

# On the profinite rigidity of triangle groups

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*For Bill Harvey, who knew Fuchsian groups well.*

## ABSTRACT

We prove that certain Fuchsian triangle groups are profinitely rigid in the absolute sense, that is, each is distinguished from all other finitely generated, residually finite groups by its set of finite quotients. We also develop a method based on character varieties that can be used to distinguish between the profinite completions of certain groups.

## 1. Introduction

A finitely generated, residually finite group  $\Gamma$  is *profinutely rigid (in the absolute sense)* if it is distinguished from all other finitely generated, residually finite groups by its set of finite quotients. More formally, if  $\Lambda$  is finitely generated and residually finite, then  $\hat{\Lambda} \cong \hat{\Gamma}$  implies  $\Lambda \cong \Gamma$  (where  $\hat{\Delta}$  denotes the profinite completion of a group  $\Delta$ ). Finitely generated abelian groups have this property, as do certain nilpotent groups, but it is hard to construct examples of profinitely rigid groups that do not satisfy a group law; indeed no such groups were known until our work in [6]. The most compelling question in the field is the conjecture that non-abelian free groups of finite rank are profinitely rigid. More generally, it seems reasonable to expect all lattices in  $\mathrm{PSL}(2, \mathbb{R})$  to be profinitely rigid. The main result of [5] shows that such lattices can at least be distinguished from each other by their finite quotients.

In [6], we proved that certain arithmetic lattices in  $\mathrm{PSL}(2, \mathbb{C})$  are profinitely rigid, including the Bianchi group  $\mathrm{PSL}(2, \mathbb{Z}[\omega])$  (where  $\omega^2 + \omega + 1 = 0$ ) and the fundamental group of the Weeks manifold, which is the closed hyperbolic 3-manifold of minimal volume. Our main purpose in the present article is to prove that certain arithmetic lattices in  $\mathrm{PSL}(2, \mathbb{R})$  are also profinitely rigid in the absolute sense.

**THEOREM 1.1.** *The following arithmetic triangle groups are profinitely rigid in the absolute sense:*

$$\begin{aligned} &\Delta(3, 3, 4), \Delta(3, 3, 5), \Delta(3, 3, 6), \Delta(2, 5, 5), \Delta(3, 5, 5), \Delta(4, 4, 4), \Delta(5, 5, 5), \\ &\Delta(2, 3, 8), \Delta(2, 3, 10), \Delta(2, 3, 12), \Delta(2, 4, 5), \Delta(2, 5, 6), \Delta(2, 4, 8), \Delta(2, 5, 10). \end{aligned}$$

We follow the convention of writing  $\Delta(p, q, r)$  to denote the orientation-preserving subgroup of index 2 in the Coxeter group generated by reflections in the sides of a hyperbolic (respectively, Euclidean or spherical) triangle with vertex angles  $\pi/p$ ,  $\pi/q$ , and  $\pi/r$ . We shall also prove (Corollary 4.5) that for the above values of  $(p, q, r)$ , these hyperbolic Coxeter groups are profinitely rigid in the absolute sense.

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In its broad outline, our strategy for proving Theorem 1.1 is the one employed in [6] to establish the existence of Kleinian groups that are profinitely rigid. Several of the key ideas developed in [6], notably that of *Galois rigidity*, will play a crucial role again here. But the end-game by which we move from the general construction of representations to profinite rigidity for specific examples is more direct in the present setting than it was in [6].

Roughly speaking, a finitely generated subgroup  $\Gamma < \mathrm{PSL}(2, \mathbb{C})$  is Galois rigid if all of the irreducible representations  $\Gamma \rightarrow \mathrm{PSL}(2, \mathbb{C})$  can be constructed from the arithmetic of the trace field of  $\Gamma$  (see §2.2). The first five groups in Theorem 1.1 arise as the image in  $\mathrm{PSL}(2, \mathbb{C})$  of the elements of norm 1 in a maximal order in a quaternion algebra over a real quadratic field, ramified at one infinite place and one finite place; see Theorem 3.2. Each of these groups is Galois rigid; see Proposition 3.3. Both  $\Delta(4, 4, 4)$  and  $\Delta(5, 5, 5)$  arise as a subgroup of index 3 in one of the previous groups; these are also Galois rigid. The other seven groups in Theorem 3.2 are index-2 extensions of the groups written above them.

A key step in the proof of Theorem 1.1 is the proof that every proper infinite subgroup of the triangle groups  $\Delta$  considered has a finite quotient that  $\Delta$  does not have. This can be established using classical techniques from the study of finitely generated Fuchsian groups (see Corollary 3.7) but is also covered by the more general and conceptually pleasing method of producing homomorphisms that we develop in §5. This method, which is based on the structure of character varieties, produces homomorphisms to groups of the form  $\mathrm{PSL}(2, \mathbb{F})$ , with  $\mathbb{F}$  a finite field, and is useful beyond the context of Theorem 1.1. For example, the following theorem applies to any group  $\Gamma$  that is the orbifold fundamental group of a non-Haken hyperbolic 3-orbifold. This allows the possibility that the orbifolds considered are non-compact, in which case the cusp cross-section will be a Euclidean orbifold of the form  $\mathbb{E}^2/\Delta(p, q, r)$ ; the Bianchi orbifold  $\mathbb{H}^3/\mathrm{PSL}(2, \mathbb{Z}[\omega])$  is an example.

We remind the reader that a finitely generated group  $\Gamma$  has Property FA if it has a global fixed point whenever it acts on a simplicial tree.

**THEOREM 1.2.** *Let  $\Gamma < \mathrm{PSL}(2, \mathbb{C})$  be a finitely generated Kleinian group that has Property FA. Suppose that  $H$  is a finitely generated non-elementary Kleinian group whose  $\mathrm{PSL}(2, \mathbb{C})$ -character variety has an irreducible component of positive dimension containing the character of a discrete faithful representation of  $H$ . Then there exists a finite field  $\mathbb{F}$  and a representation  $H \rightarrow \mathrm{PSL}(2, \mathbb{F})$  whose image is a non-abelian group that is not a quotient group of  $\Gamma$ .*

The main point of this theorem is not that  $\Gamma$  and  $H$  have different profinite completions — this can be proved more easily using either goodness (see [12, Theorem 4.11]) or the profinite invariance of the first  $\ell_2$ -betty number (see [5, Corollary 3.3]). Rather, the novelty of Theorem 1.2 is that it allows us to distinguish between  $\Gamma$  and  $H$  by looking only at their finite images in  $\mathrm{PSL}(2, \mathbb{F})$ . The groups considered are large (that is, contain finite-index subgroups surjecting a non-abelian free group, [1, 7]), so their set of finite images is vastly more complicated than their set of images in  $\mathrm{PSL}(2, \mathbb{F})$ , and the fact that they can be distinguished by the latter images alone is correspondingly stronger.

## 2. Trace-fields and Galois Rigidity

We recall some of the key ideas from [6].

### 2.1. Trace-fields

To fix notation, it will be convenient to record some basic facts about trace-fields of finitely generated subgroups of  $\mathrm{PSL}(2, \mathbb{C})$ . For concreteness, let  $\phi: \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{PSL}(2, \mathbb{C})$  be the quotient homomorphism, and  $H$  a finitely generated subgroup of  $\mathrm{PSL}(2, \mathbb{C})$ . Let  $H_1 = \phi^{-1}(H)$ .

It will be convenient to say  $H$  is *Zariski dense* in  $\mathrm{PSL}(2, \mathbb{C})$  when what we actually mean is that  $H_1$  is a Zariski dense subgroup of  $\mathrm{SL}(2, \mathbb{C})$ . The *trace-field* of  $H$  is defined to be the field

$$K_H = \mathbb{Q}(\mathrm{tr}(\gamma) : \gamma \in H_1).$$

If  $K_H$  is a number field with ring of integers  $R_{K_H}$ , we say that  $H$  has *integral traces* if  $\mathrm{tr}(\gamma) \in R_{K_H}$  for all  $\gamma \in H_1$ . The group  $H_1$  generates a  $K_H$ -quaternion algebra  $A_0H$ , and when  $H$  has integral traces,  $H_1$  generates an  $R_{K_H}$ -order  $\mathcal{O}H$  in  $A_0H$  (see [10, Chapter 3] for more details on this material). Conversely, if  $H_1$  is contained in an order of  $A_0H$ , then  $H$  has integral traces.

One well-known situation when  $K_H$  is a number field is the following (see [11, Proposition 6.6] for example). Let  $X_{\mathrm{zar}}(H, \mathbb{C})$  denote the set of Zariski dense representations up to conjugacy of  $H$  in  $(\mathrm{P})\mathrm{SL}(2, \mathbb{C})$ .

LEMMA 2.1. *Let  $H < \mathrm{PSL}(2, \mathbb{C})$  be a finitely generated group. If  $X_{\mathrm{zar}}(H, \mathbb{C})$  is finite, then  $K_H$  is a number field.*

Suppose that  $H$  is a finitely generated group and  $\rho: H \rightarrow (\mathrm{P})\mathrm{SL}(2, \mathbb{C})$  a Zariski dense representation with  $K = K_{\rho(H)}$  a number field of degree  $n_K$ . If  $K = \mathbb{Q}(\theta)$  for some algebraic number  $\theta$ , then the Galois conjugates of  $\theta$ , say  $\theta = \theta_1, \dots, \theta_{n_K}$  provide embeddings  $\sigma_i: K \rightarrow \mathbb{C}$  defined by  $\theta \mapsto \theta_i$ . These in turn can be used to build  $n_K$  Zariski dense non-conjugate representations  $\rho_{\sigma_i}: H \rightarrow (\mathrm{P})\mathrm{SL}(2, \mathbb{C})$  with the property that  $\mathrm{tr}(\rho_{\sigma_i}(\gamma)) = \sigma_i(\mathrm{tr} \rho(\gamma))$  for all  $\gamma \in H$ . We sometimes refer to these as *Galois conjugate representations*.

## 2.2. Galois rigidity

As in [6], we will be interested in groups  $\Gamma$  with the fewest possible Zariski dense representations. To be more precise, recall from the discussion in § 2.1 that if  $\Gamma$  is a finitely generated group and  $\rho: \Gamma \rightarrow (\mathrm{P})\mathrm{SL}(2, \mathbb{C})$  a Zariski dense representation whose trace field  $K = K_{\rho(\Gamma)}$  is a number field, then using the Galois conjugate representations, we have  $|X_{\mathrm{zar}}(\Gamma, \mathbb{C})| \geq n_K$ .

DEFINITION 2.2 (Galois rigid). Let  $\Gamma$  be a finitely generated group and let  $\rho: \Gamma \rightarrow (\mathrm{P})\mathrm{SL}(2, \mathbb{C})$  be a Zariski dense representation whose trace field  $K_{\rho(\Gamma)}$  is a number field. If  $|X_{\mathrm{zar}}(\Gamma, \mathbb{C})| = n_{K_{\rho(\Gamma)}}$ , we say that  $\Gamma$  is *Galois rigid* (with associated field  $K_{\rho(\Gamma)}$ ).

When we say that a subgroup of  $(\mathrm{P})\mathrm{SL}(2, \mathbb{C})$  is Galois rigid, we are implicitly taking  $\rho$  to be the inclusion map. Note that  $\Gamma$  is Galois rigid if and only if  $\rho(\Gamma)$  is Galois rigid. Note too that if  $\Gamma$  is Galois rigid, then any irreducible representation with infinite image can serve as  $\rho$ , since any such representation is a Galois conjugate of any other. In particular,  $K_{\rho(G)}$  is an intrinsic invariant of  $\Gamma$ , is the associated quaternion algebra  $A_0\Gamma := A_0\rho(\Gamma)$  and the group homomorphism  $\Gamma \rightarrow \rho(\Gamma) \hookrightarrow A_0\Gamma^1$ .

The theorem stated below can be extracted from [6, Theorem 4.8, Corollary 4.11], but the special case that we require involves some preliminaries. To that end, we fix a real quadratic number field  $K \subset \mathbb{R}$  with  $\sigma: K \rightarrow \mathbb{R}$  the non-trivial Galois embedding, a quaternion algebra  $B/K$ , a maximal order  $\mathcal{O} < B$ , and a representation  $\rho: B \rightarrow M(2, \mathbb{R})$ . Throughout,  $\mathcal{O}^1$  will denote the elements of norm 1 in the order  $\mathcal{O}$  and

$$\Gamma_{\mathcal{O}}^1 = \mathrm{P}\rho(\mathcal{O}^1) < \mathrm{PSL}(2, \mathbb{R}).$$

Since  $K$  has two real places  $v_1$  (the identity place) and  $v_2$  (associated to  $\sigma$ ), we can prescribe that  $B$  be ramified at either of  $v_1$  or  $v_2$ , and unramified at the other; denote these two possibilities by  $B_1$  and  $B_2$  respectively. If  $B_i$  is also ramified at a finite place  $\omega$  with residue field of characteristic  $p$  and  $\omega$  is the unique such place, then although  $B_1$  and  $B_2$  are not

isomorphic (over  $K$ ), there is an extension of  $\sigma$  that maps  $B_1$  to  $B_2$ . In this situation, and up to this ambiguity, we can identify  $B$  with either of  $B_i$  (for  $i = 1, 2$ ) and assume that  $\Gamma < \Gamma_{\mathcal{O}}^1$  is a finitely generated subgroup such that  $K_{\Gamma} = K$ . We identify  $A_0\Gamma$  with  $B$ . With this preamble established, the following result can readily be extracted from [6, Theorem 4.8, Corollary 4.11].

**THEOREM 2.3.** *Let  $\Gamma < \Gamma_{\mathcal{O}}^1$  be as above and assume that  $\Gamma$  is Galois rigid (with associated field  $K$ ). If  $\Delta$  is a finitely generated, residually finite group with  $\widehat{\Delta} \cong \widehat{\Gamma}$ , then:*

- (i)  $\Delta$  is Galois rigid with associated field  $K$  and Zariski dense homomorphism  $\phi': \Delta \rightarrow (\mathbb{P}) \mathrm{SL}(2, \mathbb{C})$ ;
- (ii) if  $B$  has type number 1,  $\mathrm{Ram}(B) = \{v_2, \omega\}$  where  $v_2$  is the real place described above, and  $\omega$  is a finite place as above, then the homomorphism  $\phi'$  from (i) has image contained in  $\Gamma_{\mathcal{O}}^1$ .

### 3. Triangle groups

For positive integers  $p, q, r$  with  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ , we write  $\Delta^-(p, q, r)$  for the group of isometries of the hyperbolic plane generated by reflections in the sides of a triangle  $T(p, q, r)$  with vertex angles  $\pi/p$ ,  $\pi/q$ , and  $\pi/r$ , and we write  $\Delta(p, q, r)$  for the index 2 subgroup consisting of orientation preserving isometries. The quotient orbifold  $\mathcal{O} = \mathbb{H}^2/\Delta(p, q, r)$  is a sphere with three cone points, orders  $p, q$  and  $r$ . There are standard presentations

$$\Delta(p, q, r) = \langle a, b, c \mid a^p, b^q, c^r, abc \rangle = \langle a, b \mid a^p, b^q, (ab)^r \rangle,$$

$$\Delta^-(p, q, r) = \langle x, y, z \mid x^2, y^2, z^2, (xy)^p, (yz)^q, (xz)^r \rangle,$$

where  $a, b, c$  are rotations (same orientation) at the different vertices of  $T(p, q, r)$ .

#### 3.1. Index 2 extensions of triangle groups

As we try to establish profinite rigidity for the groups in Theorem 1.1, we will need to analyse index 2 extensions of certain triangle groups.

**PROPOSITION 3.1.** *Let  $\Delta < \Gamma$  be a pair of groups with  $\Delta = \Delta(p, q, r)$  and  $[\Gamma : \Delta] = 2$ .*

- (1) *If  $p, q, r$  are distinct, then  $\Gamma \cong \Delta \times \mathbb{Z}/2\mathbb{Z}$  or  $\Gamma \cong \Delta^-(p, q, r)$ .*
- (2) *If  $q = r$ , then there four possibilities for  $\Gamma$ , up to isomorphism, all of the form  $\Delta \rtimes \mathbb{Z}/2$ ; they are  $\Delta(p, q, q) \times \mathbb{Z}/2\mathbb{Z}$ ,  $\Delta^-(p, q, q)$ ,  $\Delta(2p, q, 2)$  and*

$$\Lambda_{\rho}(p, q) := \langle a, b, c, \rho \mid 1 = a^p = b^q = c^q = \rho^2 = abc, \rho a \rho = a^{-1}, \rho b \rho = c^{-1}, \rho c \rho = b^{-1} \rangle.$$

Before proving this proposition, we note that  $\Lambda_{\rho}(p, q)$  is a discrete group of isometries of  $\mathbb{H}^2$  with quotient  $\mathbb{H}^2/\Lambda_{\rho}(p, q)$  a non-orientable 2-orbifold whose underlying space is a disc: there is a cone point of order  $q$  in the interior of the disc, while the boundary circle is mirrored (local group  $\mathbb{Z}/2\mathbb{Z}$ ) and contains a vertex where the local group is dihedral of order  $2p$ .

*Proof.* As is well-known (see, for example, [8])  $\Delta(p, q, r)$  admits a unique faithful discrete representation into  $\mathrm{PSL}(2, \mathbb{R})$ . Thus any lattice  $\Lambda < \mathrm{Isom}(\mathbb{H}^2)$  that contains a subgroup isomorphic to  $\Delta(p, q, r)$  must contain (a conjugate of) a fixed, standard copy of  $\Delta(p, q, r)$  as a subgroup of finite index.

Kerckhoff's solution to the Nielsen Realization Problem (see [9, § V, Theorem 7]) implies that for any group  $G$  containing  $\Delta$  as a subgroup of index  $m$  there is a short exact sequence

$$1 \rightarrow K \rightarrow G \rightarrow \Lambda \rightarrow 1,$$

where  $\Lambda$  is a lattice in  $\text{Isom}(\mathbb{H}^2)$  containing  $\Delta$  as a subgroup of index  $m/|K|$ . In our setting,  $m = 2$ , so either  $K = \mathbb{Z}/2\mathbb{Z}$  and  $\Lambda = \Delta$ , or else  $K$  is trivial and  $\Lambda$  contains  $\Delta$  as a subgroup of index 2. In the former case, the short exact sequence splits and  $G = \Delta \times \mathbb{Z}/2\mathbb{Z}$ . In the latter case, there is a 2-sheeted covering of Riemannian orbifolds  $\mathbb{H}^2/\Delta \rightarrow \mathbb{H}^2/\Lambda$ . In order to understand the possibilities for  $\Lambda$ , we analyse the possibilities for the deck transformation of this covering, that is, the isometric involutions of the orbifold  $O = \mathbb{H}^2/\Delta$ .

In each case, such an involution has at least one fixed point (one of the three cone points), so the corresponding outer automorphism of  $\Delta = \pi_1^{\text{orb}} O$  lifts to an involution in  $\text{Aut}(\Delta)$  and  $\Lambda$  is a semi-direct product of the form  $\Delta \rtimes \mathbb{Z}/2\mathbb{Z}$ . If  $p, q, r$  are all distinct, then any isometry of  $O$  must fix the three cone points and the geodesic arcs joining them, that is, the image in  $O$  of the boundary  $\partial T$  of the geodesic triangle  $T(p, q, r) \subset \mathbb{H}^2$ . Thus the only non-trivial isometry of  $O$  in this case is the reflection in the image of  $\partial T$ . This reflection, which we denote by  $\tau$ , is the isometry of  $O$  induced by each of the three isometries of  $\mathbb{H}^2$  that are reflections in the sides of  $T(p, q, r)$ . Thus  $\Lambda \cong \Delta \langle \tau \rangle$ . This proves (1).

To prove (2), if  $q = r \neq p$ , then each isometry of  $O$  must fix the cone point with vertex angle  $2\pi/p$ , but it can interchange the other two cone points  $v, v'$ . If  $\rho: \mathbb{H}^2 \rightarrow \mathbb{H}^2$  is the reflection in the perpendicular bisector of the edge of  $T$  opposite the vertex with angle  $\pi/p$ , then the induced isometry  $\rho_0: O \rightarrow O$  interchanges  $v$  and  $v'$  and commutes with  $\tau$ . The product  $\sigma = \tau\rho_0$  is the only other non-trivial isometry of  $O$ ; it lifts to a rotation  $s$  through  $\pi$  about the midpoint of the edge of  $T(p, q, q)$  opposite the vertex with angle  $\pi/p$ . Note that  $\langle \Delta, s \rangle = \Delta(2p, q, 2)$ .

Finally, if  $p = q = r$ , then  $\text{Isom}(O) = \text{Sym}(3) \times \mathbb{Z}/2\mathbb{Z}$  where the second factor is generated by  $\tau$ . The distinct conjugacy classes of involutions are represented by  $\tau, \rho_0$  and  $\sigma$ , so up to isomorphism (equivalently, conjugacy in  $\text{Isom}(\mathbb{H}^2)$ ) the possibilities for  $\Lambda$  are the same extensions described in the previous case with  $p = q$ .

Our description of explicit lifts for  $\tau, \rho_0$  and  $\sigma$  provides explicit presentations for the three possible lattices containing  $\Delta$  as a subgroup of index 2 in the case  $q = r$  (regardless of whether  $p = q$ ). For example, the reflection  $\rho$  conjugates the generator  $a \in \Delta(p, q, q)$  to its inverse while interchanging  $b$  and  $c^{-1}$ . Thus  $\Lambda_\rho(p, q) = \langle \Delta, \rho \rangle$  is given by the presentation in the statement of the proposition.  $\square$

### 3.2. Some arithmetic triangle groups

In Theorem 3.2, we list a subset of the arithmetic triangle groups whose invariant trace-field is a real quadratic  $\mathbb{Q}(\sqrt{d})$  (the value of  $d$  is given in the statement of the theorem); this list is taken from [16]. In each case, the invariant quaternion algebra  $A_0\Delta$  is ramified at one real place of  $\mathbb{Q}(\sqrt{d})$  and one finite place (this is listed as  $\mathcal{P}_q$  where  $q$  is the rational prime with  $\mathcal{P}_q | \langle q \rangle$ ).

**THEOREM 3.2.** *With the notation established above, each of the triangle groups shown below arises as a group  $\Gamma_{\mathcal{O}}^1$  where  $\mathcal{O}$  is a maximal order in the quaternion algebra described above.*

- (1)  $\Delta(3, 3, 4)$ ,  $d = 2$ ,  $\mathcal{P}_2$ .
- (2)  $\Delta(3, 3, 6)$ ,  $d = 3$ ,  $\mathcal{P}_2$ .
- (3)  $\Delta(2, 5, 5)$ ,  $d = 5$ ,  $\mathcal{P}_2$ .
- (4)  $\Delta(3, 5, 5)$ ,  $d = 5$ ,  $\mathcal{P}_3$ .
- (5)  $\Delta(3, 3, 5)$ ,  $d = 5$ ,  $\mathcal{P}_5$ .

In Figure 1 below, we portray the degrees of commensurabilities of the triangle groups under consideration that will be useful later (taken from [16]). Here inclusion passes upwards. The commensurability classes of  $\Delta(2, 3, 8)$ ,  $\Delta(2, 3, 12)$  and  $\Delta(2, 4, 5)$  contain additional triangle groups for which our methods are not able to establish profinite rigidity (see Remark 3.5).

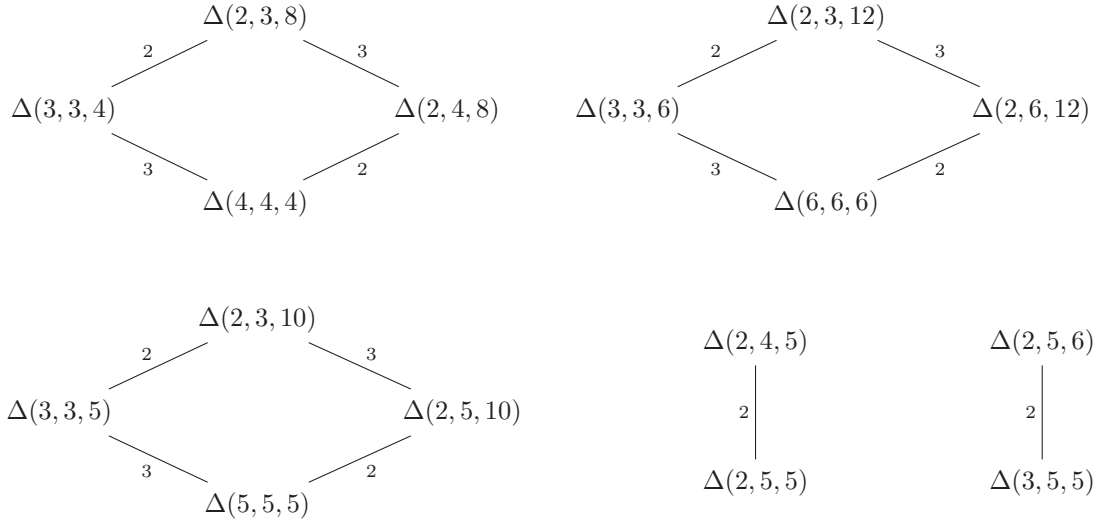


FIGURE 1. Degrees of commensurability.

### 3.3. Galois rigidity of certain triangle groups

**PROPOSITION 3.3.** *Let  $\Delta$  be one of the triangle groups  $\Delta(3, 3, 4)$ ,  $\Delta(4, 4, 4)$ ,  $\Delta(3, 3, 6)$ ,  $\Delta(2, 5, 5)$ ,  $\Delta(3, 5, 5)$ ,  $\Delta(3, 3, 5)$  or  $\Delta(5, 5, 5)$ . Then  $\Delta$  is Galois rigid.*

*Proof.* We begin with some general comments. Let  $\Delta = \Delta(p, q, r)$  and suppose that  $\rho: \Delta \rightarrow \mathrm{PSL}(2, \mathbb{C})$  is a non-trivial representation. The orders of  $\rho(a)$ ,  $\rho(b)$  and  $\rho(ab)$  divide  $p$ ,  $q$  and  $r$ , respectively. Since we are only interested in irreducible representations, we can assume that  $\rho(\Delta)$  is not conjugate into the image of the upper triangular matrices in  $\mathrm{PSL}(2, \mathbb{C})$  (see, for example, [4, §3]). Furthermore, standard properties of elements of  $\mathrm{PSL}(2, \mathbb{C})$  ensure that we can conjugate  $\rho$  so that  $\rho(a)$  fixes  $\infty$  and  $\rho(b)$  fixes 0; that is,

$$\rho(a) = \begin{pmatrix} \pm\zeta_p & 1 \\ 0 & \pm 1/\zeta_p \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} \pm\zeta_q & 0 \\ z & \pm 1/\zeta_q \end{pmatrix},$$

for some  $z \in \mathbb{C}$  with  $\zeta_p$  and  $\zeta_q$  being  $p$ th and  $q$ th roots of unity, respectively. Moreover,  $z$  is constrained by the requirement that  $\mathrm{tr}(\rho(ab)) = \zeta_p\zeta_q + 1/(\zeta_p\zeta_q) + z = \pm(\zeta_r + 1/\zeta_r)$  for some  $r$ th root of unity. Visibly, there are only finitely many possibilities for  $z$ . To go further, and establish Galois rigidity, we must analyse the possible solutions.

We do the case of  $\Delta(3, 3, 6)$  in some detail; the others are similar. As above, let  $\rho: \Delta \rightarrow \mathrm{PSL}(2, \mathbb{C})$  be an irreducible representation. Note that  $\rho(a)$  and  $\rho(b)$  must be elements of order 3, whilst the possibilities for the order of  $\rho(ab)$  are 2, 3 and 6. If  $\rho(ab)$  has order 2, the image of  $\rho$  is the alternating group  $A_4$ ; in particular, it is finite. If  $\rho(ab)$  has order 3, the image is the Euclidean triangle group  $\Delta(3, 3, 3)$ , and this can be conjugated to lie in the image in  $\mathrm{PSL}(2, \mathbb{C})$  of the upper triangular matrices; that is, it is a reducible representation. Hence  $\rho(ab)$  has order 6.

To deal with this last case, as above, we conjugate so that  $\rho(a)$  and  $\rho(b)$  have the following form (where  $\omega^2 + \omega + 1 = 0$ ):

$$\rho(a) = \begin{pmatrix} \pm\omega & 1 \\ 0 & \pm\omega^2 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} \pm\omega & 0 \\ z & \pm\omega^2 \end{pmatrix},$$

for some  $z \in \mathbb{C}$ . Consider  $\text{tr}(\rho(ab)) = \omega^2 + \omega + z = z - 1$ . Since  $\rho(ab)$  has order 6,  $z - 1 = \pm\sqrt{3}$  and this gives the two possibilities. One of these gives the faithful discrete representation ( $z = 1 + \sqrt{3}$ ) and the other gives its Galois conjugate, a representation into  $\text{PSU}(2)$ .  $\square$

REMARK 3.4. In addition to Galois rigidity, the invariant quaternion algebras of the groups listed in Theorem 3.2 all satisfy Theorem 2.3(ii). To see this, first note that they have type number 1, by [16, Proposition 3]. Since the triangle groups are arithmetic, all the defining quaternion algebras are ramified at one real place, and the finite places where they ramify are the unique places of that characteristic. Briefly, 2 is the unique ramified place in the case of  $d = 2, 3$  and similarly for 5 when  $d = 5$ . Also, 2 and 3 are inert in the case of  $d = 5$ .

REMARK 3.5. Note that in the proof of Proposition 3.3 we observed that  $\Delta(3, 3, 6)$  does admit an infinite representation into  $\text{PSL}(2, \mathbb{C})$  that does not arise as a Galois conjugate representation. In this case, the image is the Euclidean triangle group  $\Delta(3, 3, 3)$ , which of course is not Zariski dense, and so does not violate Galois rigidity. This also applies to the group  $\Delta(4, 4, 4)$ ; in this case, the image is the Euclidean triangle group  $\Delta(2, 4, 4)$ , which again does not violate Galois rigidity.

On the other hand,  $\Delta(6, 6, 6)$  admits an epimorphism to  $\Delta(2, 6, 6)$ , and this does violate Galois rigidity. Hence Proposition 3.3 does not hold for  $\Delta(6, 6, 6)$ .

### 3.4. Profinite epimorphisms among Fuchsian groups

By definition, the *Euler characteristic* of a Fuchsian group  $\Gamma$  is the orbifold Euler characteristic of  $\mathbb{H}^2/\Gamma$  (which by Gauss–Bonnet is the area of a fundamental domain for  $\Gamma$  divided by  $-2\pi$ ). It behaves multiplicatively on subgroups in the sense that  $[\Gamma : H] = d$  implies  $\chi(H) = d\chi(\Gamma)$ .

We shall need the following lemma, which could be rephrased as saying that if the area of  $\mathbb{H}^2/\Gamma_1$  is less than the area of  $\mathbb{H}^2/\Gamma_2$ , then  $\Gamma_2$  has a finite quotient that  $\Gamma_1$  does not have.

This is the first place where it is convenient for us to phrase a result in the language of profinite completions. We remind the reader that the *profinite completion* of a group  $\Gamma$  is the inverse limit of its system of finite quotients,  $\hat{\Gamma} = \varprojlim \Gamma/N$  where the limit is taken over finite index normal subgroups  $N < \Gamma$  ordered by reverse-inclusion. It is endowed with the inverse-limit topology, making it a compact topological group. If  $\Gamma$  is residually finite (as all of the groups that we consider are), then the natural map  $\Gamma \rightarrow \hat{\Gamma}$  is injective. The image of  $\Gamma$  is dense, and every epimorphism from  $\Gamma$  to a finite group  $Q$  extends to a continuous epimorphism  $\hat{\Gamma} \rightarrow Q$ .

LEMMA 3.6. *Let  $\Gamma_1$  and  $\Gamma_2$  be Fuchsian groups. If there is a continuous surjection  $\hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2$ , then  $\chi(\Gamma_1) \leq \chi(\Gamma_2)$ .*

*Proof.* To begin, given a Fuchsian group  $\Gamma$ , let  $b_1(\Gamma)$  denote the first betti number of  $\Gamma$ , that is, the rank of  $H^1(\Gamma, \mathbb{Z})$ . If  $\Gamma$  is cocompact, then  $b_1(\Gamma) \leq 2 - \chi(\Gamma)$ , and otherwise,  $b_1(\Gamma) \leq 1 - \chi(\Gamma)$ . If additionally  $\Gamma$  is torsion-free, then these are equalities. Finally, for finitely generated groups in general, if there an epimorphism  $\hat{G}_1 \rightarrow \hat{G}_2$ , then  $b_1(G_1) \geq b_1(G_2)$  (see [5, Lemma 2.10]).

Suppose there is a continuous epimorphism  $\eta: \hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2$ . We can pass to a subgroup of finite index, say  $H_2 < \Gamma_2$  of index  $d$ , so that  $H_2$  is torsion-free. Then  $H_1 := \eta^{-1}(\overline{H_2}) \cap \Gamma_1$  is a subgroup of index  $d$  in  $\Gamma_1$  and the restriction of  $\eta$  to  $\overline{H_1} \cong \hat{H_1}$  gives an epimorphism  $\hat{H_1} \rightarrow \hat{H_2}$ . (Here,  $\overline{H}$  denotes the closure of  $H < \Gamma$  in  $\hat{\Gamma}$ .) We do not assume that  $H_1$  is torsion-free.

Then,

$$2 - d\chi(\Gamma_1) \geq b_1(H_1) \geq b_1(H_2) = \epsilon - d\chi(\Gamma_2),$$



where  $\epsilon = 1$  or  $2$  according to whether  $\Gamma_2$  is cocompact or not. As  $d$  can be taken to be arbitrarily large, this implies  $\chi(\Gamma_1) \leq \chi(\Gamma_2)$ .  $\square$

A stronger form of the following result will be established by less elementary means in Section 5.

**COROLLARY 3.7.** *Let  $\Delta = \Delta(p, q, r)$  with  $1 > \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq \frac{1}{2}$ .*

(1) *If  $S$  is a non-elementary proper subgroup of  $\Delta$ , then there does not exist a continuous epimorphism  $\widehat{\Delta} \rightarrow \widehat{S}$ .*

(2) *If  $H$  is a non-elementary Fuchsian group that is not cocompact, then does there does not exist a continuous epimorphism  $\widehat{\Delta} \rightarrow \widehat{H}$ .*

*Proof.* If  $S < \Gamma_1$  has finite index, then  $\chi(S) < \chi(\Delta)$  and the lemma applies. If not, then we are in Case (2) of the corollary, and we complete using the argument below.

Assume now that  $H$  is a non-elementary Fuchsian group that is not cocompact. Hence it is a free product of a free group and some finite cyclic groups. If there is a surjection  $\widehat{\Gamma}_1 \rightarrow \widehat{H}$ , then  $H$  must be finitely generated, and since  $\Gamma_1$  has finite abelianization,  $H$  must as well. Thus we are reduced to the possibility that  $H$  is a free product of finite cyclic groups  $C_1 * \cdots * C_m$ , where  $C_i$  has order  $n_i$ , say.

Now,

$$\chi(\Delta) = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 \geq -\frac{1}{2},$$

whereas

$$\chi(H) = \frac{1}{n_1} + \cdots + \frac{1}{n_m} + 1 - m.$$

So  $\chi(H) \geq -1/2$  forces  $m = 2$  or else  $m = 3$  and  $n_1 = n_2 = n_3 = 2$ . This last possibility has abelianization  $(\mathbb{Z}/2\mathbb{Z})^3$ , which cannot be a finite image of  $\Delta$  (and hence  $\widehat{\Delta}$ ) because  $\Delta$  is generated by two elements. Thus  $H = C_1 * C_2$ .

Let  $a, b, c = (ab)^{-1}$  be the generators in the standard presentation of  $\Delta$ , and note that any two of them suffice to generate. We know from [5, Theorem 5.1] that any finite subgroup of  $\widehat{H}$  is contained in a conjugate of  $C_1$  or  $C_2$ , so for any map  $\widehat{\Delta} \rightarrow \widehat{H}$ , the image of at least two of  $a, b, c$  must lie in conjugates of the same  $C_i$ . But this means that the image of  $\Delta$  in the abelianization of  $H$  would be a proper subgroup. Since  $\Delta$  is dense in  $\widehat{\Delta}$ , this implies that no continuous map  $\widehat{\Delta} \rightarrow \widehat{H}$  can be surjective.  $\square$

#### 4. Profinite rigidity of triangle groups

In this section, we prove Theorem 1.1, whose statement we recall for the reader's convenience.

**THEOREM 1.1.** *The following arithmetic triangle groups are profinitely rigid in the absolute sense:*

$$\begin{aligned} &\Delta(3, 3, 4), \Delta(3, 3, 5), \Delta(3, 3, 6), \Delta(2, 5, 5), \Delta(3, 5, 5), \Delta(4, 4, 4), \Delta(5, 5, 5), \\ &\Delta(2, 3, 8), \Delta(2, 3, 10), \Delta(2, 3, 12), \Delta(2, 4, 5), \Delta(2, 5, 6), \Delta(2, 4, 8), \Delta(2, 5, 10). \end{aligned}$$

*Proof.* Let  $\Delta$  be one of the first five groups listed in the theorem and let  $\Lambda$  be a finitely generated, residually finite group with  $\widehat{\Lambda} \cong \widehat{\Delta}$ . We must prove that  $\Lambda \cong \Delta$ .



From Theorem 3.2, we know that  $\Delta$  can be described as  $\Delta = \Gamma_{\mathcal{O}}^1$  which is the image in  $\mathrm{PSL}(2, \mathbb{C})$  of the elements of norm 1 in a maximal order in a quaternion algebra over a real quadratic field, ramified at one infinite place and one finite place. Proposition 3.3 assures us that  $\Delta$  is Galois rigid, so by Theorem 2.3 and Remark 3.4, there exists an epimorphism  $\rho: \Lambda \twoheadrightarrow L$  onto a finitely generated, Zariski dense (hence non-elementary) subgroup  $L \leq \Delta = \Gamma_{\mathcal{O}}^1$ . This induces a continuous epimorphism  $\widehat{\Gamma} \cong \widehat{\Lambda} \rightarrow \widehat{L}$ , and Corollary 3.7 says that this is impossible unless  $L = \Delta$ . Thus  $\rho$  induces an epimorphism  $\widehat{\rho}: \widehat{\Lambda} \rightarrow \widehat{\Delta} \cong \widehat{\Lambda}$ , and since finitely generated, profinite groups are Hopfian (see [13, Proposition 2.5.2]), we conclude that  $\widehat{\rho}$  is injective, and hence  $\rho$  is an isomorphism.

We now deal with  $\Delta = \Delta(4, 4, 4)$ , which arises as a normal subgroup of index 3 in  $\Delta(3, 3, 4)$ . Let  $\Lambda$  be a finitely generated, residually finite group with  $\widehat{\Lambda} \cong \widehat{\Delta}$ . As above, using Galois rigidity we obtain an epimorphism  $\rho: \Lambda \twoheadrightarrow L$  to a non-elementary subgroup of  $\Delta(3, 3, 4) = \Gamma_{\mathcal{O}}^1$ . In fact,  $L$  must be a subgroup of  $\Delta$ , since if not, then  $L \cap \Delta$  is a normal subgroup of index 3 in  $L$ , which is impossible as the abelianization of  $\Delta$  is  $\mathbb{Z}/4\mathbb{Z}$ . From Corollary 3.7, we can now deduce that  $L = \Delta$ , and as in the previous case we conclude that  $\rho: \Lambda \rightarrow \Delta$  is an isomorphism.

The case of  $\Delta(5, 5, 5) < \Delta(3, 3, 5)$  is entirely similar to  $\Delta(4, 4, 4) < \Delta(3, 3, 4)$ .

It remains to deal with the seven groups in the bottom row of Theorem 1.1. These cases are covered by the discussion in the next section, explicitly Corollary 4.4, because each contains as a subgroup of index 2 one of the groups that we have already dealt with. In each case, the inclusion  $\Delta(p, p, q) \hookrightarrow \Delta(2, p, 2q)$  is obtained by noting that the hyperbolic isosceles triangle with vertex angles  $(\pi/p, \pi/p, \pi/q)$  is divided into two copies of  $(\pi/2, \pi/p, \pi/2q)$  by dropping a perpendicular from the vertex with angle  $\pi/q$ .  $\square$

REMARK 4.1. In the proof of Theorem 1.1, we emphasized techniques that rely on basic properties of Fuchsian groups. Some of this can be bypassed by invoking deeper parts of [6] and using Theorem 2.3(i), which implies that  $\Lambda$  is Galois rigid, which forces  $L$  to be a triangle group (cf. the proof of Corollary 5.1).

REMARK 4.2. In [6, § 1], we discussed the fact that [6, Theorem 4.8] does not apply to  $\Delta(2, 3, 7)$ , and so we cannot establish profinite rigidity in this case. The argument given in the proof of Theorem 1.1 would apply to this group if we knew that the group  $L$  constructed in the proof is actually a subgroup of  $\Delta(2, 3, 7)$ . However, as discussed in [6, § 1], we do not know this: the group  $L$  might be a subgroup of  $\mathrm{PSL}(2, R_k)$ , where  $R_k$  is the ring of integers of the field  $\mathbb{Q}(\cos \pi/7)$ . At present, we do not know how to exclude this possibility.

#### 4.1. Profinite rigidity for index 2 extensions

LEMMA 4.3. *For every Fuchsian triangle group  $\Delta = \Delta(p, q, r)$ , the index-two extensions of  $\Delta$  listed in Proposition 3.1 have pairwise non-isomorphic profinite completions.*

*Proof.* We assume first that  $p, q$  and  $r$  are all distinct. By Proposition 3.1(1), there are two extensions of  $\Delta(p, q, r)$ , namely  $\Delta(p, q, r) \times \mathbb{Z}/2\mathbb{Z}$  and  $\Delta^-(p, q, r)$ . From the presentation for  $\Delta^-(p, q, r)$  given in § 3, we deduce that  $\widehat{\Delta^-(p, q, r)}$  contains finite dihedral groups  $D_p, D_q$  and  $D_r$ . Note that at most one of  $p, q$  or  $r$  equals 2 since  $\Delta(p, q, r)$  is a Fuchsian group, and so at most one of these dihedral groups is abelian.

We now invoke [5, Theorem 5.1] which states that for any finitely generated Fuchsian group  $\Gamma$ , the inclusion  $\Gamma \rightarrow \widehat{\Gamma}$  induces a bijection between conjugacy classes of finite subgroups. In our setting, we deduce that the maximal finite subgroups of  $\widehat{\Delta(p, q, r)}$ , up to conjugacy, are  $\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}, \mathbb{Z}/r\mathbb{Z}$  and hence the maximal finite subgroups of

$\Delta(p, q, r) \times \widehat{\mathbb{Z}/2\mathbb{Z}} = \widehat{\Delta(p, q, r)} \times \mathbb{Z}/2\mathbb{Z}$  are abelian, namely  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and  $\mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . Hence  $\Delta(p, q, r) \times \mathbb{Z}/2\mathbb{Z}$  cannot be isomorphic to  $\Delta^-(p, q, r)$ .

We now deal with the groups in Proposition 3.1(2). We need to prove that the profinite completions of the following groups are distinct

$$\Delta(p, q, q) \times \mathbb{Z}/2\mathbb{Z}, \quad \Delta^-(p, q, q), \quad \Delta(2p, q, 2), \quad \Lambda_\rho(p, q).$$

We calculate their abelianizations from the explicit presentations given in §3:

$$\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/h\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad (\mathbb{Z}/2\mathbb{Z})^i, \quad \mathbb{Z}/h'\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

where  $h = \gcd(p, q)$ ,  $h' = \gcd(2p, q)$  and  $i \in \{1, 2, 3\}$  depends on the parity of  $p$  and  $q$ .

The existence of orientation reversing isometries in  $\Delta^-(p, q, q)$  and  $\Lambda$  means that the results of [5] do not apply to these groups. But, as above, for  $\Delta(p, q, q)$  and  $\Delta(2p, q, 2)$ , we can appeal to [5, Theorem 5.1]. Thus the maximal finite subgroups of  $\Delta(2p, q, 2)$  up to conjugacy are  $\mathbb{Z}/2p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ , whilst those of  $\Delta(p, q, q) \times \mathbb{Z}/2\mathbb{Z} = \widehat{\Delta(p, q, q)} \times \mathbb{Z}/2\mathbb{Z}$  are  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and two copies of  $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . In particular (noting that  $q \geq 3$ ), these two profinite completions are not isomorphic. Moreover, neither is isomorphic to the profinite completion of  $\Delta^-(p, q, q)$  or  $\Lambda_\rho(p, q)$ , since these last two groups contain finite, non-abelian dihedral groups (using the presentations in §3). Finally, the profinite completions of  $\Delta^-(p, q, q)$  and  $\Lambda_\rho(p, q)$  are different because the abelianization of the former is an elementary 2-group while from above, that of the latter is  $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and  $q \neq 2$ .  $\square$

**COROLLARY 4.4.** *If  $\Delta = \Delta(p, q, r)$  is profinitely rigid in the absolute sense, then so too is any group that contains  $\Delta$  as a subgroup of index 2.*

We highlight a special case of Corollary 4.4.

**COROLLARY 4.5.** *For each of the groups  $\Delta(p, q, r)$  listed in Theorem 1.1, the corresponding Coxeter group  $\Delta^-(p, q, r)$  is profinitely rigid in the absolute sense.*

## 5. Additional finite quotients from character varieties

In this section, we prove Theorem 1.2. The main idea in the proof is to gain control of  $\mathrm{PSL}(2, \mathbb{F})$  quotients in certain situations (here  $\mathbb{F}$  is a finite field). This in turn depends on the (P)  $\mathrm{SL}(2, \mathbb{C})$ -character variety. Recall that for any finitely generated group  $H$ , one has the  $\mathrm{PSL}(2, \mathbb{C})$ -representation variety  $\mathrm{Hom}(H, \mathrm{PSL}(2, \mathbb{C}))$ , and the  $\mathrm{PSL}(2, \mathbb{C})$ -character variety  $Y(H)$  is the algebro-geometric quotient of  $\mathrm{Hom}(H, \mathrm{PSL}(2, \mathbb{C}))$  by the conjugation action in the target. We refer the reader to [3] and [4] for definitions and further details about the  $\mathrm{PSL}(2, \mathbb{C})$ -character variety.

### 5.1. Additional finite quotients

We restate Theorem 1.2 for the convenience of the reader.

**THEOREM 1.2.** *Let  $\Gamma < \mathrm{PSL}(2, \mathbb{C})$  be a finitely generated Kleinian group that has Property FA. Suppose that  $H$  is a finitely generated non-elementary Kleinian group whose  $\mathrm{PSL}(2, \mathbb{C})$ -character variety has an irreducible component of positive dimension containing the character of a discrete faithful representation of  $H$ . Then there exists a finite field  $\mathbb{F}$  and a representation  $H \rightarrow \mathrm{PSL}(2, \mathbb{F})$  whose image is a non-abelian group that is not a quotient group of  $\Gamma$ .*

Before proving this in §5.2, we deduce a corollary of particular interest to us.

**COROLLARY 5.1.** *Let  $\Delta(p, q, r)$  be a Fuchsian triangle group, and  $H$  a non-elementary Fuchsian group that is not a triangle group. Then there exists a finite field  $\mathbb{F}$  and a representation  $H \rightarrow \mathrm{PSL}(2, \mathbb{F})$  whose image is not a quotient group of  $\Delta(p, q, r)$ .*

*Proof.* It is well-known that  $\Delta(p, q, r)$  has Property FA (see [14, Example 6.3.5]). It is also well-known that if  $H$  is any finitely generated Fuchsian group that is not itself a triangle group, then  $Y(H)$  contains a positive dimensional component (the Teichmüller component) containing the characters of faithful discrete representations of  $H$  in  $\mathrm{PSL}(2, \mathbb{C})$  (see [8]) and this proves the corollary.  $\square$

The main theorem of [5] distinguishes  $\Delta(p, q, r)$  from all other finitely generated Fuchsian groups by their finite quotients. The novel content of Corollary 5.1 is that one can find a finite quotient of  $H$  that is not a finite quotient of  $\Delta(p, q, r)$  by looking only at the subgroups of  $\mathrm{PSL}(2, \mathbb{F})$  as  $\mathbb{F}$  ranges over finite fields.

## 5.2. Proof of Theorem 1.2

We begin by explaining how Theorem 1.2 follows from the three lemmas stated and proved below.

First we fix some notation that will be used throughout this section. For  $p$  a prime,  $\mathbb{F}_p$  will denote the finite field with  $p$  elements, and  $\overline{\mathbb{F}}_p$  will be a fixed algebraic closure of  $\mathbb{F}_p$ .

With the character from Lemma 5.3 in hand, Lemma 5.4 shows that for a group  $H$  as in the statement of Theorem 1.2, we can find infinitely many primes  $p$  so that  $H$  has infinitely many conjugacy classes of irreducible representations into  $\mathrm{PSL}(2, \overline{\mathbb{F}}_p)$ . The image of such an irreducible representation is a finite subgroup of  $\mathrm{PSL}(2, \overline{\mathbb{F}}_p)$ . Hence we can construct an infinite sequence of finite quotient groups  $Q_i < \mathrm{PSL}(2, \overline{\mathbb{F}}_p)$  of  $H$  with the property that as  $i \rightarrow \infty$ ,  $Q_i$  contains an element whose trace cannot occur as the trace of an element in  $Q_{i-1}$ . By Lemma 5.2, this must eventually produce a finite group that cannot be a quotient of  $\Gamma$ . Since the representations considered are irreducible, the quotient groups cannot be abelian. Thus we have reduced the proof of Theorem 1.2 to the lemmas that follow.

**LEMMA 5.2.** *Let  $\Gamma$  be as in Theorem 1.2, and  $p$  a prime. Then  $\Gamma$  has only finitely many irreducible representations into  $\mathrm{PSL}(2, \overline{\mathbb{F}}_p)$ , up to conjugacy.*

*Proof.* This result is contained in Bass's work on finite  $n$ -representation type [2], as we shall now explain. [14, Proposition 22, p. 59] proves that if a finitely generated group  $\Gamma$  has Property FA, then in any representation  $\rho: \Gamma \rightarrow \mathrm{GL}(2, k)$  over any field  $k$ , the eigenvalues of  $\rho(\gamma)$  are integral over  $\mathbb{Z}$  for all  $\gamma \in \Gamma$ : in the terminology of [2],  $\Gamma$  has *integral 2-representation type*. [2, Proposition 5.3] and the second remark following it in [2] prove the lemma.  $\square$

**LEMMA 5.3.** *Let  $Y$  be the component defined in Theorem 1.2. Then  $Y$  contains a point corresponding to the character of an irreducible representation.*

*Proof.* Let  $\chi_\rho \in Y$  be the character of a faithful, discrete representation of  $H$ . Since  $Y$  has positive dimension,  $Q = \mathbb{H}^3/\rho(H)$  is non-compact. If  $Q$  has finite volume, it must have at least one cusp cross-section that is a torus or  $S^2$  with four marked points of cone angle  $\pi$ . If  $Q$  has infinite volume, some component of the boundary of the convex core  $C(Q)$  of  $Q$  has positive genus or is a copy of  $S^2$  with at least four cone points and with at least one cone angle less than  $\pi$  (see [3, Chapter 7]). In the former case, we can assume that  $Y$  is the Dehn surgery component, and so the generic point corresponds to the character of a Zariski dense (and hence irreducible) representation. In the latter case we can assume that  $Y$  is a positive

dimensional component that contains characters of faithful, discrete representations of  $H$  that have geometrically finite image; such representations have Zariski dense image and hence are irreducible.  $\square$

LEMMA 5.4. *Let  $H$  be a finitely generated group for which there exists an irreducible component  $Y \subset Y(H)$  of positive dimension containing the character of an irreducible representation of  $H$ . Then for infinitely many primes  $p$ ,  $H$  has infinitely many conjugacy classes of irreducible representations into  $\mathrm{PSL}(2, \overline{\mathbb{F}}_p)$ .*

Some of the ideas used below were motivated by the proofs in [17, § 8].

*Proof.* First, as  $Y$  contains the character of an irreducible representation, any representation corresponding to the character of a generic point must also be irreducible. We now fix a representation  $\rho: H \rightarrow \mathrm{PSL}(2, \mathbb{C})$  whose character is a generic point in  $Y$ . Let  $H_0 = \rho(H)$  be its image, let  $H_1 = \phi^{-1}(H_0)$  be the preimage in  $\mathrm{SL}(2, \mathbb{C})$ , let  $R$  be the subring of  $\mathbb{C}$  generated by the  $\mathbb{Z}$ -span of the traces of  $H_1$ , and let  $K$  be the fraction field of  $R$ . As  $\rho$  is a generic point of the positive dimensional component,  $K/\mathbb{Q}$  has positive transcendence degree.

As described in § 2.1, since  $H_1$  is an irreducible subgroup of  $\mathrm{SL}(2, \mathbb{C})$ , the  $K$ -span of  $H_1$  is a  $K$ -quaternion algebra  $B/K$  with  $B < \mathrm{M}(2, \mathbb{C})$ . Fixing a quadratic extension  $L/K$  which splits  $B$  (if  $B$  is already split, take  $L = K$ ), we have that  $B \otimes_K L \cong \mathrm{M}(2, L)$  and  $H_1 < \mathrm{SL}(2, L)$ . Setting  $S$  to be the subring of  $L$  generated over  $\mathbb{Z}$  by the matrix entries of  $H_1$ , we see that  $H_1 < \mathrm{SL}(2, S)$ , the  $S$ -span of  $H_1$  is  $\mathrm{M}(2, S)$ ,  $R$  is a subring of  $S$ , and the field of fractions of  $S$  is  $L$ . Since  $H_1$  is finitely generated, we see that  $S$  is finitely generated as an algebra over  $\mathbb{Z}$ . Finally, since  $L$  contains  $K$ , we see that  $L/\mathbb{Q}$  has positive transcendence degree.

Any (non-trivial) ring morphism  $\psi: S \rightarrow \overline{\mathbb{F}}_p$  induces a morphism  $\mathrm{M}(2, S) \rightarrow \mathrm{M}(2, \psi(S))$  of  $S$ -modules and a group homomorphism  $\mathrm{SL}(2, S) \rightarrow \mathrm{SL}(2, \overline{\mathbb{F}}_p)$ . As  $H_1$  generates  $\mathrm{M}(2, S)$  as an  $S$ -module, it follows that  $\psi(H_1)$  generates  $\mathrm{M}(2, \psi(S))$  as a  $\psi(S)$ -module. As  $\mathrm{M}(2, \psi(S))$  generates  $\mathrm{M}(2, \overline{\mathbb{F}}_p)$  over  $\overline{\mathbb{F}}_p$ , we see that the  $\overline{\mathbb{F}}_p$ -span of  $\psi(H_1)$  is  $\mathrm{M}(2, \overline{\mathbb{F}}_p)$ . Consequently, the induced representation  $H_1 \rightarrow \mathrm{SL}(2, \overline{\mathbb{F}}_p)$  is irreducible. Finally, as  $R$  is generated by the traces of  $H_1$ , if two homomorphisms of  $S$  to  $\overline{\mathbb{F}}_p$  do not agree on  $R$ , their induced representations of  $H_1$  in  $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$  will have distinct characters and cannot be conjugate.

We will now prove that the inclusion of  $R$  in  $S$  induces a function  $\mathrm{Hom}(S, \overline{\mathbb{F}}_p) \rightarrow \mathrm{Hom}(R, \overline{\mathbb{F}}_p)$  with infinite image. We will first prove that for all but finitely many primes  $\mathrm{Hom}(S, \overline{\mathbb{F}}_p)$  is infinite. We will complete the proof by proving that  $\mathrm{Hom}(S, \overline{\mathbb{F}}_p) \rightarrow \mathrm{Hom}(R, \overline{\mathbb{F}}_p)$  is finite-to-one.

We start by proving that  $\mathrm{Hom}(S, \overline{\mathbb{F}}_p)$  is infinite for all but finitely many primes. As  $S$  is finitely generated over  $\mathbb{Z}$  and the fraction field of  $S$  has positive transcendence degree  $r > 0$ , there exist  $y_1, \dots, y_r \in S$  such that the ring morphism  $\mathbb{Z}[x_1, \dots, x_r] \rightarrow S$  induced by sending  $x_i$  to  $y_i$  is injective and there exists  $n \in \mathbb{Z}$  such that  $S[1/n]$  is a finite ring extension of  $\mathbb{Z}[y_1, \dots, y_r][1/n]$  [15, Tag 07NA]. In particular,  $S[1/n]$  is finitely generated as a module over  $\mathbb{Z}[y_1, \dots, y_r][1/n]$ .

If  $p$  does not divide  $n$ , then the morphism  $\psi: \mathbb{Z}[y_1, \dots, y_r] \rightarrow \overline{\mathbb{F}}_p$  extends uniquely to a morphism  $\mathbb{Z}[y_1, \dots, y_r][1/n] \rightarrow \overline{\mathbb{F}}_p$ . We can tensor the ring map  $\mathbb{Z}[y_1, \dots, y_r][1/n] \rightarrow S[1/n]$  with  $\overline{\mathbb{F}}_p$  along  $\psi$  to get an inclusion  $\overline{\mathbb{F}}_p \rightarrow S[1/n] \otimes \overline{\mathbb{F}}_p$  which gives  $S[1/n] \otimes \overline{\mathbb{F}}_p$  as a finite-dimensional algebra over  $\overline{\mathbb{F}}_p$ . As  $\overline{\mathbb{F}}_p$  is algebraically closed,  $S[1/n] \otimes \overline{\mathbb{F}}_p \cong \prod \overline{\mathbb{F}}_p$  is the product of finitely many copies of  $\overline{\mathbb{F}}_p$ . Finally, we can project onto one of these factors to get a homomorphism  $S \rightarrow S[1/n] \rightarrow S[1/n] \otimes \overline{\mathbb{F}}_p \rightarrow \overline{\mathbb{F}}_p$  which restricts to  $\psi$  on  $\mathbb{Z}[y_1, \dots, y_r]$ . Thus we have shown that each homomorphism of  $\mathbb{Z}[y_1, \dots, y_r]$  to  $\overline{\mathbb{F}}_p$  extends to at least one homomorphism of  $S$  to  $\overline{\mathbb{F}}_p$ . In particular,  $\mathrm{Hom}(S, \overline{\mathbb{F}}_p)$  is infinite.

We complete the proof by showing the induced function  $\mathrm{Hom}(S, \overline{\mathbb{F}}_p) \rightarrow \mathrm{Hom}(R, \overline{\mathbb{F}}_p)$  is finite-to-one. As  $S$  is finitely generated as an  $R$ -module, the inclusion of integral domains  $R \hookrightarrow S$  makes  $S$  a finitely generated  $R$ -algebra. As the extension of fraction fields  $L/K$  is finite, there

exist  $s_1, \dots, s_m \in S$  such that each  $s_i$  is algebraic over  $R$  and  $S = R[s_1, \dots, s_m]$ . Thus, if  $F$  is any field and  $\phi: R \rightarrow F$  is a ring homomorphism, there are at most finitely many distinct homomorphisms  $\phi': S \rightarrow F$  which restrict to  $R$  to give  $\phi$ . Hence, the inclusion of  $R$  in  $S$  induces a function  $\text{Hom}(S, F) \rightarrow \text{Hom}(R, F)$  which is finite-to-one for any field  $F$ .  $\square$

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