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Generalized Hopfian Property, a Minimal Haken Manifold, and Epimorphisms Between 3-Manifold Groups

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Abstract We address the question that if π_1-surjective maps between closed aspherical 3-manifolds have the same rank on π_1 they must be of non-zero degree. The positive answer is proved for Seifert manifolds, which is used in constructing the first known example of minimal Haken manifold. Another motivation is to study epimorphisms of 3-manifold groups via maps of non-zero degree between 3-manifolds. Many examples are given.

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1 Introduction and Some Examples

Let M and N be closed 3-manifolds and $f : M \to N$ a map of non-zero degree. Then the image of f_* is a subgroup of finite index in $\pi_1(N)$. If M and N are aspherical, any homomorphism $\phi : \pi_1(M) \to \pi_1(N)$ determines a unique map $f : M \to N$ up to homotopy such that $f_* = \phi$. It seems natural to ask does there exist $f : M \to N$ of non-zero degree, given a homomorphism ϕ surjecting $\pi_1(M)$ onto a subgroup of finite index in $\pi_1(N)$?

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There are elementary constructions of examples (see below) that show, in general, that the answer is no. Before discussing some examples we make the following definition:

Definition 1.1 *A map $f : M \to N$ between 3-manifolds is π_1-surjective (resp. π_1-finite-index) if $f_* : \pi_1(M) \to \pi_1(N)$ is surjective (resp. the image of f_* is a subgroup of finite index).*

Recall that if M is an n-manifold, the rank of $\pi_1(M)$ (or just M by abuse language) is the minimal cardinality of a generating system for $\pi_1(M)$.

We first consider the situation in dimension 2, which is quite simple. Throughout the paper Σ_k will denote a closed orientable surface of genus k.

Example 1.1 It is not difficult to see that there is a π_1-surjective map $f : \Sigma_l \to \Sigma_k$ which is of degree zero when $l \geq 2k$.

On the other hand, if $f : \Sigma_l \to \Sigma_k$ is a π_1-surjective map with $0 < l < 2k$, then we claim that f is of non-zero degree. To see this, choose a 1-skeleton of Σ_k to be a one point wedge of $2k$ circles $\mathcal{V} = \vee C_i$. Fix a point x_i on C_i. If f is of degree zero, then the image of f can be deformed into $\mathcal{V}$. We assume therefore that this is the case. Since f_* is π_1-surjective, $f : \Sigma_l \to \mathcal{V}$ must be surjective. We may also assume that f is transversal to each x_i, $i = 1, 2, ..., 2k$. So $f^{-1}(\cup x_i)$ is a set of essential circles. Partition $f^{-1}(\cup x_i)$ into sets $G_1, ..., G_h$ such that two components are in the same set if and only if they are parallel. For each G_j, find an annulus A_j containing G_j. Then squeeze each A_j to an arc a_j and the part $\Sigma_k \setminus \cup A_j$ to a point. The quotient Q will be a bouquet of h circles. Since $\mathcal{V} - \{x_i, i = 1, 2, ..., 2k\}$ is contractible, the map $f : \Sigma_l \to \mathcal{V}$ factors through $q : Q \to \mathcal{V}$ which is still π_1-surjective. It follows that $h \geq 2k$. In particular, there are at least $h \geq 2k$ disjoint essential non-separating non-parallel circles. By a well-known argument in surface topology, we must have that l, the genus of Σ_l, is at least $2k$.

Returning to dimension 3, the first example illustrates the aspherical assumption.

Example 1.2 Let $f = e \circ p : S^2 \times S^1 \to S^2 \times S^1$, where p is a map which pinches $S^2 \times S^1$ to S^1, and e identifies S^1 to a fiber $* \times S^1 \subset S^2 \times S^1$. Clearly f is of zero degree but π_1-surjective.

Example 1.3 shows that if we do not require that the manifolds have the same rank, then the answer to the question is no.

Example 1.3 We construct a map $f : \Sigma_{g+1} \times S^1 \to \Sigma_g \times S^1$ of zero degree which is π_1-surjective. The map f is the composition of the following four geometric operations:

Project $\Sigma_{g+1} \times S^1$ to Σ_{g+1};

Squeeze a suitable separating circle on Σ_{g+1} to a point in such a way that the quotient space is a one point union of a torus and Σ_g;

Squeeze the torus to a circle in such a way that the quotient space is a one point union of the circle and Σ_g;

Send Σ_g and the circle to a section $\Sigma_g \times *$ and the circle fiber of $\Sigma_g \times S^1$ respectively.

Example 1.4 has the same purpose as Example 1.3, but the manifolds in this case are hyperbolic.

Example 1.4 Let M be a closed hyperbolic 3-manifold whose fundamental group surjects the free group of rank 2. Such examples are easily constructed by doing hyperbolic surgery on a null-homotopic hyperbolic knot in $\Sigma_2 \times S^1$ [Section 3, 1]. Let $\phi_1 : \pi_1(M) \to \mathbb{Z} * \mathbb{Z}$ denote such a map. Let N be any hyperbolic 3-manifold such that $\pi_1(N)$ has two generators. Then there is an epimorphism $\phi_2 : \mathbb{Z} * \mathbb{Z} \to \pi_1(N)$. If we choose N such that the volume of N is larger than the volume of M, then the map realizing the epimorphism $\phi = \phi_2 \circ \phi_1$ must be of zero degree, by the work of Gromov and Thurston. We remark that the volumes of hyperbolic 3-manifolds of rank 2 are unbounded. Briefly, it follows from the work of Adams that the volumes of hyperbolic 2-bridge knot complements are unbounded. Doing large enough hyperbolic Dehn surgeries on these gives the required family, see [2].

In fact it can be seen directly that the map realizing ϕ must be of zero degree since such a map factors through a 1-dimensional complex.

As a consequence of these examples, we state the following more refined version of the question posed above:

Question 1.5 Let M and N be closed aspherical 3-manifolds such that the rank of $\pi_1(M)$ equals the rank of $\pi_1(N)$. Assume that $\phi : \pi_1(M) \to \pi_1(N)$ is surjective or its image is a subgroup of finite index. Does ϕ determine a map $f : M \to N$ of non-zero degree?

Note that if M and N are homeomorphic and satisfy Thurston's geometrization conjecture, then a π_1-surjective map $f : M \to M$ must be of degree one. For since $\pi_1(M)$ is Hopfian, that f_* is surjective implies f_* is an isomorphism. Since M is aspherical f must be a homotopy equivalence, and so, in particular, f is of degree one. Thus the question above is a kind of generalization of the Hopfian property: the condition "homeomorphic manifolds" is replaced by "manifolds of the same rank", the condition "π_1-surjective" is replaced by "π_1-surjective" or "π_1-finite-index", and in conclusion replace "degree one" by "non-zero degree". It is easy to construct examples to show that "non-zero degree" cannot be sharpened to "degree one", see the examples in Section 3.

One of the main results of this paper is to prove that Question 1.5 has a positive answer for Seifert fibered 3-manifolds (see Theorem 2.1 and Remark 2.4).

In Section 4 we use this result to construct the first known example of a Haken 3-manifold which is minimal with respect to degree 1 mappings in Thurston's picture of 3-manifolds (Theorem 4.1). The manifold is a graph manifold built from the union of two trefoil knot complements. An orientable 3-manifold M is minimal if given a degree-one map $f : M \to N$ implies either $N = S^3$ or $M = N$. Usually it is difficult to tell if a 3-manifold is minimal. We remark that all minimal Seifert manifolds are non-Haken and that the known minimal hyperbolic 3-manifolds are also non-Haken, see [1], [3] and [4] for a further discussion of such matters.

We are also motivated by the following:

[5, Problem 1.12 (J. Simon)] Let $G_K = \pi_1(S^3 - K)$ for a knot K in S^3. Conjectures: If there is a non-trivial epimorphism $\phi : G_K \to G_L$, then

(A) rank $G_L >$ rank G_K;

(B) genus$(L) \geq$ genus(K);

(C) Given K, there is a number N_K such that any sequence of epimorphisms of knot groups $G_K \to G_{L_1} \to .... \to G_{L_n}$ with $n \geq N_K$ contains an isomorphism;

(D) Given K, there are only finitely many knot groups G for which there is an epimorphism $G_K \to G$.

These conjectures have seen little progress. On the other hand, more recently, questions similar to (C) and (D) have been raised for degree-one maps and there are already several substantial results in this setting.

[5, Problem 3.100 (Y. Rong)] Let M be a closed orientable 3-manifold.

(A) Are there only finitely many irreducible 3-manifolds N such that there exists a degree-one map $M \to N$?

(B) Does there exist an integer N_M such that given a sequence of degree-one maps $M \to M_1 \to .... \to M_k$ with $k > N_{M_0}$, the sequence contains an homotopy equivalence?

If one assumes Thurston's geometrization conjectural picture of 3-manifolds, the answer to (B) is yes if $k = \infty$ [6] by Rong, or each M_i is Haken [7] by Soma; the answer to (A) is yes if the targets have one of Thurston's eight geometries: for the hyperbolic case [8] by Soma, for the spherical case [9] by Hayat-Legrand-Wang-Zieschang and the remaining cases [10] by Wang-Zhou.

Thus it seems natural to study the conjectures of J. Simon for closed orientable 3-manifolds (Question 3.1 in Section 3). We find that the positive answer for Question 1.5 is important for studying the conjectures. This will be addressed in Section 3.

2 π_1-Surjective Maps Between Aspherical Seifert Manifolds

Theorem 2.1 *Let M_1 and M_2 be closed orientable aspherical Seifert fiber spaces with the same rank their base orbifolds be orientable. Then any π_1-surjective map $f : M_1 \to M_2$ is of non-zero degree.*

To prove Theorem 2.1, we will make use of Rong [11]; in particular, we refer the reader to [11] for the definition of a *vertical pinch*, a *squeeze*, a *squeeze torus*, and *vertical squeeze*. Suppose T is a vertical torus in a Seifert manifold M with induced S^1 fibration. Let C be a circle on T which meets each S^1 fiber in exactly one point. Then T can be parameterized as $C \times S^1$. Call the quotient map $C \times S^1 \to S^1$ by squeezing $C \times *$ to $*$ for each $* \in S^1$ a *standard squeeze.*

Also recall that any orientable Seifert manifold M with orientable base orbifold of genus g and with n singular fibers has a unique normal form $(g; b; \alpha_1, \beta_1; ...; \alpha_k, \beta_k)$, where $0 \leq \beta_i \leq \alpha_i$, $i = 1, ..., k$. The orbifold O_1 of M_1 will be denoted by $(g; \alpha_1, ..., \alpha_k)$. In the case when $g = 0$, we usually omit the reference to g.

In preparation for the proof of Theorem 2.1, we begin with two lemmas.

Lemma 2.2 *Let $f : M \to N$ be a map between aspherical Seifert manifolds and $1 \neq f_*(h) \subset h'$, where M is closed and $\partial N \neq \emptyset$, h and h' are regular fibers of M and N, respectively. Then either f admits a standard squeeze along an incompressible vertical torus, or f can be homotoped so that the image of f lies in a fiber of N.*

Proof Lemma 2.2 is exactly Lemma 3.5 of [11] but with "standard squeeze" being replaced by "vertical squeeze". However, one can verify that in Case (2) of the proof of Lemma 3.5 of [11], the vertical squeeze there is indeed a standard squeeze. Since for a fiber preserving map p from an S^1-fibered torus T to an S^1 fibered annulus A, the kernel of p and the S^1-fiber generate the $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$, therefore the circle on T generating the kernel must intersect the fiber in exactly one point.

Lemma 2.3 *Suppose $f : F \to O$ is an orbifold branch covering, where F is a surface of genus g, and O is a orbifold, both being orientable and having non-positive Euler characteristics. Then* $\mathrm{rank}(\pi_1(F)) \geq \mathrm{rank}(\pi_1(O)) - 1$ *if f is a double-branched cover over* 2*-sphere, and* $\mathrm{rank}(\pi_1(F)) \geq \mathrm{rank}(\pi_1(O))$ *if otherwise.*

Proof The proof is based on the results about the ranks of Fuchsian groups [12, Theorem 4.16.1] and the Riemann-Hurwitz formula.

Suppose O has k singular points of index v_i, $i = 1, ..., k$, with the underlying space of genus g' and the degree of f is n. Then we have

$$2 - 2g = n\left(2 - 2g' - \sum_{i=1}^{k}\left(1 - \frac{1}{v_i}\right)\right).$$

For the case $g = 1$, the verification is direct, so we assume below that $g > 1$.

If $n = 2$ then all $v_i = 2$, $k = 2m$ and we have $2 - 2g = 2(2 - 2g' - m)$, i.e., $g = 2g' + m - 1$. Now $\mathrm{rank}(\pi_1(F)) = 2g = 4g' + 2m - 2$ and the $\mathrm{rank}(\pi_1(O))$ is at most $2g' + 2m - 1$ if $g' > 0$ and is $2m - 1$ if $g' = 0$ by [12, Theorem 4.16.1]. In any case, the lemma follows.

If $n \geq 3$, then

$$2 - 2g \leq 3\left(2 - 2g' - \sum_{i=1}^{k}\left(1 - \frac{1}{v_i}\right)\right) \leq 3\left(2 - 2g' - \frac{k}{2}\right),$$

i.e., $2g \geq 6g' - 4 + \frac{3k}{2}$. If $g' > 0$, $2g \geq 2g' + \frac{3k}{2}$. But the rank of $\pi_1(O)$ is, at most, $2g' + k - 1$. If $g' = 0$, then we have $2g \geq -4 + \frac{3k}{2}$ if k is even and $g \geq -4 + \frac{3k}{2} + \frac{1}{2}$ if k is odd. The rank of $\pi_1(O)$ is, at most, $k - 1$. It follows that if $k \leq 5$, then $2g \geq k - 1$. If $k \leq 4$ we still have $2g \geq k - 1$ since we assume that $g > 1$.

Proof of Theorem 2.1 Suppose f is of zero degree. For clarity, the proof is divided into three steps.

Step (1) We prove the following

Claim: *$f(h)$ is homotopically non-trivial, where h is the regular fiber of M_1.*

Proof of Claim Let $M_1 = (g; b; a_1, b_1; ...; a_k, b_k)$ and $G_1 = \pi_1(M_1)/\langle h\rangle$, where $\langle h\rangle$ is the cyclic group generated by the regular fiber of M_1. Let $r = \mathrm{rank}(\pi_1(M_1)) = \mathrm{rank}(\pi_2(M_2))$.

By [13, Theorem 1.1] and [12, Theorem 4.16.1], one of the following cases holds:

(1) $\mathrm{rank}(\pi_1(M_1)) > \mathrm{rank}(G_1)$;

(2) $\mathrm{rank}(\pi_1(M_1)) = \mathrm{rank}(G_1)$; there is a set of generators of G_1 which realizes the rank and contains at least one torsion element;

(3) $\mathrm{rank}(\pi_1(M_1)) = \mathrm{rank}(G_1) = \mathrm{rank}(G_1/T)$, where T is the normal subgroup normally generated by the torsion elements and G_1/T is a surface group.

If $f(h)$ is homotopically trivial, then $f_* : \pi_1(M_1) \to \pi_1(M_2)$ induces an epimorphism $\phi : G_1 \to \pi_1(M_2)$.

In Case (1), the Claim is clearly true.

In Case (2), the Claim is also true since $\pi_1(M_2)$ is torsion free.

In Case (3), f_* induces an epimorphism $\phi' : G_1/T \to \pi_1(M_2)$. Let $f' : F \to M_2$ be the map which realizes ϕ'. Since ϕ' is not injective, (otherwise ϕ' would be an isomorphism and $\pi_1(M_2)$ would be a surface group), by the simple loop theorem for maps from a surface to a Seifert manifold [14], there are essential simple loops in the kernel of ϕ'. Assume first there is an essential non-separating simple loop, which we denote by α, in the kernel. Then the map f' induces a map $f'' : F' \to M_2$, where F' is a complex obtained by squeezing F along α. It is easy to see that the rank of $\pi_1(F')$ is $r-1$. We reach a contradiction. If all essential simple loops in the kernel of ϕ' are separating, let α be a maximal family of non-parallel separating essential simple closed curves in kernel ϕ'. Again f' can factor through $f'' : F' \to M_2$, where F' is a complex obtained by squeezing F along α, which is a union of closed surfaces connected by arcs. Let S be a surface in F'. Due to the maximality of α, the restriction $f''|_S$ is π_1-injective, which must be either horizontal or vertical by [14]. If $f''|_S$ is horizontal, then $p_2 \circ f''| : S \to O(M_2)$ is an orbifold branched covering, where $p_2 : M_2 \to O(M_2)$ is the fiber map. But the rank of $\pi_1(S)$ is at most $r-2$. This is also ruled out by Lemma 2.3. If $f''|_S$ is vertical for each surface S of F', then F' contains, at most, g such surfaces and each of them is a torus. Clearly the rank of $f''_*\pi_1(F')$ is at most $g+1$, which is at most $r-1$ (since $g > 1$ and $r \geq 2g$). Again we reach a contradiction.

Step (2) We will factor $f : M_1 \to X \to M_2$, where the 2-dimensional complex X is a quotient of M with rank r_X.

Since $f(h)$ is homotopically non-trivial and f_* is surjective, a standard argument in 3-manifold topology shows that $f : M_1 \to M_2$ can be deformed to be a fiber-preserving map (see [15] for example). Suppose the mapping degree is zero. We can further deform the map so that the image $f(M_1)$ misses a regular fiber h' of M_2. To see this, $f : M_1 \to M_2$ is fiber preserving. We can further deform f so that for each singular fiber of M_2, its pre-image consists of finitely many fibers of M_1. Let S_i be the union of singular fibers of M_1. Now remove $f^{-1}(f(S_1) \cup S_2)$ from M_1 and remove $f(S_1) \cup S_2$ from M_2. The restriction of f gives a proper map $f' : M_1' \to M_2'$, which is a fiber-preserving map between circle bundles. Since f is assumed to be of degree zero, f' is of zero degree. Since $f(h)$ is non-trivial, the induced proper map $\bar{f}' : F_1' \to F_2'$ between base surfaces must be of degree zero. Hence $\bar{f}'$ can be deformed so that its image misses a point of F_2'. This deformation can be lifted to the bundle map f' whose image then misses a circle fiber in M_2'. With this we reach the situation claimed above.

Now remove an open-fibered neighborhood of h', and denote the resulting manifold by N. Then we have a fiber-preserving map $f : M_1 \to N$, where $\partial N \neq \emptyset$.

According to Lemma 2.2, either $f : M_1 \to N$ admits a standard squeeze along an incompressible vertical torus, or $f(M_1) \subset$ a fiber of N. Using this we can reformulate the above so that either $f : M_1 \to M_2$ admits a standard squeeze along an incompressible vertical torus, or $f(M_1) \subset$ a fiber of M_2.

Since f is π_1-surjective, and M_2 is an closed aspherical Seifert fiber space, the situation, $f(M_1) \subset$ a fiber of M_2, cannot happen. Let $\mathcal{T}$ be a maximal family of disjoint non-parallel incompressible tori along which f admits a standard squeeze. Let $X_1 = \mathcal{Q} \cup \mathcal{A}$ be the space obtained after the squeezing, where $\mathcal{Q}$ is a union of Seifert fiber spaces with the induced Seifert fibration, $\mathcal{A}$ is a union of annuli and $\partial\mathcal{A}$ is a union of regular fibers of $\mathcal{Q}$ (due to a standard squeeze). Then f induces a π_1-surjective map $X_1 \to M_2$, which we continue to denote by f.

Suppose first $g > 0$. Then f admits a standard squeeze along a non-separating torus (Indeed by arguments before, we can assume that f is fiber preserving and the image of f misses a regular fiber of M_2. Then f induces a map $\bar{f}$ from a closed surface Σ_g to a punctured surface, and it is known that $\bar{f}$ admits a squeeze along a non-separating circle C on Σ_g, which will provide a non-separating squeeze torus of f, and then the squeeze can be chosen to be standard by Lemma 2.2). Moreover, if X_1 is obtained from M by a standard squeeze along a non-separating torus, then the rank of $\pi_1(X_1)$ is $r-1$, which will be a contradiction.

Below we assume that $g = 0$. Then every squeeze torus is a separating torus. And therefore, each annulus in $\mathcal{A}$ is separating.

Now all components of $\mathcal{Q}$ are Seifert fibered spaces with the induced Seifert fibrations, so we may assume that $Q_1, ..., Q_{k_1}$ are Seifert manifolds of $\mathcal{Q}$ which are not the trivial circle bundle over S^2 and $Q_{k_1+1}, ..., Q_{k_1+k_2}$ are trivial circle bundles over S^2. Clearly each Q_j, $j > k_1$, is $S^2 \times S^1$ and which is connected by at least three annuli in $\mathcal{A}$.

Each component Q_i of $\mathcal{Q}$ must have an infinite fundamental group, otherwise $f(h)$ is an element of finite order, which must be trivial in $\pi_1(M_2)$, and this is forbidden by Step (1). In particular, we have

Fact 1 Each Q_i contains at least 2 singular fibers, $i = 1, ..., k_1$.

We will also verify the following

Fact 2 $k_1 + k_2 \leq k - 2$.

Representing each component of $\mathcal{Q}$ by a vertex and each component of $\mathcal{A}$ by an edge, we get a connected tree G of $v = k_1 + k_2$ vertices, and at least k_2 vertices have valence at least 3. Then the number of edges e is at least $\frac{1}{2}(k_1 + 3k_2)$. By the Euler characteristic formula we have $1 = v - e \leq k_1 + k_2 - \frac{1}{2}(k_1 + 3k_2) = \frac{k_1}{2} - \frac{k_3}{2}$, i.e., $k_2 \leq k_1 - 2$. By Fact 1, we have $k_1 \leq k/2$, therefore $k_1 + k_2 \leq k - 2$.

By the maximality of $\mathcal{T}$, each Q_i contains no squeeze torus for $f|_{Q_i}$, so we have that $f(Q_i) \subset$

a fiber of M_2, and consequently we have the following

Fact 3 Each Q_i has base orbifold S^2 and has no more than 3 singular fibers (otherwise, there will be a squeeze torus).

So $f : X_1 \to M_2$ induces a π_1-surjective map $X = \mathcal{S} \cup \mathcal{A}_1 \to M_2$, where $\mathcal{S}$ is a union of $v = k_1 + k_2$ circles, which are amalgamated by e annuli. Hence $\pi_1(X)$ has a presentation of $k_1 + k_2$ generators with e relations. Therefore its rank r_X is at most $k_1 + k_2$.

Step (3) We will show that $r_X < r$ and then reach a contradiction.

Say that M_1 is of type I, if M_1 has normal form $(0; b; 2, 1; ....; 2, 1; 2\lambda + 1, b_k)$, where $k \geq 4$ is even, and $\lambda > 1$, otherwise call M_1 of type II. By [13, Theorem 1.1], then $r = k - 2$ if M_1 is of type I and $r = k - 1$ if M_1 is of type II.

Moreover, if M_1 is of type I, then at least one Q_i contains 4 singular fibers (since $\lambda > 1$ and both $(0; b; 2, 1; 2\lambda + 1, b_2)$ and $(0; b; 2, 1; 2, 1; 2\lambda + 1, b_3)$ have finite fundamental groups), which is not possible by Fact 3.

If M_1 is of type II, then $r = k - 1$ and $k \geq 3$, but

$$r_X \leq k_1 + k_2 \leq k - 2 < k - 1 = r.$$

Corollary 2.4 *Suppose M_1 and M_2 are closed orientable Seifert fiber spaces with infinite fundamental groups and the orbifold of M_1 has the underlying space S^2 and* $\mathrm{rank}(\pi_1(M_1)) < 2\,\mathrm{rank}(\pi_1(M_2))$. *If a map $f : M_1 \to M_2$ is π_1-surjective, then f is of non-zero degree.*

Proof Now $\pi_1(O_1)$ is generated by torsion elements. So Step 1 in the proof of Theorem 2.1 is passed.

Let r, k, r_X be defined as in the proof of Theorem 2.1, and r_2 be the rank of $\pi_1(M_2)$. If $f : M_1 \to M_2$ is π_1-surjective map and is of zero degree, then $r + 1 = k \geq 2r_X \geq 2r_2$ by the end of the proof of Theorem 2.1.

Remark In Theorem 2.1, the condition "f is π_1-surjective" can be replaced by "f is π_1-finite-index", and the condition "orbifolds are orientable" can be removed. For details see [16], where the proof is parallel to the proof above, but involves a more complicated case by case argument.

Also the proof in [16] needs the following

Proposition 2.5 *Let M_1 and M_2 be closed orientable aspherical Seifert fiber spaces with orientable base orbifolds of the same genus g. Suppose there is a π_1-surjective map $f : M_1 \to M_2$, then f is of non-zero degree.*

Proof We may assume that the map f has been deformed to be a fiber-preserving map.

(a) Special Case. We first prove that when both M_1 and M_2 are circle bundles, and $f : M_1 \to M_2$ is π_1-surjective or of π_1-finite-index, f is of non-zero degree.

If f is of π_1-finite index, then we have a π-surjective lift $\tilde{f} : M_1 \to \tilde{M}_2$ of f. By the rank consideration, the genus of the orbifold of $\tilde{M}_2$ must be also g. So below we assume that f is π_1-surjective.

Let t_1 and t_2 be circle fibers of M_1 and M_2, respectively. The restriction $f|$ on the fiber t_1 is not null-homotopy. Otherwise we have an epimorphism $\phi = \hat{f}_* \circ p : \pi_1(F) = \pi_1(M_1)/\langle t_1\rangle \to \pi_1(M_2) \to \pi_1(M_2)/\langle t_2\rangle\pi_1(F)$, where $\hat{f} : F \to M_2$ is the induced map. It follows that the genus of F_1 is at least the genus of F_2. By the Hopfian property of surface group, ϕ must be an isomorphism. It follows that $\bar{f}_* : \pi_1(F) \to \pi_1(M_2)$ is an isomorphism. This is impossible since M_2 is a closed 3-manifold. It follows that indeed $f| : t_1 \to t_2$ must be of degree 1. Then $f : M_1 \to M_2$ induces a map $\bar{f} : F \to F$ which is also π_1-surjective. $\bar{f}$ must be of degree 1. A conclusion is that f itself must be of degree 1.

(b) General Case. We equip each regular fiber with the structure of a unit circle in the complex plane and the length of each singular fiber of type (α, β) is $\frac{2\pi}{\alpha}$. Furthermore, we assume that the map f has been deformed so that the restriction $f| : C_1 \to C_2$ on each regular fiber is given by $e^{i\zeta} \mapsto e^{ik\zeta}$, where the constant k is the mapping degree on regular fibers, and on the regular neighborhood of singular fiber is the "linear extension". Suppose the singular fibers of M_1 are of type $(\alpha_1, \beta_1; ...; \alpha_m, \beta_m)$ and the singular fibers of M_2 are of type $(\alpha'_1, \beta'_1; ...; \alpha'_n, \beta'_n)$. Let $l = \prod \alpha_i \alpha'_j$. Let $\overline{M}_i = M_i/\mathbb{Z}_l$, where the cyclic group $\mathbb{Z}_l$ acts on M_i such that (1) $\mathbb{Z}_l$ keeps each fiber invariant, (2) on each regular fiber, the action is given by $e^{i\zeta} \mapsto e^{i\zeta + \frac{2\pi}{l}}$. Then one can verify (as we did in Example 3) that (1) both $\overline{M}_1$ and $\overline{M}_2$ are circle bundles over F, and (2) $f : M \to N$ induces a fiber-preserving map, $\bar{f} : \overline{M} \to \overline{N}$, and the following diagram is commutative:

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ p_1 \downarrow & & \downarrow p_2 \\ \overline{M}_1 & \xrightarrow{\bar{f}} & \overline{M}_2. \end{array}$$

Since both p_1, p_2 and f are of π_1-finite index, $\bar{f}$ is of π-finite index. By the special case we considered, $\bar{f}$ is of non-zero degree. So f is of non-zero degree.

3 On Epimorphisms Between 3-Manifold Groups

In this section we study the following questions:

Question 3.1 Let M_i be closed orientable aspherical 3-manifolds. Suppose there is an non-trivial epimorphism $\phi : \pi_1(M_1) \to \pi_1(M_0)$.

(A) Is rank $\pi_1(M_1) >$ rank $\pi_1(M_0)$?

(B) Is the Heegaard genus of $M_1 \geq$ Heegaard genus M_0?

Moreover, given M_0:

(C1) Is there a number N_M such that any sequence of epimorphisms $\pi_1(M_0) \to \pi_1(M_1) \to .... \to \pi_1(M_n)$ with $n \geq N_M$ contains an isomorphism?

(C2) Does any infinite sequence of epimorphisms $\pi_1(M_0) \to \pi_1(M_1) \to .... \to \pi_1(M_n) \to ...$ contain an isomorphism?

(D) Are there only finitely many M_i with the same first Betti number, or the same π_1-rank, as that of M_0, for which there is an epimorphism $\pi_1(M_0) \to \pi_1(M_i)$?

We remark that a positive answer for (B) of Question 3.1 implies a positive solution to the Poincaré Conjecture. From Example 1.4 of the Introduction, the answer to (D) is negative if we remove the condition on the first Betti number or π_1-rank on (D) of Question 3.1.

We describe, first, some examples of non-trivial π_1-surjective maps between two 3-manifolds of the same rank, which give a negative answer to (A) of Question 3.1. Clearly those examples are all of non-zero degrees.

Example 3.1. Let M be a Seifert manifold of normal form $(0;0;6,b_1;5,b_2;7,b_3)$. Let $\mathbb{Z}_2$ be a cyclic group acting on M such that it induces the identity on the base space and standard rotation on each regular fiber. Then one verifies that $M/\mathbb{Z}_2$ is a Seifert manifold of normal form $(0;0;3,b_1;5,2b_2;7,2b_3)$. Now

$$\pi_1(M) = \langle s_1, s_2, s_3, h \mid [s_j, h], s_1^6 h^{b_1}, s_2^5 h^{b_2}, s_3^7 h^{b_3}, s_1 s_2 s_3 \rangle$$

and

$$\pi_1(M/\mathbb{Z}_2) = \langle t_1, t_2, t_3, h' \mid [t_j, h'], t_1^3 {h'}^{b_1}, t_2^5 {h'}^{2b_2}, t_3^7 {h'}^{2b_3}, t_1 t_2 t_3 \rangle.$$

The quotient map $p : M \to M/\mathbb{Z}_2$ is a branched covering of degree 2 and p_* sends $s_j \mapsto t_j$ and $h \mapsto {h'}^2$. Since $(2, b_1) = 1$, p_* is surjective. By [13] these manifolds have rank 2.

Examples 3.2. We now give some examples of π_1-surjective non-zero degree maps between hyperbolic manifolds of the same π_1 ranks.

Let M be a closed orientable 3-manifold and $k \subset M$ be any hyperbolic fibered knot. Suppose the fiber F has genus g. Let M_n be the n-fold cyclic branched cover of M over the knot k. Then the rank of $\pi_1(M_n)$ is bounded by $2g+1$ for all n and M_n is hyperbolic when n is large. If $k|n$, then $M_n \to M_k$ is a branched cover, which is π_1-surjective. So there must be infinitely many π_1-surjective branched coverings $M_n \to M_k$ between hyperbolic 3-manifolds of the same ranks.

A well-studied case is when M_n is the n-fold cyclic branched cover of the figure-eight knot. Then for $n \geq 3$ the fundamental groups are all 2-generator, in fact, they are the Fibonacci groups $F(2, 2n)$ (see [17] for example), which are all hyperbolic if $n \geq 4$. By abelianizing $F(2, 2n)$ we see that all M_n have their first Betti number zero (see [17] for example).

Example 3.3 (1) Let $M(n,k) = (0;0;2^k 3, b_1; 5, 2^{n-k} b_2; 7, 2^{n-k} b_3)$. Similarly to Example 3.1, we have a sequence of degree 2 branched coverings $M(n,n) \to ... \to M(n,1) \to M(n,0)$ of length $n+1$, which induces a sequence of epimorphisms of groups $\pi_1(M(n,n)) \to ... \to \pi_1(M(n,1)) \to \pi_1(M(n,0))$ of rank 2. Let M be $\Sigma_2 \times S^1$. Clearly $\pi_1(M)$ surjects onto $\mathbb{Z} * \mathbb{Z}$, then we have the sequence of epimorphisms

$$\pi_1(M) \to \pi_1(M(n,n)) \to ... \to \pi_1(M(n,1)) \to \pi_1(M(n,0))$$

of length $n+2$, where n can be arbitrarily large.

Moreover, suppose we choose b_1, b_2, b_3 such that the Euler number of $M(n,n)$ is non-zero. Since each $M(N,k)$ has infinite π_1 and is the image of $M(n,n)$ under non-zero degree map, the Euler number of $M(n,k)$ is non-zero [18, Theorem 2]. It follows that $M(n,k)$ has neither a horizontal nor a vertical incompressible surface, and therefore all $M(n,k)$ are non-Haken [15].

(2) Let M_n be the n-fold cyclic branched covering of S^3 over figure-eight knot as at the end of Example 3.2. Then we have a sequence of branched coverings of hyperbolic rational homology spheres $M_{4k} \to ... \to M_8 \to M_4$ of length l which induces a sequence of epimorphisms of groups $\pi_1(M_{4k}) \to ... \to \pi_1(M_8) \to \pi_1(M_4)$ with rank 2. Let M be a hyperbolic 3-manifold with $\pi_1(M)$ surjecting $\mathbb{Z} * \mathbb{Z}$ (as in Example 1.4). Then we have the sequence of epimorphisms

$$\pi_1(M) \to \pi_1(M_{4k}) \to ... \to \pi_1(M_8) \to \pi_1(M_4)$$

of length $l+1$, where l can be arbitrarily large.

The next result gives a partial positive answer to (C2) of Question 3.1.

Theorem 3.4 *Given M_0, and a sequence M_i of closed orientable aspherical Seifert manifolds with epimorphisms $\pi_1(M_0) \to \pi_1(M_1) \to .... \to \pi_1(M_n) \to ...$, this sequence contains an isomorphism.*

Proof By passing to an infinite subsequence, we may assume that all groups in the sequence have the same rank (each epimorphism in the subsequence is the composition of epimorphisms involved). Then each epimorphism $\phi_i : \pi_1(M_i) \to \pi_1(M_{i+1})$ in the sequence can be realized by a map $f_i : M_i \to M_{i+1}$ of non-zero degree by Theorem 2.1. Moreover, the Seifert fibrations of the M_i's can be arranged so that each f_i is fiber-preserving. Let O_i be the orbifold of M_i. Then $\chi(O_i) \leq 0$ and we have the induced sequence of epimorphisms

$$\pi_1(O_0) \to \pi_1(O_1) \to .... \to \pi_1(O_n) \to ...$$

of Fuchsian groups. We therefore have a decreasing sequence

$$-\chi(M_0) \geq -\chi(M_1) \geq .... - \chi(M_n) \geq \dots.$$

The $\{-\chi(O)\}$ forms a well-ordered subset of reals, where O runs over compact 2-orbifolds, $\chi(O_k) = \chi(O_{k+1})$ for k larger than a given N (see [6, Lemmas 2.5 and 2.6] for details). Since there are at most finitely many orbifolds O with given χ, by passing to a subsequence, we may assume that all O_i are the same.

Let $O_i = (g; \alpha_{1,}, ..., \alpha_n)$. Then $M_i = (g; b_i; \alpha_{1,}, \beta_{1,i}; ...; \alpha_{n,}, \beta_{n,i})$.

Since $0 < \beta_{l,i} < \alpha_l$ for $l = 1, ...n$, by passing to a further subsequence, we may assume that $\beta_{l,i} = \beta_l$, and finally we get $M_i = (g; b_i; \alpha_1, \beta_1; ...; \alpha_n, \beta_n)$. Moreover we may assume that all $b_i \neq 0$. Note that by [3, p. 680], all M_i have the same first Betti number and the torsion part of $H_1(M_i, \mathbb{Z})$ is unbounded if b_i unbounded. Since epimorphisms on π_1 induce epimorphisms on the first homology groups, it follows that b_i's are bounded. Now we have $b_i = b_j$ for some i, j, then $M_i = M_j$ and by the Hopfian property of Seifert manifold groups, the epimorphism $\pi_1(M_i) \to \pi_1(M_j)$ is an isomorphism. Then in the sequence above there must be an isomorphism. Theorem 3.4 follows.

We have seen that Theorem 2.1 plays an important role in the proof of Theorem 3.4. If the answer to Question 1.5 is also yes for hyperbolic 3-manifolds, this will lead to a positive answer to (C2) and (D) for hyperbolic 3-manifolds.

Proposition 3.5 *Suppose Question* 1.5 *has a positive answer for hyperbolic* 3*-manifolds. Then for a given closed orientable hyperbolic* 3*-manifold* M_0*:*

(1) *Any infinite sequence of epimorphisms* $\pi_1(M_0) \rightarrow \pi_1(M_1) \rightarrow .... \rightarrow \pi_1(M_n) \rightarrow ...$ *contains an isomorphism, where all* M_i *are closed orientable hyperbolic* 3*-manifolds.*

(2) *There are only finitely many closed orientable hyperbolic* 3*-manifolds* M_i *with the same* π_1*-rank as that of* M_0*, for which there is an epimorphism* $\pi_1(M_0) \rightarrow \pi_1(M_i)$*.*

Proof (1) By passing to an infinite subsequence we may assume all $\pi_1(M_i)$ have the same rank. Since we assume that Question 1.5 has a positive answer for hyperbolic 3-manifolds, this sequence is realized by a sequence of non-zero degree maps $M_0 \rightarrow M_1 \rightarrow ... \rightarrow M_n \rightarrow \dots$.

The rest of the proof is now standard. Since all maps $f_i : M_i \rightarrow M_{i+1}$ in the sequence are of non-zero degree, by Gromov's Theorem [19, Chapter 6], $v(M_i) \geq v(M_{i+1})$, where $v(M_i)$ is the hyperbolic volume of M_i. By Thurston-Jøgenson's Theorem [19, Chapter 6], $v(M_k)$ must be a constant when k is larger than a given integer N. Then by Gromov-Thurston's Theorem [19, Chapter 6], f_k is homotopic to a homeomorphism, $k > N$, so f_{k*} is an isomorphism.

For (2), since we again assume that Question of 1.5 has a positive answer for hyperbolic 3-manifolds, each $\phi_i : \pi_1(M_0) \rightarrow \pi_i(M_i)$ can be realized by a map of non-zero degree. By Soma's theorem [20], there are only finitely many such M_i.

Remark Without the assumption that Question 1.5 has positive answer for hyperbolic 3-manifold, Proposition 3.5 (1) has been proved in [21]. Indeed, after discussing with D. Cooper, we tend to believe that Question 1.5 should have a negative answer for hyperbolic 3-manifolds, but we do not have complete argument yet.

We also note the following partial positive answer to (D) of Question 3.1 follows easily from the methods of Reid and Wang [4]:

Theorem 3.6 *Suppose* M *is a non-Haken hyperbolic* 3*-manifold. Then there are only finitely many closed orientable hyperbolic* 3*-manifolds* M_i *for which there is an epimorphism* $\pi_1(M) \rightarrow \pi_1(M_i)$*.*

Proof As in [4] we use the character variety of M to control possible target groups. Briefly if there are an infinite number of M_i as in the statement of the theorem, we can produce a curve of characters C in the character variety of M. By degenerating to a point at infinity of C we getting a splitting of the group and hence an embedded incompressible surface in M (see [22] and [4] for more on this). This is a contradiction.

4 A Minimal Haken Manifold

Let E be the exterior of a trefoil knot with m the meridian and l the longitude. E has a unique

Seifert fibration with two singular fibers of indices 2 and 3, over the disc. Via this Seifert structure, we have a presentation

$$\pi_1(E) = \langle a, b, c, t \mid a^2 t, b^3 t, abc \rangle,$$

where t is the regular Seifert fiber. Let E_1 and E_2 be homeomorphic to E with meridian and longitudes (m_i, l_i), $i = 1, 2$. Now glue E_1 to E_2 via a homeomorphism $h : \partial E_1 \to \partial E_2$ such that $h(l_1) = m_2$ and $h(m_1) = l_2^{-1}$. Let M denote the resulting manifold, which is a closed graph manifold. The main theorem of this section is:

Theorem 4.1 *M is a minimal closed Haken 3-manifold among all 3-manifolds satisfying Thurston's geometric conjecture.*

We begin the proof by collecting some elementary facts.

Lemma 4.2 (1) *For any representation $\phi : \pi_1(E) \to \mathrm{SL}(2, \mathbb{C})$, if $\phi(t) \neq 1$, then the image $\phi(\pi_1(E))$ is a cyclic group $\langle \lambda \rangle$. Moreover, we must have $\phi(a) = \lambda^{-2}$, $\phi(b) = \lambda^{-3}$, $\phi(c) = \lambda^5$, and $\phi(t) = \lambda^6$.*

(2) *In $\pi_1(T)$, where $T = \partial E$, we have $m = tc^{-1}$ and $l = t^{-5}c^6$. (Equivalently, $t = 6m + l$ and $c = 5m + l$.) Hence $h(t_1^{-5}c_1^6) = t_2 c_2^{-1}$ and $h(t_1 c_1^{-1}) = t_2^5 c_2^{-6}$.*

(3) *M is an integral homology 3-sphere.*

(4) *The only 2-sided incompressible surface is the incompressible torus T, which separates M into E_1 and E_2.*

Proof The main part of (1) follows from [23, Prop. 3] and the fact that $H_1(E, \mathbb{Z})$ is cyclic. (2) and (3) and the remaining parts of (1) are just direct calculations. Finally, to establish (4) we observe the following. Since the trefoil knot is 2-bridge E cannot contain a closed embedded essential surface by [24]. If M contained an embedded incompressible surface $\neq T$, it would follow from the remark above and the gluing homeomorphism that E would have a boundary slope $1/0$. However, [22, Theorem 2.0.3] then implies the existence of a closed embedded essential surface in E.

To show that M is minimal, we assume to the contrary that there is a degree one map $f : M \to N$, where N is irreducible, $N \neq M$, and $N \neq S^3$. First, since M is a graph manifold, its Gromov norm is zero, so N cannot be a hyperbolic 3-manifold by [Theorem, Chapter 6]. Moreover, it is well-known that N must be an integer homology sphere, [4, Lemma 3.1]. The proof of Theorem 4.1 will be finished by Lemmas 4.3, 4.4 and 4.5 below.

Lemma 4.3 *N is non-Haken.*

Proof Suppose N is Haken, and let $F \subset N$ be an embedded incompressible surface. We may deform f so that $f^{-1}(F)$ is an incompressible surface in M. By (4) of Lemma 4.2, $f^{-1}(F)$ must consist of parallel copies of T. By standard 3-manifold topology, we can further deform f so that $f^{-1}(F) = T$. It follows that F is a 2-sphere or torus. Since N is irreducible, F must be a torus separating N into two parts N_1 and N_2. Furthermore, the map f can be decomposed into two proper degree-one maps $f| : E_i \to N_i$. However, E_i is a minimal 3-manifold among knot complements in 3-manifolds via proper degree-one maps [1]. Thus, each $f|$ is a homeomorphism, and it follows that f itself is homotopic to a homeomorphism.

Lemma 4.4 *N is not a Seifert manifold with finite fundamental group* (*other than possibly* S^3).

Proof By (3) of Lemma 4.2, if N is a Seifert fibered manifold of finite fundamental group and $N \neq S^3$, it must be the Poincaré Homology 3-sphere P. Note $\pi_1(P)$ surjects onto A_5, the alternating group on 5 letters. In particular, (as is well known) A_5 is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$—since SO(3) can be identified with PSU(2), and the latter is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$. To prove the lemma, it suffices to prove that the image group of any representation of $\phi : \pi_1(M) \mapsto \mathrm{PSL}(2,\mathbb{C})$ cannot be A_5.

Case (1) If $\phi(t_1) \neq 1$ and $\phi(t_2) \neq 1$, by (1) of Lemma 4.2, the whole image $\phi(\pi_1(M))$ must be a cyclic group (actually trivial).

Case (2) Without loss of generality, we may assume that $\phi(t_1) = 1$ and $\phi(t_2) \neq 1$. By (1) and (2) of Lemma 4.2, $\phi : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ factors as $\nu : \pi_1(M) \to G$ and $\mu : G \to \mathrm{PSL}(2,\mathbb{C})$, where G is generated by two groups described in (a) and (b) below:

(a) $\nu(\pi_1(E_1)) = \langle a_1, b_1, c_1 | a_1^2, b_1^3, a_1 b_1 c_1\rangle$, (b) A cyclic group $\langle \lambda_2\rangle$ such that $\nu(c_2) = \lambda_2^5$, $\nu(t_2) = \lambda_2^6$.

Since $h(t_1 c_1^{-1}) = t_2^5 c_2^{-6}$, we have $\nu(h(c_1^{-1})) = \nu(h(t_1 c_1^{-1})) = \nu(t_2^5 c_2^{-6}) = 1$. It follows that

$$G = \langle a_1, b_1, c_1 | a_1^2, b_1^3, a_1 b_1 c_1, c_1\rangle = \langle a_1, b_1 | a_1^2, b_1^3, a_1 b_1\rangle,$$

which is the trivial group.

Case (3) $\phi(t_1) = 1$ and $\phi(t_2) = 1$. In this case $\phi : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ factors through a group G via a map $\nu : \pi_1(M) \to G$, with $\nu(\pi_1(E_i))$ being the quotient of $G_i = \langle\, a_i,\, b_i,\, c_i | a_i^2,\, b_i^3,\, a_i\ b_i\ c_i\,\rangle$, $i = 1, 2$. Moreover, by (2) of Lemma 4.2 we have that in the quotient $c_1 = c_2^6$ and $c_2 = c_1^{-6}$. Immediately we have that $c_1^{37} = 1$ and $c_2^{37} = 1$ and finally

$$G = \langle a_i, b_i, c_i, i = 1, 2 | a_i^2, b_i^3, a_i b_i c_i, c_i^{37}, c_1 = c_2^6, i = 1, 2\rangle.$$

Suppose there is a homomorphism $\mu : G \to A_5$. Since the order of c_i is 37, and A_5 has order 60, under the homomorphism μ the images of c_1 and c_2 must be trivial. It follows that $\mu : G \to A_5$ can factor through the group G',

$$G' = \langle a_1, b_1, | a_1^2, b_1^3, a_1 b_1\rangle * \langle a_2, b_2, | a_2^2, b_2^3, a_2 b_2\rangle,$$

but as said above, this is trivial.

Lemma 4.5 *N is not a Seifert manifold with infinite π_1.*

The proof of Lemma 4.5 requires a sequence of additional lemmas. We suppose below that N is a Seifert manifold with infinite π_1. By Lemma 4.3, we may assume that N is non-Haken. Hence N must be a Seifert manifold with three singular fibers over S^2.

We begin by establishing:

Lemma 4.5.1 (1) *Suppose* $\Delta \subset \mathrm{PSL}(2,\mathbb{R})$ *is a triangle group and* $\phi : \pi_1(2,3,l) \to \Delta$ *is of finite index. Then the image of* ϕ *is a hyperbolic triangle group isomorphic to* $\pi_1(2,3,k)$, *where* $k|l$.

(2) *Suppose a Seifert manifold* N *is an integer homology* 3*-sphere with infinite* π_1 *and orbifold* $O = (a_1, a_2, a_3)$. *Then* $\gcd(a_i, a_j) = 1$ *for* $i, j = 1, 2, 3$, *and* O *is a hyperbolic orbifold.*

Proof (1) Let x', y' be the order 2 and order 3 elements which generate $\pi_1(2,3,l)$ such that $x'y'$ is of order l. Use x and y to denote their images in $\mathrm{PSL}(2,\mathbb{R})$, then x and y generate the image of ϕ. Since the image of ϕ is of finite index in Δ, it must be co-compact and of rank 2. By the well-known facts, the image is a triangle group with $x^2 = y^3 = (xy)^k = 1$, where $k|l$.

(2) follows from [3, p. 680 (d)].

Lemma 4.5.2 *There is a simple closed curve in the kernel of* $f| : T \to N$.

Proof Since $\pi_1(N)$ is torsion free and T is a torus, to prove the lemma, we need only to show that the kernel of $f| : T \to N$ is non-trivial.

Suppose $f(t_1) \neq 1$; otherwise, the claim is proved. Note that all elements in $f(\pi_1(E_1))$ commute with $f(t_1)$. If $f(t_1)$ is not the fiber t of N, then either

$$f(\pi_1(E_1)) = f(t_1) \quad \text{or} \quad f(\pi_1(E_1)) = \langle f(t_1), f(c_1)\rangle = \mathbb{Z} \oplus \mathbb{Z}.$$

The second case is not possible since $H_1(E_1;\mathbb{Z}) = \mathbb{Z}$. In the first case we deduce that $\ker(f|_T)_*$ is non-trivial. Similarly if $f(t_2)$ is not the fiber t of N, then $\ker(f|_T)_*$ is non-trivial. Suppose $f(t_1) = t = f(t_2)$. Since t_1 and t_2 do not coincide up to an isotopy, still we have that $\ker(f|_T)_*$ is non-trivial.

Let C be the simple closed curve provided by Lemma 4.5.2. Suppose $C = pm_1 + ql_1$ on ∂E_1. Then $C = -qm_2 + pl_2$. By (1) of Lemma 4.2 we have $pm + gl = (p - 5q)t + (-p + 6q)c$ and $-qm + pl = (-q - 5p)t + (q + 6p)c$. So the degree 1 map f factors through $f : M \to N_1 \cup_{S^1} N_2 \to N$, where N_1 and E_2' are Seifert manifolds whose normal forms are given by $(2, 1; 3, 1; -p + 6q, p - 5q)$ and $(2, 1; 3, 1; q + 6p, -5p - q)$, respectively, and the two cores of the surgery solid tori are identified. If $f|_*(\pi_1(N_1)) \neq \pi_1(N)$ and $f|_*(\pi_1(N_2)) \neq \pi_1(N)$ then $\pi_1(N)$ can be presented as a non-trivial free product with amalgamation by the classical result (see [22] for example). It follows that N will be Haken, contrary to Lemma 4.3. Thus without loss of generality, we assume that $f|_*(\pi_1(N_1)) = \pi_1(N)$.

Lemma 4.5.3 $f|_{N_2}$ *is of non-zero degree.*

Proof Let $\tilde{E}$ be the covering of N corresponding to $f|_*(\pi_1(N_2))$. Then $f : N_2 \to N$ lifts to $\tilde{f} : N_2 \to \tilde{E}$, which is π_1-surjective. If $f|_*(\pi_1(N_2)) \subset \pi_1(N)$ is of finite index, then $\tilde{E}$ is a closed Seifert manifold. Since both $\pi_1(N_1)$ and $\pi_1(N)$ are of rank 2, $\pi_1(\tilde{E})$ must also be of rank 2. Then $\tilde{f}$ is of non-zero degree by Theorem 2.1. Hence $f|_{N_2}$ is of non-zero degree.

Below we show $f|_*(\pi_1(N_2)) \subset \pi_1(N)$ must be of finite index. Otherwise $\tilde{E}$ is a non-compact, aspherical Seifert manifold, for which it is known that either the rank of $H_1(\tilde{E})$ is positive or $\pi_1(\tilde{E})$ is trivial. Since $f|_*(\pi_1(N_2))$ is not trivial and N_2 is a rational homology sphere, all of the above cases are ruled out. So $f|_*(\pi_1(N_2))$ must be of finite index in $\pi_1(N)$.

Since N_1 and N_2 are in symmetry position, we have both $f|N_1$ and $f|N_2$ are of non-zero degree.

By Lemma 4.5.3, we may assume that $f|N_i$ is fiber preserving. Then $f|N_i$ induces an homomorphism $\phi_i : \pi_1(O_i) \to \pi_1(O)$; in particular, ϕ_1 is surjective and ϕ_2 is of finite index, where $O_1 = (2, 3, 6q - p)$, $O_2 = (2, 3, 6q + p)$ and $O = (a_1, a_2, a_3)$ are orbifolds of N_1, N_2 and N, respectively. Since N is an integral homology 3-sphere of infinite π_1, it follows that $\pi_1(O)$ is isomorphic to a hyperbolic triangle group. Since $\phi_1 : G_1 \to G$ is surjective, it follows that $O = (2, 3, k)$, where $k|6q - p$ by Lemma 4.5.1(1). Since ϕ_2 is of finite index, the image of ϕ_2 is a hyperbolic triangle group $\pi_1(2, 3, k')$ with $k'|6q + p$ by Lemma 4.5.1(1); moreover, $k'|k$. It is easy to see that k' is a divisor of both $12q$ and $2p$. Since p and q are coprime, the great common divisor of $12q$ and $2p$ is 12. So k' is either 2, or 3, or 4, or 6, or 12. Then N can not be an integer homology sphere by Lemma 4.5.1(2).

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