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Some remarks on group actions on hyperbolic 3-manifolds

Colin Maclachlan
Department of Mathematical Sciences, University of Aberdeen, Meston Bldg, Aberdeen, AB24 3UE, U.K.

Alan Reid*
Department of Mathematics, Rice University, Houston, TX 77005, USA

Antonio Salgueiro†
Departamento de Matemática, Universidade de Coimbra, Largo D. Dinis, 3000 Coimbra, Portugal

Dedicated to Marston Conder on the occasion of his 65th birthday

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Abstract

We prove that there are infinitely many non-commensurable closed orientable hyperbolic 3-manifolds X, with the property that there are finite groups G_1 and G_2 acting freely by orientation-preserving isometries on X with X/G_1 and X/G_2 isometric, but G_1 and G_2 are not conjugate in $\mathrm{Isom}(X)$. We provide examples where G_1 and G_2 are non-isomorphic, and prove analogous results when G_1 and G_2 act with fixed-points.

Keywords: Hyperbolic manifold, finite group action, arithmetic group.

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E-mail addresses: (Colin Maclachlan), alan.reid@rice.edu (Alan Reid), ams@mat.uc.pt (Antonio Salgueiro)

1 Introduction

The study of finite group actions, both free and with fixed points, on closed Riemannian manifolds has a long and rich history. In the context of low-dimensional geometry and topology, two notable examples of this are Kerckhoff's solution to the Nielsen Realization Problem for surfaces [19] and in the setting of geometric 3-manifolds, it is known that any such action is always conjugate to an isometric action (see [7, 12, 14], and [25]) which formed part of Thurston's geometrization program for 3-manifolds and 3-orbifolds. In this paper we will be concerned with (isometric) finite group actions on hyperbolic manifolds and in particular in dimension 3.

More specifically we will be interested in the following situation: *two finite groups G_1 and G_2 acting freely or with fixed points by (orientation-preserving) isometries on a closed orientable hyperbolic 3-manifold X with $X/G_1 \cong X/G_2$* (here and throughout the symbol $\cong$ in the context of manifolds or orbifolds will denote *isometric*). By volume considerations, it is clear that in this setting, G_1 and G_2 must have the same order, and it is also clear that if G_1 and G_2 are conjugate in $\mathrm{Isom}(X)$ then the quotients will be isometric. The aim of this article is to provide constructions, both general and explicit, of examples of closed orientable hyperbolic 3-manifolds X, and groups G_1 and G_2 *that are not conjugate* in $\mathrm{Isom}(X)$ with $X/G_1 \cong X/G_2$. In fact, our methods provide examples of groups that are not even isomorphic.

Our first result deals with free actions.

Theorem 1.1. *There are infinitely many non-commensurable closed orientable hyperbolic 3-manifolds X, with the property that there are finite groups G_1 and G_2 satisfying:*

(1) *G_1 and G_2 act freely on X by orientation-preserving isometries on X with $X/G_1 \cong X/G_2$.*

(2) *$|G_1| = |G_2|$, but G_1 and G_2 are not conjugate in $\mathrm{Isom}(X)$.*

As mentioned above, since $X/G_1 \cong X/G_2$, it is immediate that $|G_1| = |G_2|$. However, by making the construction explicit we can actually exhibit examples of manifolds X as in Theorem 1.1 for which G_1 is an elementary abelian group of order p^3 (for certain primes p), and G_2 is the non-abelian group of order p^3 containing an element of order p^2 (see §4.1).

Examples of closed orientable hyperbolic 3-manifolds X that are fibered over S^1 admitting free actions by finite groups G_1 and G_2 with $X/G_1 \cong X/G_2$ and for which G_1 and G_2 are not conjugate in $\mathrm{Isom}(X)$ are given [21] (although they are unable to determine whether these examples fall into infinitely many commensurability classes). By focusing on very explicit examples, we can also impose topological conditions on the manifold X and the quotients $X/G_1 \cong X/G_2$, namely we prove the following

Corollary 1.2. (1) *There are infinitely many examples of (commensurable) closed orientable hyperbolic 3-manifolds X that fiber over the circle with the property that there are finite groups G_1 and G_2 as in the conclusion of* Theorem 1.1 *such that $X/G_1 \cong X/G_2$ also fiber over the circle.*

(2) *There is a hyperbolic 3-manifold X which is a rational homology 3-sphere with the property that there are finite groups G_1 and G_2 as in the conclusion of* Theorem 1.1 *such that $X/G_1 \cong X/G_2$ is also a rational homology 3-sphere.*

The finite groups G_1 and G_2 in both cases of Corollary 1.2 are of the type described before Corollary 1.2 (i.e. elementary abelian p-groups and non-abelian p-groups of order p^3 for certain primes p). As far as the authors are aware, the example of Corollary 1.2(2) is the first such example of a hyperbolic rational homology 3-sphere as in the conclusion of Corollary 1.2(2).

By way of contrast, results in [28] and [29] consider the question to what extent G_1 and G_2 acting *with fixed points* must be conjugate in $\mathrm{Isom}(X)$ (for certain closed Riemannian 3-manifolds X not necessarily hyperbolic), and for example [28, Theorem 8 and Proposition 13] provides a uniqueness statement in certain settings (e.g. rational homology 3–spheres and most cyclic group actions).

Our methods also provide a construction when the action is not non-free.

Theorem 1.3. *There are infinitely many non-commensurable closed orientable hyperbolic 3-manifolds X, with the property that there are finite groups G_1 and G_2 satisfying:*

(1) *G_1 and G_2 act by orientation-preserving isometries on X, have non-empty fixed-point set, and with $X/G_1 \cong X/G_2$.*

(2) *$|G_1| = |G_2|$, but G_1 and G_2 are not conjugate in* $\mathrm{Isom}(X)$.

The examples constructed in the proofs of Theorems 1.1, 1.3 and Corollary 1.2 come from the class of arithmetic hyperbolic 3-manifolds (see Section 3 and [23] for further details), and exploit the fact that such manifolds have fundamental groups with large commensurator. The advantages of the arithmetic nature of the construction are first, it provides infinitely many commensurability classes of examples, and second, the groups G_1 and G_2 can be made explicit.

As will be clear, the method of proof of Theorem 1.1 (see Section 2) is very general for arithmetic groups (given a description of maximal groups in the commensurability class), and although our main focus is hyperbolic 3-manifolds, we sketch some variations of Theorem 1.1 in other dimensions; for example, we provide examples of Riemann surfaces which admit actions of distinct finite p-groups with conformally equivalent quotients. Although there is a vast literature on (p-)group actions on Riemann surfaces, we were unable to find results which have precise overlap with ours, although questions of a similar nature have been addressed (see [17, 18] and [20] to name a few). However, we do note that the method of proof of [21] also provides examples of Riemann surfaces with finite groups G_1 and G_2 acting freely on X with X/G_1 conformally equivalent to X/G_2.

More care is needed in generalizing the proof of Theorem 1.3, but we expect that this can also be done.

This paper has its origins in a visit of the third author to the second in the Spring of 2011 whilst the second author held a position at U.T. Austin. The first author sadly passed away in November 2012, and the paper remained stubbornly unfinished since that time. Because of Marston's close personal and mathematical connection to Colin Maclachlan through his visits to the U.K., and Colin's to New Zealand, it seemed an appropriate opportunity to finish the paper, and submit as part of the celebration of Marston's 65th birthday. The second author has no doubts that Colin would have been very pleased with this arrangement, and would have very much enjoyed raising a glass of Scotland's national drink in Marston's honor!

As was remarked upon above, results similar to Theorems 1.1, 1.3 have since appeared in the literature (see [21], [22]), but our methods are very different, and so still seem worthy of publication. The reader will likely note the influence of the first author in this work. However, since the first author was neither able to verify nor influence the final version of the paper, any errors in the paper should be attributed to the other two authors.

2 A basic construction

The basic idea of our construction is contained in Proposition 2.1 below. Although our main focus is in dimension 3, we state it for hyperbolic manifolds of arbitrary dimension. In Section 3 we provide a detailed discussion how to construct examples of hyperbolic 3-manifolds satisfying the hypothesis of Proposition 2.1 using arithmetic techniques in dimension 3, and in Section 7 construct examples in dimension 2. In Section 8 we discuss applying Proposition 2.1 to higher dimensional arithmetic hyperbolic manifolds, but we have decided to only offer a sketch of a proof. The complete proof requires a detailed discussion of maximal arithmetic lattices in this setting.

Recall that if $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ is a lattice, the commensurator of Γ is the group

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom}^+(\mathbb{H}^n) \mid g\Gamma g^{-1} \text{ is commensurable with } \Gamma\}. \tag{2.1}$$

It was proved by Margulis [24], that Γ is an arithmetic lattice if and only if $\mathrm{Comm}(\Gamma)$ is dense in $\mathrm{Isom}^+(\mathbb{H}^n)$. Furthermore, it is known in this case (see [8], [23, Theorem 11.4], and Theorem 3.2 below for dimension 3 and [9] more generally) that there are infinitely many distinct maximal arithmetic lattices commensurable with Γ. On the other hand, if $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ is a non-arithmetic lattice, then $[\mathrm{Comm}(\Gamma) : \Gamma] < \infty$. Moreover, in this case, $\mathrm{Comm}(\Gamma)$ is the unique maximal discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$ containing Γ.

Notation: For $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$, we denote by Γ^+ the subgroup of index at most 2 obtained as $\Gamma \cap \mathrm{Isom}^+(\mathbb{H}^n)$.

Proposition 2.1. *Let $\Gamma_0 \subset \mathrm{Isom}(\mathbb{H}^n)$ be a maximal arithmetic lattice, and let Γ_1 be a normal torsion-free subgroup of finite index in Γ_0 which is contained in Γ_0^+. Assume that there exists $g \in \mathrm{Comm}(\Gamma_0^+) \setminus \Gamma_0^+$ such that $g\Gamma_1 g^{-1} \subset \Gamma_0^+$.*

Then there exists $\Delta \subset \Gamma_0^+$ for which $X = \mathbb{H}^n/\Delta$ satisfies the conclusion of Theorem 1.1.

Proof. Since $g\Gamma_1 g^{-1} \subset \Gamma_0^+$, we deduce, by volume considerations, that $g\Gamma_1 g^{-1}$ has finite index in Γ_0^+ equal to $[\Gamma_0^+ : \Gamma_1]$. By maximality, the normalizer of Γ_1 in $\mathrm{Isom}(\mathbb{H}^n)$ is Γ_0. Note also that $g \notin \Gamma_0$, otherwise $g \in \Gamma_0 \cap \mathrm{Comm}(\Gamma_0^+) \subset \Gamma_0 \cap \mathrm{Isom}^+(\mathbb{H}^n) = \Gamma_0^+$ contradicting the hypothesis on g. It follows that $g\Gamma_1 g^{-1} \neq \Gamma_1$.

Since the subgroup $\Gamma_1 \cap g\Gamma_1 g^{-1}$ has finite index in Γ_0, it contains a finite index subgroup Δ, normal in Γ_0, namely the core (i.e. the intersection of all conjugates of $\Gamma_1 \cap g\Gamma_1 g^{-1}$ in Γ_0). Then $X = \mathbb{H}^n/\Delta$ is a hyperbolic n-manifold which admits free finite group actions by the groups $G_1 = \Gamma_1/\Delta$ and $G_2 = g\Gamma_1 g^{-1}/\Delta$ with quotients $\mathbb{H}^n/\Gamma_1 \cong M \cong \mathbb{H}^n/g\Gamma_1 g^{-1}$. Clearly $|G_1| = |G_2|$ and G_1 and G_2 act by orientation-preserving isometries. Furthermore, $\mathrm{Isom}(X) = \Gamma_0/\Delta$ since Γ_0 is maximal. Now $G_1 \neq G_2$ and G_1 is a normal subgroup of $\mathrm{Isom}(X)$. It follows that G_1 and G_2 cannot be conjugate in $\mathrm{Isom}(X)$. □

Remark 2.2. A version of this Proposition still holds if we assume that Γ_1 is not torsion-free. In this case, to obtain the manifold X as in the conclusion of Theorem 1.3 we must further require that Δ as in the proof, is torsion-free. However, standard methods makes this easy to arrange.

Remark 2.3. In the non-arithmetic setting, the discussion of the commensurator above shows that there is a unique maximal lattice in the commensurability class; i.e. Γ_0 in this case. Hence, if in Proposition 2.1 we were to assume that Γ_0 is non-arithmetic, then there could be no element g as stated. However, a variation of Proposition 2.1 can be formulated:

Let $\Gamma_0 \subset \mathrm{Isom}(\mathbb{H}^n)$ be a maximal non-arithmetic lattice, and $\Gamma < \Gamma_0^+$ a proper torsion-free subgroup of finite index containing a normal subgroup Δ of finite index that is not normal in Γ_0. Suppose that there exists a subgroup Γ_1 with $\Delta \subset \Gamma_1 \subset \Gamma$ and $g \in \Gamma_0$ such that $\Delta \subset g\Gamma_1 g^{-1} \subset \Gamma$ with $G_1 = \Gamma_1/\Delta$ and $G_2 = g\Gamma_1 g^{-1}/\Delta$ not isomorphic. Then the conclusion of Theorem 1.1 holds.

This construction is very much in the spirit of that given in [21]. Indeed, it is likely that "many" of the examples in [21] are non-arithmetic, so the above statement would cover the construction of their examples.

Remark 2.4. We could have defined the commensurator of Γ in $\mathrm{Isom}(\mathbb{H}^n)$. The group $\mathrm{Comm}(\Gamma)$ defined at (2.1) is a subgroup of index at most 2 in this larger group. However, we did not want to constantly keep distinguishing the "orientation-preserving" commensurator and so we retain $\mathrm{Comm}(\Gamma)$ to be as defined at (2.1).

3 Preliminaries on arithmetic hyperbolic 3-manifolds

As mentioned in Section 1, our examples are built using arithmetic hyperbolic manifolds. Here we focus on the case of arithmetic Kleinian groups and arithmetic hyperbolic 3-orbifolds, and recall some relevant results and facts that will be needed (see [23] for further details).

3.1 Arithmetic Kleinian groups

Arithmetic Kleinian groups are obtained as follows. Let k be a number field having exactly one complex place, R_k its ring of integers and B a quaternion algebra over k which ramifies at all real places of k. Let $\rho\colon\ B \to M(2, \mathbb{C})$ be an embedding, $\mathcal{O}$ an order of B, and $\mathcal{O}^1$ the elements of norm one in $\mathcal{O}$. Then $\mathrm{P}\rho(\mathcal{O}^1) \subset \mathrm{PSL}(2, \mathbb{C})$ is a finite co-volume Kleinian group, which is co-compact if and only if B is a division algebra, which in turn is equivalent to B not being isomorphic to $M(2, \mathbb{Q}(\sqrt{-d}))$, where d is a square-free positive integer. Following [8], we denote the group $\mathrm{P}\rho(\mathcal{O}^1)$ by $\Gamma^1_{\mathcal{O}}$.

An *arithmetic Kleinian group* Γ is a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ commensurable with a group $\Gamma^1_{\mathcal{O}}$. In addition, we call $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ arithmetic if it is commensurable with an arithmetic Kleinian group. We call $Q = \mathbb{H}^3/\Gamma$ *arithmetic* if Γ is arithmetic.

The wide (i.e. up to conjugacy) commensurability class of an arithmetic Kleinian group is determined by the isomorphism class of B (see e.g. [23, Theorem 8.4.1]). We can refine this further by noting that if $\mathrm{Ram}_f(B)$ denotes the finite set of prime ideals $\mathcal{P}$ of k where B is ramified, i.e. $B_{\mathcal{P}} = B \otimes_k k_{\mathcal{P}}$ is a division algebra, then the isomorphism class of B (as in the definition of an arithmetic Kleinian group given above) is determined by $\mathrm{Ram}_f(B)$. In particular, using the previous remark, to construct infinitely many commensurability classes of arithmetic hyperbolic 3-manifolds, it is sufficient to fix the field k and vary $\mathrm{Ram}_f(B)$.

Our arguments crucially depend on a fine understanding of maximal arithmetic Kleinian groups defined using maximal and Eichler orders (intersections of maximal orders) in the quaternion algebra B (see [8] or [23, Chapter 11] for more details). We will discuss this further below, however, for convenience, we first provide a "warm-up" version of the general construction that may be useful as a template for the reader to bear in mind. This will construct finite volume non-compact examples that satisfy the conclusion of Theorem 1.1.

3.2 Warm-up construction

The group $\Gamma = \mathrm{PGL}(2, \mathbb{Z}[i])$ is a maximal arithmetic Kleinian group, although it is not maximal in $\mathrm{Isom}(\mathbb{H}^3)$. The maximal group is obtained as the group generated by $< \tau, \mathrm{PGL}(2, \mathbb{Z}[i]) >$ where τ is the reflection on $\mathbb{H}^3$ obtained by extension of complex conjugation on $\mathbb{C}$. We will let this group be denoted by Γ_0. Let $A \subset \mathbb{Z}[i]$ be an ideal, and let

$$\Gamma(A) = \ker\{\phi_A \colon \mathrm{PGL}(2, \mathbb{Z}[i]) \to \mathrm{PGL}(2, \mathbb{Z}[i]/A)\},$$

which will be a subgroup of the Picard group $\mathrm{PSL}(2, \mathbb{Z}[i])$ for most ideals $A \subset \mathbb{Z}[i]$, e.g. those of odd norm.

We will focus on ideals A of the form $p\mathbb{Z}[i]$ where $p \in \mathbb{Z}$ is a prime congruent to $3 \mod 4$, and so the ideal $p\mathbb{Z}[i]$ remains prime in $\mathbb{Z}[i]$. It is easy to check that $\Gamma(p) = \Gamma(p\mathbb{Z}[i])$ is torsion-free. Note that since complex conjugation preserves the ideal $p\mathbb{Z}[i]$, $\Gamma(p)$ is also a normal subgroup of Γ_0.

Now it is a simple matter to check that the element $\sigma_p = \mathrm{P}\begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$ normalizes the subgroup $\Gamma_0(p) = \phi_p^{-1}(B_p)$ where B_p is the group of upper triangular matrices in $\mathrm{PSL}(2, \mathbb{Z}[i]/p\mathbb{Z}[i])$. In particular, $\sigma_p \in \mathrm{Comm}(\mathrm{PGL}(2, \mathbb{Z}[\mathrm{i}]))$. Moreover, and most importantly in our situation, observe that

$$\sigma_p \Gamma(p) \sigma_p^{-1} = \mathrm{P}\{\begin{pmatrix} 1+ap & b \\ cp^2 & 1+dp \end{pmatrix} : a, b, c, d \in \mathbb{Z}[i], \text{ and determinant } 1\}.$$

With this in hand, we can now apply Proposition 2.1 to the groups Γ_0 and $\Gamma_1 = \Gamma(p)$, with $g = \sigma_p$.

From above, the group $g\Gamma_1 g^{-1}$ contains the group $\Delta = \Gamma(p^2)$ which is also normal in Γ_0. Now take $X = \mathbb{H}^3/\Delta$, and the groups G_1 and G_2 to be given by Γ_1/Δ and $g\Gamma_1 g^{-1}/\Delta$ respectively. This finishes the construction of the manifold X as in Theorem 1.1 in this setting.

To get a version of Theorem 1.3, we continue to use the group Γ_0 as above, and tweak the above construction as follows. In this case we choose the ideal $< 1+i >$ and construct the group $\Gamma(1+i)$. Again, since complex conjugation preserves the ideal $< 1+i >$, $\Gamma(1+i)$ is a normal subgroup of Γ_0. However, $\Gamma(1+i)$ is not torsion-free since the element $\mathrm{P}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \Gamma(1+i)$. Thus we cannot take this group to be Γ_1. However, setting $g = \mathrm{P}\begin{pmatrix} 0 & -1/\sqrt{1+i} \\ \sqrt{1+i} & 0 \end{pmatrix}$, the group Γ_1 can be constructed by passing to a torsion-free subgroup of $\Gamma(1+i) \cap g\Gamma(1+i)g^{-1}$, and taking its core in Γ_0. This is the group defining the manifold X in Theorem 1.3. We refer the reader to the end of the proof of Theorem 1.3 in Subsection 6.2 for details of why the groups G_1 and G_2 are not conjugate in this case.

The discussion that follows in Subsections 3.3 and 3.4 provides the necessary generalization of the framework described above that will allow us to pass to the closed case, and thereby prove Theorem 1.1.

Remark 3.1. In the argument above in the case of $\Gamma(1+i)$ and

$$g = \mathrm{P}\begin{pmatrix} 0 & -1/\sqrt{1+i} \\ \sqrt{1+i} & 0 \end{pmatrix},$$

then as in the first case of $\Gamma(p)$, the intersection $\Gamma(1+i) \cap g\Gamma(1+i)g^{-1}$ contains the principal congruence subgroup $\Gamma((1+i)^2) = \Gamma(2)$ which is a normal torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{Z}[i])$ (being the fundamental group of a link complement in S^3 [5]), so in fact, in this case one can take $\Delta = \Gamma(2)$ and so there is no need to pass to any further subgroup of finite index.

3.3 Orders and maximal groups

The argument of Subsection 3.2 provides the template, and so to arrange that the manifolds are closed we use the discussions in Subsection 3.1 to replace $M(2,\mathbb{Q}(i))$ with quaternion division algebras over a number field with one complex place, generalize the groups $\Gamma(p)$ and $\Gamma(1+i)$ using the principal congruence subgroups described in Subsection 3.4 below, and generalize the group $\Gamma_0(p)$ and element σ_p using the theory of Eichler orders and their normalizers as we now describe.

Thus let $\mathcal{O} \subset B$ be a maximal order and $\mathcal{E} \subset \mathcal{O}$ an Eichler order, i.e. the intersection $\mathcal{O} \cap \mathcal{O}'$, for some maximal order $\mathcal{O}' \neq \mathcal{O}$. The Eichler order $\mathcal{E}$ is said to be of *square-free level* S, where S is a finite set of prime ideals of k, disjoint from $\mathrm{Ram}_f(B)$, if, locally at each finite place, $\mathcal{E}_\mathcal{P} = \mathcal{O}_\mathcal{P} = \mathcal{O}'_\mathcal{P}$ if $\mathcal{P} \notin S$ and, if $\mathcal{P} \in S$, then $\mathcal{E}_\mathcal{P} = \mathcal{O}_\mathcal{P} \cap \mathcal{O}'_\mathcal{P}$ has level $\mathcal{P}$ so that $\mathcal{O}_\mathcal{P}$ and $\mathcal{O}'_\mathcal{P}$ are adjacent maximal orders in the tree of maximal orders in $B_\mathcal{P} \cong M_2(k_\mathcal{P})$ (see [23, Chapter 6.5]). When $S = \emptyset$, we simply get the maximal order $\mathcal{O}$.

Indeed, it can always be arranged that for square-free level S and $\mathcal{P} \in S$ we have $\mathcal{E}_\mathcal{P} = \mathcal{T}_\mathcal{P}$ where

$$\mathcal{T}_\mathcal{P} = \left\{ \begin{pmatrix} a & b \\ \pi c & d \end{pmatrix} \mid a,b,c,d \in R_\mathcal{P} \right\}, \tag{3.1}$$

and $R_\mathcal{P} \subset k_\mathcal{P}$ is the valuation ring with uniformizer π (see [23, Chapter 6.5]).

Let $N(\mathcal{E})$ and $N(\mathcal{O})$ denote the normalizers of $\mathcal{E}$ and $\mathcal{O}$ respectively in B^*. Their images, $\mathrm{P}\rho(N(\mathcal{E}))$ and $\mathrm{P}\rho(N(\mathcal{O}))$ in $\mathrm{PGL}(2,\mathbb{C}) \cong \mathrm{PSL}(2,\mathbb{C})$, denoted by $\Gamma_\mathcal{E}$ and $\Gamma_\mathcal{O}$ respectively, are arithmetic Kleinian groups and any arithmetic Kleinian group is conjugate to a subgroup of some such $\Gamma_\mathcal{E}$ or $\Gamma_\mathcal{O}$ (see [8] and [23, Chapter 11.4]).

Note that since conjugation preserves the norm, the groups $\Gamma_\mathcal{O}$ and $\Gamma_\mathcal{E}$ normalize the groups $\Gamma^1_\mathcal{O}$ and $\Gamma^1_\mathcal{E}$ (the image in $\mathrm{PSL}(2,\mathbb{C})$ of $\mathcal{E}^1$ the elements of norm one in $\mathcal{E}$). For convenience we state the following result of Borel [8](see also [23, Chapter 11.4]).

Theorem 3.2. *Fix a maximal order $\mathcal{O} \subset B$. Then there exist infinitely many distinct sets of prime ideals $S_i \subset R_k$ and Eichler orders $\mathcal{E}_i \subset \mathcal{O}$ of level S_i such that $\Gamma_{\mathcal{E}_i}$ are distinct maximal arithmetic Kleinian groups.*

Remark 3.3. We single out two cases of Theorem 3.2 that we will make use of: if k is quadratic imaginary or cubic with one complex place and has class number 1, $\mathcal{P}$ a k-prime ideal and $S = \{\mathcal{P}\}$ then the group $\Gamma_\mathcal{E}$ can be proved to be maximal (see [8] and [23, Chapters 6.7 and 11.4]).

3.4 Principal congruence subgroups

For $\mathcal{O}$ a maximal order, we now describe a distinguished class of subgroups of $\Gamma^1_{\mathcal{O}}$, known as *principal congruence subgroups*. To that end, let $I \subset B$ be a 2-sided integral ideal (see [23, Chapter 6] for further details). Then $I \subset \mathcal{O}$ and $\mathcal{O}/I$ is a finite ring. Define:

$$\mathcal{O}^1(I) = \{\alpha \in \mathcal{O}^1 \mid \alpha - 1 \in I\}.$$

The corresponding *principal congruence subgroup* of $\Gamma^1_{\mathcal{O}}$ is $\Gamma(\mathcal{O}(I)) = \mathrm{P}\rho(\mathcal{O}^1(I))$.

Since I is a 2-sided ideal, it follows that $\Gamma(\mathcal{O}(I))$ is a normal subgroup of finite index in $\Gamma^1_{\mathcal{O}}$. Indeed, we can say more.

Lemma 3.4. *The principal congruence subgroups $\Gamma(\mathcal{O}(I))$ are normal subgroups of $\Gamma_{\mathcal{O}}$. Thus the normalizer of $\Gamma(\mathcal{O}(I))$ in $\mathrm{PSL}(2, \mathbb{C})$ is $\Gamma_{\mathcal{O}}$.*

Proof. The second statement follows from the first since, as noted above, $\Gamma_{\mathcal{O}}$ is maximal. Now let $\alpha \in \mathcal{O}^1(I)$ and $x \in N(\mathcal{O})$. Then $\alpha \in 1 + I$ and $x(1 + I)x^{-1} = 1 + xIx^{-1}$. Now I and $x\mathcal{O}$ are elements of the set of two-sided integral ideals, which is an abelian group (see [23, Chapter 6.7]). This, together with integrality of I gives:

$$xIx^{-1} = (x\mathcal{O})I(\mathcal{O}x^{-1}) = (x\mathcal{O})I(x\mathcal{O})^{-1} = I,$$

and we deduce that $x\alpha x^{-1} \in \mathcal{O}^1(I)$. □

In addition, for most ideals I, the groups $\Gamma(\mathcal{O}(I))$ are torsion-free. We record the following for convenience which needs some additional notation (see [23, Chapters 6.5, 6.6] for details). Apart from a finite set of k-primes $T(I)$, $I_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}}$. For $\mathcal{P} \in T(I)$, we have $I_{\mathcal{P}} = \pi^{n_{\mathcal{P}}}\mathcal{O}_{\mathcal{P}}$ where π is a uniformizer for $k_{\mathcal{P}}$.

Proposition 3.5. *In the notation above, suppose for $\mathcal{P} \in T(I)$ the following holds:*

(1) *$\mathcal{P} \notin \mathrm{Ram}_f(B)$;*

(2) *$\mathcal{P}$ does not ramify in $k \mid \mathbb{Q}$;*

(3) *$\mathcal{P}$ is not dyadic (i.e. $\mathcal{P}$ does not divide 2).*

Then $\Gamma(\mathcal{O}(I))$ is torsion-free.

A group $\Gamma \subset \Gamma^1_{\mathcal{O}}$ is called *a congruence subgroup* if there is some ideal I such that $\Gamma(\mathcal{O}(I)) \subset \Gamma$.

As an example of a congruence subgroup that we will exploit (and the analogue of $\Gamma_0(p)$ in Subsection 3.2), it is shown in [1] that if $\mathcal{E} \subset \mathcal{O}$ is an Eichler order of square-free level S, then $\Gamma^1_{\mathcal{E}} \subset \Gamma^1_{\mathcal{O}}$ is a congruence subgroup. Using (3.1), we can be more explicit. If $\Gamma_{\mathcal{E}}$ is of square-free level S, then for each $\mathcal{P} \notin S$, we have $\mathcal{E}_p = \mathcal{O}_{\mathcal{P}} = M_2(R_{\mathcal{P}})$ and for $\mathcal{P} \in S$ we have $\mathcal{E}_{\mathcal{P}} = \mathcal{T}_{\mathcal{P}}$ where $\mathcal{T}_{\mathcal{P}}$ is as defined at (3.1).

Then in the notation above, defining I locally by $I_{\mathcal{P}} = \pi\mathcal{O}_{\mathcal{P}}$ for each $\mathcal{P} \in S$ and $I_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}}$ otherwise, it follows that, $T(I) = S$. Hence, for $\mathcal{P} \in S$ and $\alpha \in \mathrm{SL}(2, R_{\mathcal{P}})$ is such that $\alpha - 1 \in I_{\mathcal{P}}$ then $\alpha \in \mathcal{T}^1_{\mathcal{P}}$. It now follows that $\Gamma(\mathcal{O}(I)) \subset \Gamma^1_{\mathcal{E}}$.

We will next exploit the feature of having a "large" commensurator which requires an additional piece of terminology.

For a maximal order $\mathcal{O}$ in B, choose an Eichler order $\mathcal{E}$ of square-free level S such that $\Gamma_{\mathcal{E}}$ is maximal (using Theorem 3.2). By maximality, we can find elements $g \in \Gamma_{\mathcal{E}}$ such that $g \notin \Gamma_{\mathcal{O}}$. Call such an element *admissible* and note that $g \in \mathrm{Comm}(\Gamma_{\mathcal{O}})$.

Lemma 3.6. *Suppose that $\Gamma^1_{\mathcal{E}}$ contains a principal congruence subgroup $\Gamma(\mathcal{O}(I))$ and that g is an admissible element of $\Gamma_{\mathcal{E}}$. Then $g\Gamma(\mathcal{O}(I))g^{-1}$ is commensurable with, but distinct from, $\Gamma(\mathcal{O}(I))$.*

Proof. Commensurability is straightforward since $g \in \Gamma_{\mathcal{E}}$. If $g\Gamma(\mathcal{O}(I))g^{-1} = \Gamma(\mathcal{O}(I))$, then $g \in \Gamma_{\mathcal{O}}$ by Lemma 3.4. But this contradicts the admissibility of g. □

4 Proof of Theorem 1.1

We now focus on constructing explicit examples in dimension 3 using the material from Section 3.

4.1 General construction

Let k be a number field with exactly one complex place. It will be convenient in the construction that follows to show that, for certain k, maximal arithmetic subgroups Γ of $\mathrm{Isom}(\mathbb{H}^3)$ are Kleinian groups, i.e. $\Gamma = \Gamma^+$.

Lemma 4.1. *If $[k : \mathbb{Q}]$ is odd or $[k : \mathbb{Q}]$ is even but k has no subfield of index 2, then any maximal arithmetic Kleinian group defined over k will be maximal in $\mathrm{Isom}(\mathbb{H}^3)$.*

Proof. Suppose that Γ is a maximal arithmetic Kleinian group which is properly contained in Γ_0, a maximal discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$. Note that it follows that $[\Gamma_0 : \Gamma] = 2$, otherwise Γ is properly contained in Γ_0^+ which contradicts maximality.

Let $\Gamma^{(2)}$ denote the subgroup of Γ generated by squares of elements of Γ. Then $\Gamma^{(2)}$ is a finite index characteristic subgroup of Γ, and so is normal in Γ_0. Hence the orbifold $\mathbb{H}^3/\Gamma^{(2)}$ admits an orientation-reversing isometry.

By [23, Theorem 8.3.1], the field k can be identified with the field $\mathbb{Q}(\mathrm{tr}\gamma : \gamma \in \Gamma^{(2)})$, and by [27, Proposition 3.4], the existence of the orientation-reversing isometry on $\mathbb{H}^3/\Gamma^{(2)}$ implies that k is stable under complex conjugation. As k is a field with one complex place, it follows that $[k : k \cap \mathbb{R}] = 2$. This contradicts the choice of k. □

Proof of Theorem 1.1: In the light of Lemma 4.1, we now assume that $[k : \mathbb{Q}]$ is odd, and let B be a quaternion algebra defined over k, ramified at all real places. The discussion in Subsection 3.1 immediately implies that all arithmetic hyperbolic 3-manifolds arising from B/k are closed. We make an additional assumption about $\mathrm{Ram}_f(B)$ to ensure torsion-freeness in principal congruence subgroups.

Thus let R denote the finite set of prime ideals $\mathcal{P}$ of k such that, either $\mathcal{P} \in R$ is a dyadic prime of k or, if $p\mathbb{Z} = \mathcal{P} \cap \mathbb{Z}$, then p is ramified in k. We assume that $\mathrm{Ram}_f(B)$ contains R.

Let $\mathcal{O}$ be a maximal order in B so that $\Gamma_{\mathcal{O}}$ is a maximal arithmetic Kleinian group and, by Lemma 4.1, also maximal in $\mathrm{Isom}(\mathbb{H}^3)$. Let $\mathcal{E} \subset \mathcal{O}$ be an Eichler order of square-free level S, chosen so that $\Gamma_{\mathcal{E}}$ is maximal (e.g. as in Theorem 3.2). Then as noted in Subsection 3.4, $\Gamma^1_{\mathcal{E}}$ contains a principal congruence subgroup $\Gamma(\mathcal{O}(I))$ where I is defined by S. By definition S is disjoint from $\mathrm{Ram}_f(B)$ (and so from R), so it follows from Proposition 3.5 that $\Gamma(\mathcal{O}(I))$ is torsion-free.

Choose an admissible element g in $\Gamma_{\mathcal{E}}$ (which as noted in Subsection 3.4 exists and is an element of $\mathrm{Comm}(\Gamma_{\mathcal{O}})$). By Lemma 3.4, $\Gamma(\mathcal{O}(I))$ is a normal subgroup of finite index in $\Gamma_{\mathcal{O}}$. Using the fact (recall Subsection 3.3) that $\Gamma^1_{\mathcal{E}}$ is a normal subgroup of $\Gamma_{\mathcal{E}}$ we deduce:

$$g\Gamma(\mathcal{O}(I))g^{-1} \subset g\Gamma^1_{\mathcal{E}}g^{-1} = \Gamma^1_{\mathcal{E}} \subset \Gamma^1_{\mathcal{O}} \subset \Gamma_{\mathcal{O}}.$$

Furthermore, by Lemma 3.6, $g\Gamma(\mathcal{O}(I))g^{-1} \neq \Gamma(\mathcal{O}(I))$. Now apply Proposition 2.1 with $\Gamma_0 = \Gamma_{\mathcal{O}}$ and $\Gamma_1 = \Gamma(\mathcal{O}(I))$.

By Theorem 3.2 we can choose infinitely many sets S, giving infinitely many examples where Theorem 1.1 holds in a *fixed* commensurability class. From Subsection 3.1, the commensurability class of $\Gamma_{\mathcal{O}}$ is determined by the isomorphism class of B which, in turn, is determined by its ramification set. Hence we have an infinite number of choices of $\mathrm{Ram}_f(B)$ subject to the restriction that $R \subset \mathrm{Ram}_f(B)$, and Theorem 1.1 now follows. □

4.2 Specific examples

We now refine the construction in Subsection 4.1 to provide more specific examples of finite groups. In particular we will be able to gain extra control in the construction of a normal subgroup Δ as in the proof of Proposition 2.1, and this will allow us to get control of G_1 and G_2.

We fix $k = \mathbb{Q}(x)$ where $x^3+x-1 = 0$ so that k has one complex place, has discriminant -31, and has class number 1. Indeed, in what follows, k can be any cubic number field k with one complex place having class number 1. Let B be a quaternion algebra defined over k with $\mathrm{Ram}_f(B) = \{\mathcal{Q}\}$ where the prime ideal $\mathcal{Q}$ does not divide any prime p that splits completely to k (e.g. in this case we can take $\mathcal{Q}$ to be the unique prime dividing 5 of norm 5^3).

Let $p \in \mathbb{Z}$ be an odd prime that splits completely to k: that there are infinitely many such primes p is a well-known consequence of Cebotarev Density theorem (see for example [23, Chapter 0]). Let $\mathcal{P} \subset R_k$ be a prime dividing such a p, and so in particular $N\mathcal{P} = p$.

Define the two-sided integral ideal $I = \mathcal{P}\mathcal{O}$. This can be done locally as follows: For all k-prime ideals $\mathcal{Q}$ let $\mathcal{O}_{\mathcal{Q}} = M_2(R_{\mathcal{Q}})$, and then set $I_J = \mathcal{O}_J$ for all primes $J \neq \mathcal{P}$ and $I_{\mathcal{P}} = \pi\mathcal{O}_{\mathcal{P}}$. The Eichler order $\mathcal{E}$ is then defined locally by $\mathcal{E}_J = \mathcal{O}_J$ for $J \neq \mathcal{P}$ and $\mathcal{E}_{\mathcal{P}} = \mathcal{T}_{\mathcal{P}}$ as at (3.1) so that $S = \{\mathcal{P}\}$ and $\Gamma^1_{\mathcal{E}}$ contains $\Gamma(\mathcal{O}(I))$. Since p is odd and $p \neq 31$, $\mathcal{P}$ is unramified in $k \mid \mathbb{Q}$, and Proposition 3.5 applies to show that $\Gamma(\mathcal{O}(I))$ is torsion-free.

Applying Remark 3.3, the group $\Gamma_{\mathcal{E}}$ is maximal. Indeed, an examination of Borel's construction [8] and [23, Chapters 6.7 and 11.4] provides an element $\alpha \in N(\mathcal{E})$ with $g = \mathrm{P}\rho(\alpha)$ so that locally at $\mathcal{P}$, g acts as conjugation by the element $\begin{pmatrix} 0 & 1 \\ \pi_{\mathcal{P}} & 0 \end{pmatrix}$, that is to say, it acts locally on $\Gamma(\mathcal{O}(I))$ by (cf. the discussion in Subsection 3.2)

$$g\begin{pmatrix} 1+a\pi_{\mathcal{P}} & b\pi_{\mathcal{P}} \\ c\pi_{\mathcal{P}} & 1+d\pi_{\mathcal{P}} \end{pmatrix}g^{-1} = \begin{pmatrix} 1+d\pi_{\mathcal{P}} & c \\ b\pi^2_{\mathcal{P}} & 1+a\pi_{\mathcal{P}} \end{pmatrix}. \tag{4.1}$$

With this in place, we now define a two-sided $\mathcal{O}$-ideal I' by $I'_J = \mathcal{O}_J$ for $J \neq \mathcal{P}$ and $I'_{\mathcal{P}} = \pi^2_{\mathcal{P}}\mathcal{O}_{\mathcal{P}}$ so that $\Gamma(\mathcal{O}(I)) \cap g\Gamma(\mathcal{O}(I))g^{-1} \supset \Gamma(\mathcal{O}(I'))$. Now by Lemma 3.4, $\Gamma(\mathcal{O}(I'))$ is a normal subgroup of $\Gamma_{\mathcal{O}}$. Thus the manifold $X = \mathbb{H}^3/\Gamma(\mathcal{O}(I'))$ admits free actions by the groups $G_1 = \Gamma(\mathcal{O}(I))/\Gamma(\mathcal{O}(I'))$ and $G_2 = g\Gamma(\mathcal{O}(I))g^{-1}/\Gamma(\mathcal{O}(I'))$.

We now identify the groups G_1 and G_2 explicitly. Note first that since B is unramified at $\mathcal{P}$ we have:

$$\Gamma^1_{\mathcal{O}}/\Gamma(\mathcal{O}(I')) \cong \mathrm{PSL}(2, R_{\mathcal{P}}/\pi^2_{\mathcal{P}}R_{\mathcal{P}})$$

and G_1 is the kernel of the reduction homomorphism $\mathrm{PSL}(2, R_{\mathcal{P}}/\pi_{\mathcal{P}}^2 R_{\mathcal{P}}) \to \mathrm{PSL}(2, R_{\mathcal{P}}/\pi_{\mathcal{P}} R_{\mathcal{P}})$. Since $N\mathcal{P} = p$, G_1 is an elementary abelian group of order p^3 generated by the images of the matrices

$$\begin{pmatrix} 1 & \pi_{\mathcal{P}} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pi_{\mathcal{P}} & 1 \end{pmatrix}, \begin{pmatrix} 1+\pi_{\mathcal{P}} & 0 \\ 0 & 1-\pi_{\mathcal{P}} \end{pmatrix}.$$

The group G_2 has the same order p^3 as G_1. To identify G_2, we consider the reduction of $g\Gamma(\mathcal{O}(I))g^{-1}$ locally modulo $\pi_{\mathcal{P}}^2$. From (4.1), the image modulo the ideal $\pi_{\mathcal{P}}^2 R_{\mathcal{P}}$ consists of matrices of the form $\begin{pmatrix} 1+d\pi_{\mathcal{P}} & c \\ 0 & 1+a\pi_{\mathcal{P}} \end{pmatrix}$. From this description it is easy to check that G_2 is non-abelian and contains the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of order p^2. We summarize this discussion as follows.

Corollary 4.2. *For infinitely primes p, we can find a closed hyperbolic 3-manifold X_p such that X_p admits free, orientation-preserving actions by finite groups of isometries, G_1 and G_2 such that*

(1) *G_1 and G_2 are finite groups of order p^3 with $X_p/G_1 \cong X_p/G_2$.*

(2) *G_1 is isomorphic to an elementary abelian group of order p^3, and G_2 is the unique non-abelian group of order p^3 which contains an element of order p^2.*

By varying the prime $\mathcal{Q}$ ramifying the quaternion algebra B/k we also obtain infinitely many commensurability classes of manifolds as in Corollary 4.2. This again follows from an application of the Cebotarev Density theorem which provides infinitely many rational primes whose inertial degrees are greater than 1.

5 A rational homology 3-sphere and fibered examples

In the construction of the examples stated in Corollary 1.2, we will make use of the cubic number field $k = \mathbb{Q}(x)$ where $x^3 - x^2 + 1 = 0$. This has one complex place, discriminant -23, and class number 1. Let B be the quaternion algebra defined over k with $\mathrm{Ram}_f(B) = \{\mathcal{Q}\}$ where $\mathcal{Q}$ is the unique k-prime ideal of norm 5. This determines the commensurability class of the Weeks manifold (see [23, Chapters 4.8.3 and 9.8.2]), which is the smallest volume closed orientable hyperbolic 3-manifold [16].

As we describe below, what is important for us is that if $\mathcal{O} \subset B$ is the unique (up to B^*-conjugacy) maximal order, the group $\Gamma^1_{\mathcal{O}}$ contains subgroups of index 24 that are congruence subgroups of certain levels which are the fundamental groups of a fibered hyperbolic 3-manifold or of a rational homology 3-sphere. In Section 9 we include some Magma [10] computations, which shows, amongst other things, that $\Gamma^1_{\mathcal{O}}$ has 11 conjugacy classes of subgroups of index 24, and unique conjugacy classes with abelianization $\mathbb{Z} \oplus \mathbb{Z}/11\mathbb{Z}$ and $\mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/42\mathbb{Z}$. We will make use of these in what follows. Note that the presentation for $\Gamma^1_{\mathcal{O}}$ that is used in the Magma calculations comes from a description of the orbifold $\mathbb{H}^3/\Gamma^1_{\mathcal{O}}$ as $(3,0)$ Dehn surgery on the knot 5_2 (see [11, Subsection 5.4]).

5.1 Fibered examples

From [11, Proof of Lemma 9.3], the manifold M arising as the double cover of 0-surgery on the knot 6_2 has fundamental group that is a subgroup of $\Gamma^1_{\mathcal{O}}$ of index 24 and has

$H_1(M,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/11\mathbb{Z}$. As we remarked upon above, $\Gamma^1_{\mathcal{O}}$ has a unique such subgroup (up to conjugacy), which we denote by Γ. As shown in Section 9, Magma computes the core C of Γ in $\Gamma^1_{\mathcal{O}}$ and is the kernel of a homomorphism onto $\mathrm{PSL}(2,\mathbb{F}_{23})$, the finite simple group of order 6072. As is also discussed in [11, Subsection 9.1] there are two k-primes of norm 23, one of which is ramified in $k \mid \mathbb{Q}$ (recall the discriminant is -23) and one unramified. Denote these primes by $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively, and these give rise to the 2-sided integral ideals $I_1 = \mathcal{P}_1\mathcal{O}$ and $I_2 = \mathcal{P}_2\mathcal{O}$. Since B is unramified at both of $\mathcal{P}_1$ and $\mathcal{P}_2$, it follows that $\Gamma^1_{\mathcal{O}}/\Gamma(\mathcal{O}(I_j)) \cong \mathrm{PSL}(2,\mathbb{F}_{23})$ for $j = 1, 2$. Putting all of this together we may deduce that $C = \Gamma(\mathcal{O}(I_j))$ for one of $j = 1, 2$. We will not need to explicitly identify which one, and simply denote the relevant subgroup as $\Gamma(\mathcal{O}(I))$. In particular, since $\mathbb{H}^3/\Gamma(\mathcal{O}(I)) \to M = \mathbb{H}^3/\Gamma$ it is fibered over the circle. We now apply Proposition 2.1 together with Lemma 4.1, using an admissible element from $\Gamma_{\mathcal{E}}$ where $\mathcal{E}$ is the Eichler order of level $S = \{\mathcal{P}_j\}$ for $j = 1$ or 2 (using Remark 3.3).

To pass to infinitely many examples, we can use the principal congruence subgroups $\Gamma(\mathcal{O}(I^n))$ for $n \geq 2$ an integer. Arguing as in Subsection 4.2, in particular the local conjugation given by (4.1), shows that $\Gamma(\mathcal{O}(I^{n+1})) \subset \Gamma(\mathcal{O}(I^n)) \cap g\Gamma(\mathcal{O}(I^n))g^{-1}$ and so the existence of infinitely many fibered examples follows since $\mathbb{H}^3/\Gamma(\mathcal{O}(I^n))$ is fibered for all $n \geq 1$.

Remark 5.1. At present we do not know how to produce infinitely many commensurability classes. Given [2] we know that every closed hyperbolic 3-manifold has a finite cover that fibers over the circle. What is needed is that the finite covers can be identified as congruence subgroups (as in the example above). In principal this should be possible, and note the work of Agol and Stover [4] in this direction.

5.2 A rational homology 3-sphere

As can be checked there is a unique k-prime ideal $\mathcal{P}$ of norm 7, which is unramified in B. Taking I to be the two-sided integral ideal $\mathcal{P}\mathcal{O}$ it follows (as in Subsection 4.2) that $\Gamma^1_{\mathcal{O}}/\Gamma(\mathcal{O}(I)) \cong \mathrm{PSL}(2,\mathbb{F}_7)$. In particular, the Magma computations in Section 9 shows that there is a unique index 24 subgroup, `l[9]`, which has core a normal subgroup of index 168 with quotient group being simple. Hence the group `l[9]` is a congruence subgroup of $\Gamma^1_{\mathcal{O}}$ containing $\Gamma(\mathcal{O}(I))$ of index 7. From Section 9, we also find that the abelianization of $\Gamma(\mathcal{O}(I))$ is $\mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/42\mathbb{Z}$. Hence there is a unique homomorphism $\phi : \Gamma(\mathcal{O}(I)) \to \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$. Now arguing as in Subsection 4.2, it follows that $\Gamma(\mathcal{O}(I))/\Gamma(\mathcal{O}(I^2)) \cong \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$ and so we take as the manifold $X = \mathbb{H}^3/\Gamma(\mathcal{O}(I^2))$. Choosing an admissible element $g \in \Gamma_{\mathcal{E}}$ where $\mathcal{E}$ is the Eichler order of level $S = \{\mathcal{P}\}$ (using Remark 3.3), the local conjugation argument given by (4.1) again shows that $\Gamma(\mathcal{O}(I^2)) \subset \Gamma(\mathcal{O}(I)) \cap g\Gamma(\mathcal{O}(I))g^{-1}$. Applying Proposition 2.1 together with Lemma 4.1 and note from Section 9 that X is a rational homology 3-sphere since the abelianization of $\Gamma(\mathcal{O}(I^2))$ (viewed as $\ker\phi$) is still finite (albeit gigantic!). This completes the proof.

6 Proof of Theorem 1.3

Proposition 3.5 shows that the groups $\Gamma(\mathcal{O}(I))$ are typically torsion-free. The proof of Theorem 1.3 will exploit a situation where these groups are *not torsion-free*, and apply Proposition 2.1 (and see Remark 2.2 following it). To construct examples in this setting, we begin with some preliminary discussion that will help locate elements of order 2 in principal congruence subgroups.

6.1 Elements of order 2 in $\Gamma^1_{\mathcal{O}}$

First we recall the following (see [23, Theorem 12.5.4] for details). Let B be a quaternion algebra over a number field k, let $L = k(i)$ and let $\mathcal{O}$ be a maximal order in B. Then $\mathrm{P}(B^1) < \mathrm{PSL}(2,\mathbb{C})$ contains an element of order 2 if and only if $i \notin k$ and no finite place at which B is ramified, splits in $L \mid k$. Moreover, $\Gamma^1_{\mathcal{O}}$ will contain an element of order 2 unless B is unramified at all finite places, and every dyadic place of k splits in $L \mid k$.

Indeed, if we consider $L \hookrightarrow B$ being given as the subfield $k(u) \subset B$ where u is the image of i, then for a suitable choice of maximal order $\mathcal{O}$, u (or simply abusing notation i) will project to an element of order 2 in $\Gamma^1_{\mathcal{O}}$.

6.2 Explicit commensurability classes of examples

We now fix $k = \mathbb{Q}(\sqrt{-2})$, with ring of integers R_k and p a rational prime with $p \equiv 3 \mod 8$. Hence, $pR_k = \mathcal{P}_p\mathcal{P}'_p$. Let B_p/k be the quaternion algebra ramified at exactly the places corresponding to $\mathcal{P}_p$ and $\mathcal{P}'_p$. This can be described explicitly by the Hilbert Symbol (see [23, Chapter 2.1])

$$\left(\frac{-1\,,\,-p}{k}\right).$$

In addition, it can be checked that a maximal order $\mathcal{O}_p$ can be explicitly described as the R_k-submodule of B_p given by $R_k[1, i, (i+j)/2, (1+ij)/2]$.

Given the discussion in Subsection 6.1 with $L = k(i) = \mathbb{Q}(\mathrm{e}^{\pi i/4})$ we see that L embeds in B_p. In addition, R_L embeds in $\mathcal{O}_p$. The reason for this is that it embeds in some maximal order by [15], and since k has class number 1 there is a unique type of maximal order in all of the quaternion algebras B_p.

For convenience we suppress the subscript p in what follows. By construction, $\Gamma^1_{\mathcal{O}}$ has an element of order 2 given as the image of i. Let J_L (resp. J) denote the principal L-ideal (resp. k-ideal) generated by $i-1$ (resp. $\sqrt{-2}$). Notice that $\sqrt{-2} = (1-i)v$ where $v = \frac{(-\sqrt{2}+\sqrt{-2})}{2} \in R^*_L$, so $\sqrt{-2} \in J_L$, from which it follows that $J_L \cap R_k = J$. Note also that 2 is totally ramified in $L \mid k$ (i.e. $2R_L = \mathcal{P}_2^4$ for some ideal $\mathcal{P}_2 \subset R_L$ of norm 2), and $(i-1)R_L = JR_L$. Now take $I \subset \mathcal{O}$ to be the two-sided ideal $J\mathcal{O}$. The previous discussion shows that $i - 1 \in I$, and so i determines an element of order 2 in $\Gamma(\mathcal{O}(I))$.

We also note that as in the discussion in Subsection 3.2, there is an orientation-reversing involution (an extension of complex conjugation in $\mathbb{C}$ to $\mathbb{H}^3$) that normalizes $\Gamma^1_{\mathcal{O}}$. This follows since, by the explicit nature of $\mathcal{O}$, the involution on B given by the extension of complex conjugation preserves $\mathcal{O}$.

Proof of Theorem 1.3: Using the notation above, we see that B is a division algebra, and so $\Gamma^1_{\mathcal{O}}$ is cocompact. Since there are infinitely many primes $p \equiv 3 \mod 8$, there are infinitely many isomorphism classes of quaternion algebras and so as before, Subsection 3.1 provides infinitely many commensurability classes of arithmetic hyperbolic 3-manifolds.

As discussed above, $\Gamma^1_{\mathcal{O}}$ is normalized by an orientation-reversing involution, so that unlike the proof of Theorem 1.1, the group $\Gamma_{\mathcal{O}}$ is maximal in $\mathrm{PSL}(2,\mathbb{C})$ but is not maximal in $\mathrm{Isom}(\mathbb{H}^3)$. Denote the maximal group in $\mathrm{Isom}(\mathbb{H}^3)$ containing $\Gamma_{\mathcal{O}}$ by $G_{\mathcal{O}}$ (so that

$G^+_{\mathcal{O}} = \Gamma_{\mathcal{O}}$). With I as above, the group $\Gamma(\mathcal{O}(I))$ will again play the role of Γ_1. The explicit description given of I and $\mathcal{O}$ implies that $\Gamma(\mathcal{O}(I))$ is also normal in $G_{\mathcal{O}}$.

We now argue as in Subsection 4.2: apply Lemma 3.6 for a suitable choice of Eichler order $\mathcal{E}$; namely the Eichler order of square-free level $\{J\}$, and just as in Subsection 3.4, $\Gamma^1_{\mathcal{E}}$ contains $\Gamma(\mathcal{O}(I))$. Remark 3.3 shows that $\Gamma_{\mathcal{E}}$ is a maximal arithmetic Kleinian group. Choose an admissible element $g \in \Gamma_{\mathcal{E}}$ and apply Proposition 2.1. As in Subsection 4.2, $\Gamma(\mathcal{O}(I^2)) \subset \Gamma(\mathcal{O}(I)) \cap g\Gamma(\mathcal{O}(I))g^{-1}$. This remains normal in $G_{\mathcal{O}}$ since $I^2 = 2\mathcal{O}$. If $\Gamma(\mathcal{O}(I^2))$ is torsion-free we use this group as we did previously. Otherwise we can pass to a torsion-free subgroup and then to its core Δ in $G_{\mathcal{O}}$, as in the proofs of Proposition 2.1 and Theorem 1.1.

With this we have constructed a hyperbolic 3-manifold $X = \mathbb{H}^3/\Delta$ that admits actions by groups of orientation-preserving isometries G_1 and G_2 acting on X with fixed points and with X/G_1 isometric to X/G_2. That G_1 and G_2 are not conjugate subgroups of $\mathrm{Isom}(X)$, or equivalently $\Gamma(\mathcal{O}(I))$ and $g\Gamma(\mathcal{O}(I))g^{-1}$ are not conjugate in $G_{\mathcal{O}}$, follows as in the proof of Theorem 1.1 since $\Gamma(\mathcal{O}(I))$ is normal in $G_{\mathcal{O}}$. This completes the proof. □

Remark 6.1. With reference to the proof of Theorem 1.3, it seems likely that the group $\Gamma(\mathcal{O}(I^2))$ is torsion-free, but we will not pursue this here.

7 Dimension 2

We now discuss a specific analogue of Theorem 1.1 and the arithmetic constructions given in Subsection 4.2 in the context of hyperbolic surfaces, or equivalently, on emphasizing complex structures, Riemann surfaces. We prove the following.

Theorem 7.1. *For infinitely many primes p, there exist compact hyperbolic (Riemann) surfaces X_p with the property that there are finite p-groups $G_{1,p}$ and $G_{2,p}$ which are non-isomorphic, which act freely on X_p with $X/G_{1,p} \cong X/G_{2,p}$ (or equivalently $X/G_{1,p}$ and $X/G_{2,p}$ are conformally equivalent Riemann surfaces).*

Borel's work [8] applies in this setting, and the structure of maximal arithmetic lattices in $\mathrm{Isom}(\mathbb{H}^2)$ is entirely analogous to what is described in Subsections 3.3, 3.4. So, although the proof given below can be carried out in more generality, we believe it is instructive to simply focus on one commensurability class: that given by the indefinite quaternion algebra $B/\mathbb{Q}$ ramified at the primes 2 and 3. If $\mathcal{O} \subset B$ is a maximal order, then it is known that $\Gamma^+_{\mathcal{O}} = \Gamma_{\mathcal{O}} \cap \mathrm{PSL}(2,\mathbb{R})$ is the $(2,4,6)$ Fuchsian triangle group and $\Gamma_{\mathcal{O}}$ is the group generated by reflections in the faces of this triangle (see [23, Chapter 13.3]). Note the difference with the Fuchsian case in comparison to the case of arithmetic Kleinian groups is that $N(\mathcal{O})$ can contain elements of determinant -1, thereby giving elements in $\mathrm{PGL}(2,\mathbb{R})$. However, a version of Remark 3.3 continues to hold in this setting.

Proof. In the notation established above, we let $\Gamma_0 = \Gamma_{\mathcal{O}}$, and for $p \in \mathbb{Z}$ a prime different from $2, 3$ we construct the principal congruence subgroup $\Gamma(\mathcal{O}(I(p)))$ where $I(p)$ is the two-sided integral ideal defined as $p\mathcal{O}$. We now follow the argument in Subsection 4.2: using a version of Remark 3.3, we may build an Eichler order $\mathcal{E}_p$ of level $S = \{p\}$, a maximal group $\Gamma_{\mathcal{E}_p}$, an admissible element g_p, and then follow the argument in Subsection 4.2 to build the required Riemann surface X_p with the free actions of groups G_1 and G_2. That the covering groups G_1 and G_2 are finite p-groups (of order p^3) follows as in Subsection 4.2. □

Example 7.2. We take the case of $p = 5$ in Theorem 7.1. In this case, it can be shown that $\Gamma^1_{\mathcal{O}}$ is a Fuchsian group of signature $(0; 2, 2, 3, 3)$ (see for example [30]) which has co-area $2\pi/3$. By construction, $\Gamma^1_{\mathcal{O}}/\Gamma(\mathcal{O}(I(5))) \cong \mathrm{PSL}(2, 5)$, and using the co-area computed above, determines a 60-fold Riemann surface cover $\Sigma = \mathbb{H}^2/\Gamma(\mathcal{O}(I(5)))$ of $\mathbb{H}^2/\Gamma^1_{\mathcal{O}}$. Hence Σ has genus 7. The surface X_5 we want is then a $(\mathbb{Z}/5\mathbb{Z})^3$ cover of Σ, so of genus 751, which also admits a free action of a non-abelian 5 group of order 5^3 with quotient isometric to Σ.

8 Higher dimensions

As mentioned in Section 2 we will only sketch the proof of an analogous result to Theorem 1.1 in higher dimensions. This will make use of so-called *arithmetic groups of simplest type* (we refer the reader to [26, Subsection 6.8] for a fuller discussion of these arithmetic lattices). For convenience, we restrict to one particular family in each dimension $n \geq 4$, the generalizations will be clear. We emphasize that this is a *sketch of a proof*, and so we will not designate with "Theorem" as some additional discussion of maximal groups is required to give a complete proof:

For each $n \geq 4$, there are infinitely many non-commensurable closed orientable hyperbolic n-manifolds X, with the property that there are finite groups G_1 and G_2 satisfying:

(1) *G_1 and G_2 act freely by orientation-preserving isometries on X with $X/G_1 \cong X/G_2$.*

(2) *$|G_1| = |G_2|$, but G_1 and G_2 are not conjugate in* $\mathrm{Isom}(X)$.

Sketch Proof: We recall some background on certain arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. Let d be a square-free positive integer, and let $f_{n,d}$ denote the quadratic form:

$$f_{n,d} = x_0^2 + x_1^2 + \ldots + x_{n-1}^2 - \sqrt{d}x_n^2.$$

This has signature $(n, 1)$, and after applying the non-trivial Galois automorphism σ given by $\sqrt{d} \mapsto -\sqrt{d}$ the resultant quadratic form $f^\sigma_{n,d}$ has signature $(n+1, 0)$; i.e. the quadratic form $f_{n,d}$ is equivalent over $\mathbb{R}$ to the quadratic form $J_n = x_0^2 + x_1^2 + \ldots + x_{n-1}^2 - x_n^2$, and the quadratic form $f^\sigma_{n,d}$ is equivalent over $\mathbb{R}$ to $x_0^2 + x_1^2 + \ldots + x_{n-1}^2 + x_n^2$.

Let $F_{n,d}$ be the symmetric matrix associated to the quadratic form $f_{n,d}$ and let $\mathrm{O}(f_{n,d})$ (resp. $\mathrm{SO}(f_{n,d})$) denote the linear algebraic groups defined over k described as:

$$\mathrm{O}(f_{n,d}) = \{X \in \mathrm{GL}(n+1, \mathbb{C}) : X^t F_{n,d} X = F_{n,d}\} \text{ and}$$
$$\mathrm{SO}(f_{n,d}) = \{X \in \mathrm{SL}(n+1, \mathbb{C}) : X^t F_{n,d} X = F_{n,d}\}.$$

For a subring $L \subset \mathbb{C}$, we denote the L-points of $\mathrm{O}(f_{n,d})$ (resp. $\mathrm{SO}(f_{n,d})$) by $\mathrm{O}(f_{n,d}, L)$ (resp. $\mathrm{SO}(f_{n,d}, L)$). Let $R_d \subset \mathbb{Q}(\sqrt{d})$ denote the ring of integers.

Note that, given this set-up, there exists $T \in \mathrm{GL}(n+1, \mathbb{R})$ such that $T^{-1}\mathrm{O}(f_{n,d}, \mathbb{R})T = \mathrm{O}(n, 1)$, in which case, $\mathrm{Isom}(\mathbb{H}^n)$ can be identified with the group $\mathrm{O}^+(J_n, \mathbb{R}) = \mathrm{O}^+(n, 1)$, which is the subgroup of $\mathrm{O}(n, 1)$ preserving the upper-half sheet of the hyperboloid $J_n = -1$. A similar discussion holds for $T^{-1}\mathrm{SO}(f_{n,d}, \mathbb{R})T = \mathrm{SO}(n, 1)$ and groups of orientation-preserving isometries. In particular this conjugation provides subgroups $\Lambda \subset \mathrm{O}(f_{n,d}, R_d)$ and $\Lambda^+ \subset \mathrm{SO}(f_{n,d}, R_d)$ whose images lie in $\mathrm{O}^+(n, 1)$ and $\mathrm{SO}^+(n, 1)$ respectively.

A subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ commensurable with the image in $\mathrm{Isom}(\mathbb{H}^n)$ of the subgroup of $\mathrm{O}(f_{n,d}, R_d)$ (under the conjugation map described above) is an example of an arithmetic lattice of simplest type. The corresponding arithmetic hyperbolic n-manifold $M = \mathbb{H}^n/\Gamma$ is also called arithmetic of simplest type. By construction, all the arithmetic lattices of simplest type we have described above are cocompact.

For an ideal $I \subset R_d$, let $\Gamma(I)$ denote the principal congruence subgroup of $\mathrm{O}(f_{n,d}, R_d)$ obtained as the kernel of the homomorphism:

$$\pi_I \colon \mathrm{O}(f_{n,d}, R_d) \to \mathrm{O}(f_{n,d}, R_d/I),$$

and note that so long as I is not a dyadic prime ideal, $\Gamma(I) \subset \mathrm{SO}(f_{n,d}, R_d)$.

We now fix the ideal I to be considered, namely let $p \in \mathbb{Z}$ be an odd prime that is inert to R_d; i.e. pR_d remains a prime ideal which we denote by $\mathcal{P}$. In this case, $\Gamma(\mathcal{P})$ is torsion-free (see [26, Subsection 4.8]), and the groups Λ and $\Gamma(\mathcal{P}) \cap \Lambda^+$ will play the roles of Γ_0 and Γ_1 in Proposition 2.1.

As before, the key point now is a detailed understanding of maximal arithmetic lattices in this setting. As in the case of arithmetic Kleinian groups, these maximal arithmetic groups arise as normalizers of certain number theoretically defined arithmetic lattices (using Bruhat-Tits theory), and are again congruence subgroups (see [9] and also [3] and [6] that deal explicitly with the case of $\mathrm{O}^+(n,1)$). In particular our group Λ is a maximal discrete subgroup of $\mathrm{O}^+(n,1)$ and using the description of maximal discrete groups in the commensurability class of Λ given in [9] it is possible to find an element g as required by Proposition 2.1.

9 Magma calculations

In what follows `g` is the group $\Gamma^1_{\mathcal{O}}$ of Section 5. The presentation was computed using SnapPy [13]. The group Γ from Subsection 5.1 is `l[2]`, and the index 24 subgroup from Subsection 5.2 is `l[9]`. The routine `CosetAction` produces a finite image group `i` and a kernel `k` which is the core of the relevant subgroup. The routine `pQuotient` is used to compute the kernel of a homomorphism a to the elementary abelian 7-group $(\mathbb{Z}/7\mathbb{Z})^3$.

```
> g<a,b>:=Group<a,b|b^3,a^2*b^-1*a^-2*b^-1*a^2*b*a^-1*b>;
> print AbelianQuotientInvariants(g);
[ 3 ]
> l:=LowIndexSubgroups(g,<24,24>);
> print #l;
11
> print AbelianQuotientInvariants(l[1]);
[ 30 ]
> print AbelianQuotientInvariants(l[2]);
[ 11, 0 ]
> print AbelianQuotientInvariants(l[3]);
[ 3, 6, 0 ]
> print AbelianQuotientInvariants(l[4]);
[ 2, 2, 6 ]
> print AbelianQuotientInvariants(l[5]);
[ 5, 30 ]
```

```
> print AbelianQuotientInvariants(l[6]);
[ 3, 6, 6 ]
> print AbelianQuotientInvariants(l[7]);
[ 66 ]
> print AbelianQuotientInvariants(l[8]);
[ 3, 6, 6 ]
> print AbelianQuotientInvariants(l[9]);
[ 7, 42 ]
> print AbelianQuotientInvariants(l[10]);
[ 3, 6, 0 ]
> print AbelianQuotientInvariants(l[11]);
[ 2, 2, 6 ]
> f,i,k:=CosetAction(g,l[2]);
> print Order(i);
6072
> IsSimple(i);
true
> f,i,k:=CosetAction(g,l[9]);
> print Order(i);
168
> IsSimple(i);
true
> print AbelianQuotientInvariants(k);
[ 7, 7, 42 ]
> K:=Rewrite(g,k);
> F,a,b:=pQuotient(K,7,1:Print:=1);
Lower exponent-7 central series for K

Group: K to lower exponent-7 central class 1 has order 7^3
> K1:=Kernel(a);
> print AbelianQuotientInvariants(K1);
[ 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13,
13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13,
13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13,
13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13,
13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13,
13, 13, 13, 13, 13, 13, 13, 13, 2743, 2743, 2743, 2743,
2743, 2743, 2743, 2743, 2743, 2743, 2743, 2743, 2743,
2743, 79547, 79547, 79547, 79547, 79547, 79547, 79547,
79547, 79547, 79547, 79547, 79547, 79547, 79547, 79547,
79547, 79547, 79547, 79547, 79547, 79547, 79547, 79547,
79547, 79547, 79547, 79547, 79547, 79547, 79547, 79547,
79547, 79547, 79547, 79547, 79547, 79547, 79547, 79547,
3897803, 3897803, 23386818 ]
> f,i,k:=CosetAction(g,l[1]);
> print Order(i);
1320
```

```
> f,i,k:=CosetAction(g,l[3]);
> print Order(i);
2204496
> f,i,k:=CosetAction(g,l[4]);
> print Order(i);
504
> f,i,k:=CosetAction(g,l[5]);
> print Order(i);
504
> f,i,k:=CosetAction(g,l[6]);
> print Order(i);
2204496
> f,i,k:=CosetAction(g,l[7]);
> print Order(i);
6072
> IsSimple(i);
true
> a:=AbelianQuotientInvariants(k);
> print a;
[ 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 161,
161, 161, 161, 966, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
> print Multiplicity(a,0);
44
> f,i,k:=CosetAction(g,l[8]);
> print Order(i);
2204496
> f,i,k:=CosetAction(g,l[10]);
> print Order(i);
2204496
> f,i,k:=CosetAction(g,l[11]);
> print Order(i);
504
```

Referring to Subsection 5.1, there is an index 24 subgroup of $\Gamma^1_{\mathcal{O}}$ whose core is the principal congruence subgroup arising from "the other" k-prime of norm 23. This corresponds to the subgroup `l[7]` of the Magma output above. Although this gives rise to a manifold that is also a rational homology 3-sphere, the manifold with fundamental group which is the principal congruence subgroup (i.e. the core) has first Betti number equal to 44 (as shown in the Magma output above), and so does not provide rational homology 3-sphere examples as in Corollary 1.2.

ORCID iDs

Alan Reid https://orcid.org/0000-0003-0367-9453
Antonio Salgueiro https://orcid.org/0000-0002-7290-0708

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