and in fact by the Strong Approximation Theorem, $\overline{\Delta}\cong\prod_{l\neq p}\mathrm{SL}(4,\mathbb{Z}_l)$, which is a compact, open subgroup of $\prod'_{l\neq p}\mathrm{SL}(4,\mathbb{Q}_l)$. That Δ has the Congruence Subgroup Property is equivalent to $\widehat{\Delta}\cong\overline{\Delta}$.

2.2. **The second group.** Before defining the group Γ itself, we need to define and establish some facts about certain matrix groups over quaternion algebras. We refer the reader to [13, Chapters 1.4 & 2.3] for more details on the material described here.

For K a field and A a quaternion algebra over K, we can define the algebra $\mathrm{M}(2,A)$ of 2×2 matrices over A. Choosing a quadratic field extension F/K which splits A, we can embed A into $\mathrm{M}(2,F)$, and hence $\mathrm{M}(2,A)$ embeds into $\mathrm{M}(4,F)$ as K-algebras. Restricting the determinant from $\mathrm{M}(4,F)$ to $\mathrm{M}(2,A)$ induces the *reduced norm map* $\mathrm{Nrd}:\mathrm{GL}(2,A)\to K^\times$ where $\mathrm{GL}(2,A)$ is the group of invertible elements of $\mathrm{M}(2,A)$. It can be shown that Nrd does not depend on the choice of F nor the embedding of A, and that the image of Nrd lies in $K^\times$ as stated. We now define

$$\mathrm{SL}(2,A)=\{g\in\mathrm{GL}(2,A):\mathrm{Nrd}(g)=1\}.$$

Note that as described in [13, Chapter 2.3.1], if A is a division algebra of quaternions, $\mathrm{SL}(2,A)$ is the group of K-rational points of a simple, simply connected algebraic group defined over K of K-rank 1.

Finally we observe that when A is the split quaternion algebra $\mathrm{M}(2,K)$, the field F could be chosen as K itself. So in this case, $\mathrm{M}(2,A)=\mathrm{M}(4,K)$, $\mathrm{GL}(2,A)=\mathrm{GL}(4,K)$, and Nrd is just the usual determinant of $\mathrm{GL}(4,K)$ so that $\mathrm{SL}(2,A)=\mathrm{SL}(4,K)$.

To construct the group Γ we first specialize the above discussion, and to that end let $A/\mathbb{Q}$ be the definite quaternion algebra ramified at the infinite place and the place associated to our fixed rational prime p (so that A is a division algebra). As noted above, $\mathrm{SL}(2,A)$ is the group of rational points of a simple, simply connected algebraic group over $\mathbb{Q}$, for which the $\mathbb{R}$-points (resp. the $\mathbb{Q}_p$-points) of this algebraic group are $\mathrm{SL}(2,\mathbb{H})$ (resp. $\mathrm{SL}(2,\mathbb{H}_p)$) where $\mathbb{H}$ is the usual Hamilton quaternions and $\mathbb{H}_p$ is the unique division quaternion algebra over $\mathbb{Q}_p$. Since A is unramified at any prime $l\neq p$ we have $A\otimes_{\mathbb{Q}}\mathbb{Q}_l\cong\mathrm{M}(2,\mathbb{Q}_l)$, and so as described above, the $\mathbb{Q}_l$-points can be identified with $\mathrm{SL}(4,\mathbb{Q}_l)$.

We now consider the S-adelic embedding of $\mathrm{SL}(2,A)$ into $\prod'_{l\neq p}\mathrm{SL}(4,\mathbb{Q}_l)$. We will sometimes suppress the embedding and continue to simply refer to $\mathrm{SL}(2,A)$. Since $\mathrm{SL}(2,\mathbb{H})\times\mathrm{SL}(2,\mathbb{H}_p)$ is not compact, by the Strong Approximation Theorem, the image of $\mathrm{SL}(2,A)$ under this embedding is dense [13, Theorem 7.12]. We define the group Γ as:

$$\Gamma=\mathrm{SL}(2,A)\cap\prod_{l\neq p}\mathrm{SL}(4,\mathbb{Z}_l).$$

Hence Γ is an S-arithmetic subgroup of $\mathrm{SL}(2,A)$ and so is finitely presented by [13, Theorem 5.11]. Moreover, Γ is an irreducible lattice in $\mathrm{SL}(2,\mathbb{H})\times\mathrm{SL}(2,\mathbb{H}_p)$. Because $\prod_{l\neq p}\mathrm{SL}(4,\mathbb{Z}_l)$ is an open subgroup of $\prod'_{l\neq p}\mathrm{SL}(4,\mathbb{Q}_l)$, in which $\mathrm{SL}(2,A)$ is dense, Γ is also dense in $\prod_{l\neq p}\mathrm{SL}(4,\mathbb{Z}_l)$. Since Γ is not integral at p, the congruence completion of Γ is its closure in $\prod'_{l\neq p}\mathrm{SL}(4,\mathbb{Q}_l)$. Thus we have for the congruence completion $\overline{\Gamma}=\prod_{l\neq p}\mathrm{SL}(4,\mathbb{Z}_l)$. Finally, since the $\mathbb{Q}$-rank of $\mathrm{SL}(2,A)$ is 1, and the S-rank is 2 with $S=\{\infty,p\}$, a result of Raghunathan [15] (see also [14]) applies to show that the S-arithmetic group Γ also has the Congruence Subgroup Property, so that $\widehat{\Gamma}\cong\overline{\Gamma}$.

3. Completing the proof of Theorem 1.1

It is immediate from §2.1 and §2.2 that $\widehat{\Gamma} \cong \widehat{\Delta}$ since both have the Congruence Subgroup Property and $\overline{\Delta} \cong \overline{\Gamma}$. Thus to complete the proof we must show that Γ does not have Property FA. We now describe how this is done, and to that end briefly recall the construction of a tree on which Γ acts non-trivially (following [16, Chapter II]).

3.1. **The Bruhat-Tits tree X associated to $\mathrm{SL}(2, \mathbb{H}_p)$.** We include a quick description of the tree X associated to $G = \mathrm{SL}(2, \mathbb{H}_p)$. This is similar to the construction of the tree of SL_2 over a local field in [16, Chapter II], and indeed, it is implicit in [16, Chapter II] (see in particular the discussion on [16, p. 74-75]) that the local field can be replaced by the case at hand, the division algebra $\mathbb{H}_p$. We briefly comment on the construction of X following [16, Chapter II].

The quaternion algebra $\mathbb{H}_p$ has a p-adic valuation $\alpha : \mathbb{H}_p^* \to \mathbb{Z}$ obtained by composing the reduced norm on $\mathbb{H}_p$ with the p-adic valuation on $\mathbb{Q}_p$. In particular, since $\mathbb{H}_p$ is a division algebra, there is a uniformizer for $\mathbb{H}_p$; i.e. an element $\pi \in \mathbb{H}_p$ with $\alpha(\pi) = 1$. Furthermore, $\mathbb{H}_p$ has a unique maximal order $\mathcal{O}_p = \{x \in \mathbb{H}_p \,|\, \alpha(x) \geq 0\}$ (where we set $\alpha(0) = \infty$). The tree X is constructed using the lattices in the left $\mathbb{H}_p$-vector space $\mathbb{H}_p^2$, that is the finitely generated left $\mathcal{O}_p$ submodules of $\mathbb{H}_p^2$ which generate all of $\mathbb{H}_p^2$ over $\mathbb{H}_p$. The vertices of the tree X are the $\mathbb{H}_p$-homothety classes of lattices, and two classes of lattices are connected by an edge in X exactly when they can be represented by lattices L and L' with $L' < L$ and $L/L' \cong \mathbb{F}_p$ as $\mathcal{O}_p$-modules. This gives rise to a $(p+1)$-regular tree X.

Lemma 3.1. *The action of Γ on the tree X is non-trivial. In particular Γ does not have Property FA nor Property T.*

Proof. The group Γ is a dense subgroup of $\mathrm{SL}(2, \mathbb{H}_p)$ which inherits the transitive action of $\mathrm{SL}(2, \mathbb{H}_p)$ on the edges of X. Thus, there is no global fixed point for the action of Γ on X and therefore, Γ does not have Property FA. That Γ does not have Property T follows as above using [18]. ◻

Putting everything from §2 and §3 together completes the proof of Theorem 1.1. ◻

4. Final remarks

Inspired by Corollary 1.3, one might be tempted to reproduce this construction to find other S-arithmetic groups which have the same profinite completions as $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ or $\mathrm{SL}(3, \mathbb{Z}[\frac{1}{p}])$. However the analogous construction does not produce the desired results in these cases. For $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$, one would want to consider an S-arithmetic subgroup of A^1, the group of norm 1 elements in the quaternion algebra A which we defined previously. Unfortunately, such an S-arithmetic group would be a lattice in the compact group $\mathbb{H}^1 \times \mathbb{H}_p^1$, and hence would be finite.

In fact, this is the only other case that needs to be considered when searching for an S-arithmetic group with the same profinite completion as $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$. Because $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ has the Congruence Subgroup Property, any S-arithmetic subgroup with the same profinite completion would have to have the same congruence completion as well. If it did not, Strong Approximation could be used to provide a finite quotient that $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ does not have. One can then check that A^1 is the only other potential $\mathbb{Q}$-group whose local behavior is consistent with this congruence completion.

A similar situation occurs in the case of $\mathrm{SL}(3, \mathbb{Z}[\frac{1}{p}])$, which excludes the possibility of a different S-arithmetic group having the same profinite completion. Using the "road map" set out in [4], one can show that to establish whether or not these groups are profinitely rigid actually reduces to answering the following question.

Question 4.1. *If Γ is $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ or $\mathrm{SL}(3, \mathbb{Z}[\frac{1}{p}])$, can Γ contain a finitely generated proper subgroup H whose inclusion induces an isomorphism $\widehat{H} \cong \widehat{\Gamma}$?*

If such an H existed, (Γ, H) would be a Grothendieck pair (see [5], for example). The case of $\Gamma = \mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ is especially provocative because it splits as an amalgam, which would have significant consequences for the structure of the potential subgroup H.

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