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We study existence and lack thereof of closed, embedded, orientable, codimension one, totally geodesic submanifolds of minimal volume, cusped, orientable, hyperbolic manifolds.

57M50; 57K40

1 Introduction

Let W^n denote a minimal volume, orientable, cusped hyperbolic n–manifold. We will be concerned with the existence of closed, embedded, orientable, totally geodesic hyperbolic submanifolds $M^{n-1} \hookrightarrow W^n$.

When $n = 2$, W^2 is either the thrice-punctured sphere or a once-punctured torus, and in the former case there are no such submanifolds, whilst in the latter there are many. In dimension 3, by work of Cao and Meyerhoff [5], the manifolds in question are the complement of the figure-eight knot and its sister. These do not contain any closed, embedded, orientable, totally geodesic surfaces (although they do contain infinitely many immersed, closed, totally geodesic surfaces; see Maclachlan and Reid [22, Chapter 9]). Indeed, they do not contain any closed, embedded essential surfaces (see Thurston [32], Culler, Jaco and Rubinstein [8] and Floyd and Hatcher [9]). These 3–manifolds are arithmetic hyperbolic 3–manifolds, as is the case for the thrice-punctured sphere. The once-punctured torus has a positive-dimensional Teichmüller space, but there are a finite number of examples which are arithmetic.

This paper is concerned with the following conjecture. As we discuss in Section 2.2, Conjecture 1.1 is easily reduced to the only nontrivial case, that of dimension 4.

Conjecture 1.1 *Let W^n denote a minimal volume, orientable, cusped, arithmetic hyperbolic n–manifold. If W^n contains a codimension one, closed, embedded,*

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orientable, totally geodesic submanifold, then $n = 2$ and W^2 is an arithmetic, once-punctured torus.

In [28], Ratcliffe and Tschantz provide a census of 1171 so-called integral congruence two, hyperbolic 4–manifolds that are all obtained from face-pairings of the ideal 24–cell in $\mathbb{H}^4$. These are all commensurable cusped, arithmetic hyperbolic 4–manifolds of Euler characteristic 1 (ie minimal volume). Amongst these, only 22 are orientable, and these are listed in Section 8 together with some information that we will make use of. Towards a positive resolution of Conjecture 1.1 we prove the following result:

Theorem 1.2 *Let W denote one of the 22 manifolds mentioned above. Then W does not contain a closed, embedded, orientable, totally geodesic hyperbolic 3–manifold.*

As remarked upon, the 1171 integral congruence two, hyperbolic 4–manifolds are all commensurable. In a private communication, J Ratcliffe and S Tschantz have informed us that there are many more manifolds obtained by side-pairings of the ideal 24–cell in this commensurability class, namely 13 108 side-pairings of the ideal 24–cell (up to symmetry of the 24–cell) yield a cusped hyperbolic 4–manifold of Euler characteristic 1. Only 675 of these side-pairings provide orientable manifolds (which include the 22 orientable ones of Theorem 1.2). In addition, Riolo and Slavich [29] show that there is at least one more commensurability class of cusped, arithmetic hyperbolic 4–manifolds that contains an orientable, cusped hyperbolic 4–manifold with Euler characteristic 1.

At present we cannot say anything about Conjecture 1.1 for these other orientable examples, nor do we have a classification of the finite number of commensurability classes of cusped, arithmetic hyperbolic 4–manifolds that contain a manifold with Euler characteristic 1 (although in principle this should be doable).

Our methods also apply to another situation. In [14], Ivanšić provides an example of a cusped, orientable hyperbolic 4–manifold of Euler characteristic 2 that is the complement of five 2–tori in S^4 (with the standard smooth structure; see Ivanšić [15]). This link complement arises as the orientable double cover of the nonorientable manifold 1011 in the census of integral congruence two, hyperbolic 4–manifolds mentioned above (see also [14]). We prove the following result:

Theorem 1.3 *Let W be the link complement in S^4 described above. Then W contains an embedded orientable, totally geodesic, cusped hyperbolic 3–manifold* *isometric to the complement of the link* 8^3_9 (shown in Figure 1) *in* S^3, *but no closed*, *orientable*, *embedded*, *totally geodesic hyperbolic* 3*–manifold.*

A simple but elegant argument (see Ivanšić [13, Proposition 4.10]) shows that if X is a hyperbolic link complement of 2–tori in S^4, then $\chi(X) = \chi(S^4) = 2$, and so there are *only finitely many* hyperbolic link complements of 2–tori in S^4. A similar statement holds more generally for link complements of 2–tori and Klein bottles in other fixed 4–manifolds. Ivanšić, Ratcliffe and Tschantz [16] found four additional examples of link complements of 2–tori in manifolds homeomorphic to S^4. These arise as the orientable double covers of the nonorientable manifolds in the census of [28] with numbers 23, 71, 1091 and 1092. In a forthcoming paper [6], we will address the existence of closed, embedded, totally geodesic hyperbolic 3–manifolds in these examples. This requires additional techniques.

By way of comparison, Thurston's hyperbolization theorem shows many links in S^3 have hyperbolic complements, and although it is known that many hyperbolic link complements in S^3 do not contain a closed, embedded, totally geodesic surface (eg alternating links; see Menasco and Reid [23]), examples do exist (see Leininger [20] and [23]).

Finally, we point out that if one merely asks for a smooth embedding of a closed, orientable 3–manifold into S^4 then there are obstructions; for example, it is a result of Hantzsche [11] that if a closed, orientable 3–manifold M embeds in S^4, then $\mathrm{Tor}(H_1(M,\mathbb{Z})) \cong A \oplus A$ for some finite abelian group A. In fact, the Kirby problem list [18, Question 3.20] asks: *Under what conditions does a closed*, *orientable* 3*–manifold* M *smoothly embed in* S^4? We refer the reader to Budney and Burton [4] for examples, more discussion of this question and additional references.

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2 Cusped, arithmetic hyperbolic manifolds

We will mainly work with the hyperboloid model of $\mathbb{H}^n$ defined using the quadratic form j_n defined as $x_0^2 + x_1^2 + \cdots + x_{n-1}^2 - x_n^2$, ie

$$\mathbb{H}^n = \{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : j_n(x) = -1,\ x_n > 0\}$$

equipped with the Riemannian metric induced from the Lorentzian inner product associated to j_n. The full group of isometries of $\mathbb{H}^n$ is then identified with $\mathrm{O}^+(n,1)$, the subgroup of

$$\mathrm{O}(n,1) = \{A \in \mathrm{GL}(n+1,\mathbb{R}) : A^t J_n A = J_n\}$$

preserving the upper sheet of the hyperboloid $j_n(x) = -1$, and where J_n is the symmetric matrix associated to the quadratic form j_n. The full group of orientation-preserving isometries is given by $\mathrm{SO}^+(n,1) = \{A \in \mathrm{O}^+(n,1) : \det(A) = 1\}$.

2.1 Constructing cusped, arithmetic hyperbolic manifolds

Cusped, arithmetic hyperbolic n–manifolds are constructed as follows (see [33] for example). Suppose that $X = \mathbb{H}^n/\Gamma$ is a finite-volume, cusped hyperbolic n–manifold. Then X is arithmetic if Γ is commensurable with a group $\Lambda < \mathrm{SO}^+(n,1)$ as described below.

Let f be a nondegenerate quadratic form defined over $\mathbb{Q}$ of signature $(n,1)$, which we can assume is diagonal and has integer coefficients. Then f is equivalent over $\mathbb{R}$ to the form j_n defined above; ie there exists $T \in \mathrm{GL}(n+1,\mathbb{R})$ such that $T^t F T = J_n$, where F and J_n denote the symmetric matrices associated to f and j_n, respectively. Then $T^{-1}\mathrm{SO}(f,\mathbb{Z})T \cap \mathrm{SO}^+(n,1)$ defines the arithmetic subgroup $\Lambda < \mathrm{SO}^+(n,1)$.

Note that the form f is anisotropic (ie does not represent 0 nontrivially over $\mathbb{Q}$) if and only if the group Γ is cocompact; otherwise the group Γ is noncocompact (see [3]). By Meyer's theorem [31, Section IV.3.2, Corollary 2], the case that f is anisotropic can only occur when $n = 2, 3$.

2.2 Reducing Conjecture 1.1 to dimension 4

We include a quick proof of the following, likely well-known result. Clearly Theorem 2.1 then reduces Conjecture 1.1 to dimension 4.

Theorem 2.1 *Let $X = \mathbb{H}^n/\Gamma$ be a cusped, arithmetic hyperbolic n–manifold with $n \geq 5$. Then X does not contain any codimension one, immersed, closed, totally geodesic hyperbolic manifold. On the other hand, it contains infinitely many codimension one, immersed, cusped, totally geodesic hyperbolic manifolds.*

Proof The first part now follows easily from the discussion in Section 2.1, since any codimension one, immersed, totally geodesic hyperbolic manifold is arithmetic arising from a quadratic form of signature $(n-1, 1)$. Since $n \geq 5$, such a form is isotropic by Meyer's theorem, and so the submanifold in question is noncompact.

For the last claim we argue as follows. Assume that f is a diagonal form defined over $\mathbb{Q}$ of signature $(n, 1)$ with $n \geq 5$. Then, we can restrict to a subquadratic form f_1 of signature $(n-1, 1)$, which by hypothesis is isotropic. Using Section 2.1 we can use f_1 to build a noncocompact subgroup $H < \Gamma$ with H an arithmetic subgroup of $\mathrm{O}^+(n-1, 1)$. Now use the density of the commensurator of Γ to construct infinitely many such groups H. □

3 Some background from [28]

3.1 Integral congruence two, hyperbolic 4–manifolds

For convenience we now set $J = j_4$. The manifolds of Theorem 1.2 are all obtained by face-pairings of the regular ideal 24–cell in $\mathbb{H}^4$ $\left(\text{with all dihedral angles } \frac{\pi}{2}\right)$, and arise as regular $(\mathbb{Z}/2\mathbb{Z})^4$ covers of the orbifold $\mathbb{H}^4/\Gamma(2)$, where $\Gamma(2)$ is the level two congruence subgroup of the group $\mathrm{O}^+(J, \mathbb{Z}) = \mathrm{O}^+(4, 1) \cap \mathrm{O}(J, \mathbb{Z})$. These are the manifolds referred to as *integral congruence two, hyperbolic 4–manifolds* in [28].

It will be useful to describe the $(\mathbb{Z}/2\mathbb{Z})^4$ action, and this is best described in the ball model as follows. Locate the 24–cell in the ball model of hyperbolic space with vertices

$$(\pm 1, 0, 0, 0), \quad (0, \pm 1, 0, 0), \quad (0, 0, 0 \pm 1, 0), \quad (0, 0, 0, \pm 1), \quad \left(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\right).$$

The four reflections in the coordinate planes of $\mathbb{R}^4$ can be taken as generators of this $(\mathbb{Z}/2\mathbb{Z})^4$ group of isometries. Passing to the hyperboloid model, these reflections are elements of $\Gamma(2)$ and are listed as the first four matrices in [28, page 110]. Following [28], we denote this $(\mathbb{Z}/2\mathbb{Z})^4$ group of isometries by $K < \Gamma(2)$. Note, from Table 2 of [28], only one of the 22 examples under consideration admits a larger group of isometries (of order 48) than that given by K. In particular, none of these 22 manifolds are regular covers of the orbifold $\mathbb{H}^4/\mathrm{O}^+(J, \mathbb{Z})$ (since $[\mathrm{O}^+(J, \mathbb{Z}) : \Gamma(2)] = 120$).

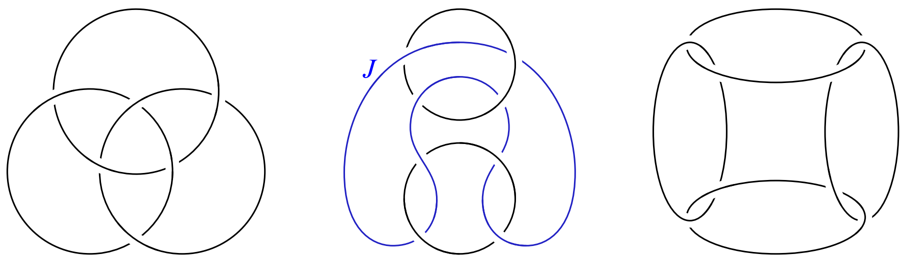

Figure 1: The links 6^3_2, 8^3_9 and 8^4_2.

As noted in [28] (see also [27]) all of the face-pairings of *any* of the integral congruence two, hyperbolic 4–manifolds are invariant under the group K. This implies that each of the coordinate hyperplane cross-sections of the 24–cell extends in each of the integral congruence two, hyperbolic 4–manifolds to a totally geodesic hypersurface which is the fixed-point set of one of the reflections described above. Following [27], we call these hypersurfaces *cross-sections*. As described in [27], these cross-sections can be identified with integral congruence two, hyperbolic 3–manifolds, which are also described in [28]. Moreover, it is possible to use [28] to identify these explicitly in any given example. The following can be deduced from [28] or [27]:

Lemma 3.1 *Any orientable cross-section is isometric to one of the complement in* S^3 *of the link* 6^3_2 (*the Borromean rings*), *the link* 8^3_9 *or the link* 8^4_2 (see Figure 1).

Proof Each of the 3–dimensional cross-sections must be isometric to one of the 107 possibilities encoded in [28, page 115]. However, these 107 are classified into 13 equivalence classes corresponding to isometry classes of the corresponding 3–manifolds [28, Theorem 5], out of which only three are orientable 3–manifolds. The three orientable possibilities are described in [28, pages 108–109] and are the complement in S^3 of the link 6^3_2, the link 8^3_9 or the link 8^4_2. □

3.2 More about the links 6^3_2, 8^3_9 and 8^4_2

Let L denote one of the links 6^3_2, 8^3_9 or 8^4_2. The complements of these links share the same 3–dimensional hyperbolic volume, which is approximately $7.3277247\ldots$.

Lemma 3.2 $S^3 \setminus L$ *does not contain a closed*, *embedded*, *totally geodesic surface.*

Proof It is shown in [21] that the complement of the Borromean rings is small (ie it does not contain any closed, embedded, essential surface), and in particular does not contain a closed, embedded, totally geodesic surface.

The link 8_2^4 is the Montesinos link $K\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ (in the notation of [26]) and so Theorem 1 of [26] implies that any closed, embedded, essential surface in the complement of 8_2^4 arises from tubing the obvious 4–punctured spheres separating pairs of tangles. In particular, such a surface carries an accidental parabolic, and so cannot be totally geodesic. Indeed, in this case, it can be shown that in fact 8_2^4 is small, as these tubed surfaces compress.

Now consider the case of $L = 8_9^3$, and suppose that $S^3 \setminus L$ contains a closed, embedded, totally geodesic surface S. Trivial filling on the component J in Figure 1 provides a split link, and hence S compresses in this filling. In addition, ± 1 fillings yield a manifold homeomorphic to the complement of the Whitehead link, which, being a 2–bridge link, does not contain any closed, embedded, essential surface [12]. Thus S compresses in both ± 1 fillings. Now the surface S is totally geodesic (so does not carry an accidental parabolic element), and so an application of [34, Theorem 1] (following [7]) provides a contradiction since the slopes ± 1 have distance 2. □

4 Codimension one, closed, totally geodesic submanifolds in cusped, arithmetic hyperbolic 4–manifolds

In dimension 3, any cusped, arithmetic hyperbolic 3–manifold contains infinitely many immersed, closed, totally geodesic surfaces (see [22, Chapter 9]). In this section, we show that the situation in dimension 4 is similar, providing a contrast with Theorem 2.1 in dimensions ≥ 5.

4.1 Immersed, closed, totally geodesic hyperbolic 3–manifolds in integral congruence two, hyperbolic 4–manifolds

We first show that the integral congruence two, hyperbolic 4–manifolds of [28] all contain many immersed, closed, totally geodesic hyperbolic 3–manifolds (indeed any manifold in the commensurability class of these integral congruence two, hyperbolic 4–manifolds). To that end, let $p \equiv -1 \bmod 8$ be a prime, and q_p the quadratic form (over $\mathbb{Q}$) given by $x_1^2 + x_2^2 + x_3^2 - p x_4^2$. The congruence condition on p implies that this form is anisotropic, and so, as in Section 2.1, the group $\mathrm{SO}(q_p, \mathbb{Z})$ determines a cocompact arithmetic lattice in $\mathrm{SO}^+(3,1)$.

Proposition 4.1 *With p as above, and given any integral congruence two, hyperbolic 4–manifold N, there is a finite-index subgroup $\Lambda_N < \mathrm{SO}(q_p, \mathbb{Z})$ such that $\mathbb{H}^3/\Lambda_N \hookrightarrow \mathbb{H}^4/N$ is an immersed, closed, totally geodesic hyperbolic 3–manifold. In particular, any such N contains infinitely many commensurability classes of immersed, closed, totally geodesic hyperbolic 3–manifolds.*

Proof The proof of the first claim will follow using standard arguments on equivalences of quadratic forms over $\mathbb{Q}$ yielding commensurable arithmetic lattices. In particular, we need to show that the quadratic form $q_p \perp \langle p\rangle$ is equivalent to the form J of Section 3.1 over $\mathbb{Q}$ (see [1, Sections 5–6], for example). In this case, the equivalence can be seen directly, as follows.

Let $p = 8k-1$, and let

$$A_p = \begin{pmatrix} 4k & 4k-1 \\ 4k-1 & 4k \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad D_p = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}.$$

A simple calculation shows that $A_p D A_p^t = D_p$, and from this the required equivalence can be deduced.

The second part follows from the fact that there are infinitely many primes $\equiv -1 \mod 8$, and, as noted above, all these quadratic forms, being anisotropic over $\mathbb{Q}$, provide closed hyperbolic 3–manifolds. □

4.2 Immersed, closed, totally geodesic hyperbolic 3–manifolds in arithmetic hyperbolic 4–manifolds

In this section we prove that the conclusion of Proposition 4.1 holds much more broadly for cusped, arithmetic hyperbolic 4–manifolds.

Theorem 4.2 *Let W be a cusped, arithmetic hyperbolic 4–manifold. Then W contains infinitely many commensurability classes of immersed, closed, totally geodesic hyperbolic 3–manifolds.*

We begin with some preliminaries on nondegenerate diagonal quaternary quadratic forms

$$f_p = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$$

over the local field $\mathbb{Q}_p$ for p a prime or $p = -1$, with the understanding that $\mathbb{Q}_{-1} = \mathbb{R}$.

Let $c_p(f)$ denote the Hasse–Minkowski invariant, which is defined as

$$c_p(f) = \prod_{i<j} (a_i, a_j)_p, \tag{1}$$

where $(\cdot,\cdot)_p$ denotes the Hilbert symbol. This invariant depends only on the equivalence class of f and not on the choice of orthogonal basis.

We collect some useful statements about Hilbert symbols and quadratic forms over local fields taken from [31]. Throughout, $\left(\frac{u}{p}\right) = (u, p)_p$ denotes the Legendre symbol, which, as in [31, Section II.3.3], is extended to be defined for $u \in \mathbb{Z}_p^*$.

Lemma 4.3 (a) *If $p \neq 2$, the image of the integer $x = p^n u$ is a square in $\mathbb{Q}_p^*$ if and only if n is even and $\left(\frac{u}{p}\right) = 1$* [31, Section II.3.3, Theorem 3, page 17].

(b) *The image of the integer $x = 2^n u$ is a square in $\mathbb{Q}_2^*$ if and only if n is even and $u \equiv 1 \bmod 8$* [31, Section II.3.3, Theorem 4, page 18].

(c) *The Hilbert symbol satisfies the formulas* [31, Section III.1.1, Proposition 2, page 19]

  (i) $(a,b)_p = (b,a)_p$,

  (ii) $(a,b^2)_p = 1$,

  (iii) $(a,-a)_p = 1$.

  (iv) $(-1,-1)_p = \begin{cases} -1 & \text{if } p = -1, 2, \\ 1 & \text{if } p \text{ is odd.} \end{cases}$

(d) *If $a = p^{\alpha} u$ and $b = p^{\beta} v$, then*

$$(a,b)_p = \begin{cases} (-1)^{\alpha\beta\epsilon(p)} \left(\frac{u}{p}\right)^{\beta} \left(\frac{v}{p}\right)^{\alpha} & \text{if } p \neq 2, \\ (-1)^{\epsilon(u)\epsilon(v)+\alpha\omega(v)+\beta\omega(u)} & \text{if } p = 2, \end{cases}$$

where $\epsilon(u)$ denotes the class modulo 2 of $\frac{1}{2}(u-1)$ and $\omega(u)$ denotes the class modulo 2 of $\frac{1}{8}(u^2-1)$ [31, Section III.1.2, Theorem 1, page 20].

(e) *By Dirichlet's theorem, if a and m are relatively prime positive integers, there exist infinitely many primes q such that $q \equiv a \bmod m$; see* [31, Section III.2.2, Lemma 3, page 25].

(f) *A quadratic form f_p over $\mathbb{Q}_p$ is anisotropic if and only if its determinant $d(f)$ is in $(\mathbb{Q}_p^*)^2$ and $c_p(f_p) = -(-1,-1)_p$* [31, Section IV.2.2, Theorem 6, page 36].

(g) *By the Hasse principle, a quadratic form f over $\mathbb{Q}$ is anisotropic if there is some prime p for which the local form f_p over $\mathbb{Q}_p$ is anisotropic* [31, Section IV.3, Theorem 8, page 41].

Proof of Theorem 4.2 Let $W = \mathbb{H}^4/\Lambda$ be such that Λ is an arithmetic group commensurable to a group $\mathrm{SO}(f,\mathbb{Z})$ for some nondegenerate quadratic form f defined over $\mathbb{Q}$ of signature $(4,1)$. According to [25, Theorems 6 and 8], the commensurability class of $\mathrm{SO}(f,\mathbb{Z})$ is uniquely determined by the projective equivalence class of f, which, in turn, is itself determined by an invariant S which is a product of s distinct odd primes, and another invariant $e_{-1}(F)$ (which we will not define here). In our case, because every f has signature $(4,1)$, this invariant is always 2 and so can be ignored. Hence the projective equivalence class of f is completely determined by S.

Using [25, Theorem 10] (actually Claim 2 of the proof of Theorem 10), we may take f to be the diagonal form (with basis $\{e_0,e_1,e_2,e_3,e_4\}$)

$$f = \begin{cases} \langle -1,1,1,aS,a\rangle & \text{if } S \equiv 1 \bmod 4, \\ \langle 1,1,1,aS,-a\rangle & \text{if } S \equiv -1 \bmod 4, \end{cases} \tag{2}$$

where a is an odd prime such that $a \nmid S$, $a \equiv (-1)^s \bmod 4$ if $S \equiv 1 \bmod 4$ or $a \equiv (-1)^{s+1} \bmod 4$ if $S \equiv -1 \bmod 4$, and $\left(\frac{-a}{p}\right) = -1$ for all $p \mid S$. The proof will be completed as a consequence of Lemmas 4.4 and 4.5, stated and proved below. □

Lemma 4.4 *Suppose* $f = \langle -1,1,1,aS,a\rangle$ *with* $S \equiv 1 \bmod 4$ *and* $a \nmid S$, $a \equiv (-1)^s \bmod 4$ *with* $\left(\frac{-a}{p}\right) = -1$ *for all* $p \mid S$ *as in equation* (2)*. Then* f *contains infinitely many projectively inequivalent, anisotropic quadratic subforms over* $\mathbb{Q}$ *of signature* $(3,1)$.

Proof Suppose first that s is even, so, from the description of a given above, $a \equiv 1 \bmod 4$. By Lemma 4.3(e), there exist infinitely many odd primes q such that $q \equiv -S \bmod 8$. Let $u = \left(\frac{1}{2}(q+1)\right)e_0 + \left(\frac{1}{2}(q-1)\right)e_1$, so $f(u) = -q$. Then the diagonal form $f' = \langle -q,1,aS,a\rangle$ is a subform of f with orthogonal basis $\{u,e_2,e_3,e_4\}$.

Over the local field $\mathbb{Q}_2$, the determinant $d(f') = -qa^2S$ is in $(\mathbb{Q}_2^*)^2$ since q was chosen so that $-qS \equiv 1 \bmod 8$ (see Lemma 4.3(b)). Using Lemma 4.3(c), $c_2(f')$ simplifies to $(-q,S)_2(a,-S)_2$. However, since $S \equiv a \equiv 1 \bmod 4$, it follows from Lemma 4.3(d) that $c_2(f') = 1 = -(-1,-1)_2$. Therefore f' is anisotropic over $\mathbb{Q}_2$ and thus also over $\mathbb{Q}$. Since different choices of q yield projectively inequivalent forms, we get infinitely many $\mathbb{Q}$–inequivalent anisotropic quadratic subforms of signature $(3,1)$.

Now suppose that s is odd, so $a \equiv -1 \bmod 4$. Since s is odd and $S \equiv 1 \bmod 4$, we can find a prime $p \mid S$ with $p \equiv 1 \bmod 4$. Let $u = \left(\frac{1}{2}(S+1)\right)e_0 + \left(\frac{1}{2}(S-1)\right)e_1$, so $f(u) = -S$. Let $v = \left(\frac{1}{2}(S-1)\right)e_0 + \left(\frac{1}{2}(S+1)\right)e_1 + me_2$, where m is any positive integer such that p does not divide m. Then $f(v) = S + m^2 = pS' + m^2$ and, by

Lemma 4.3(a), $f(v) \in (\mathbb{Q}_p^*)^2$. Therefore the diagonal form $f' = \langle -S, S+m^2, aS, a\rangle$ is a subform of f with orthogonal basis $\{u, v, e_3, e_4\}$. Over the local field $\mathbb{Q}_p$, the determinant $d(f') = -a^2S^2(S+m^2)$ is in $(\mathbb{Q}_p^*)^2$ since $\left(\frac{-1}{p}\right) = 1$ for $p \equiv 1 \bmod 4$ (see Lemma 4.3(a)). As in the previous case, using Lemma 4.3(c), $c_p(f')$ simplifies to

$$c_p(f') = (-S, aS)_p(-S, a)_p(aS, a)_p = (-S, a)_p(-S, a)_p(-S, a)_p = (-S, a)_p.$$

Now, since $p \mid S$, $p \equiv 1 \bmod 4$ and $\left(\frac{-a}{p}\right) = -1$, Lemma 4.3(d) implies that $(-S, a)_p = -1 = -(-1, -1)_p$. Therefore f' is anisotropic over $\mathbb{Q}_p$ and thus also over $\mathbb{Q}$. Since different choices of m yield projectively inequivalent forms, the conclusion follows as before. □

Lemma 4.5 *Suppose* $f = \langle 1, 1, 1, aS, -a\rangle$ *with* $S \equiv -1 \bmod 4$ *and* $a \nmid S$, $a \equiv (-1)^{s+1} \bmod 4$ *with* $\left(\frac{-a}{p}\right) = -1$ *for all* $p \mid S$ *as in equation* (2). *Then* f *contains infinitely many projectively inequivalent anisotropic quadratic subforms over* $\mathbb{Q}$ *of signature* $(3, 1)$.

Proof Suppose first that s is even, so $a \equiv -1 \bmod 4$. Pick $\alpha > \beta \geq 0$ such that $aS(\alpha^2 S - \beta^2) \equiv -1 \bmod 8$. Such α and β always exist. To see this, if $a \equiv -1 \bmod 8$, let $\alpha = 1$ and $\beta = 0$; if $a \equiv 3 \bmod 8$ and $S \equiv -1 \bmod 8$, let $\alpha = 2$ and $\beta = 0$; and if $a \equiv 3 \bmod 8$ and $S \equiv 3 \bmod 8$, let $\alpha = 3$ and $\beta = 2$. Note also that $aS(\alpha^2 S - \beta^2) > 0$.

Set $u = \beta e_3 + \alpha S e_4$, so that $f(u) = -aS(\alpha^2 S - \beta^2) < 0$. Set $m = -f(u) > 0$. Then the diagonal form $f' = \langle 1, 1, 1, -m\rangle$ is a subform of f with orthogonal basis $\{u, e_2, e_3, e_4\}$. Since $m = -f(u) \equiv -1 \bmod 8$, it is not the sum of three squares and so f' is anisotropic. Since any other choice of α and β congruent to the particular α and β given as examples would still work, and different choices yield infinitely many projectively inequivalent forms, the conclusion follows as before.

Now suppose that s is odd, so $a \equiv 1 \bmod 4$. Fix a prime $p \mid S$ with $p \equiv -1 \bmod 4$. By Lemma 4.3(e), there exist infinitely many primes $q \equiv 1 \bmod 4$ such that $\left(\frac{q}{p}\right) = -1$. Since $q \equiv 1 \bmod 4$, it can be written as a sum of two squares $q = \alpha^2 + \beta^2$. Let $w_1 = \alpha e_1 + \beta e_2$, so $f(w_1) = q$. Consider the diagonal quadratic form $g = \langle 1, q, -p^2\rangle$.

Claim 1 *g represents S over* $\mathbb{Q}$.

Assuming the claim for now, there exists an integer solution $x^2 + qy^2 - p^2z^2 = Sm^2$. Let $w_2 = xe_0 + \beta y e_1 - \alpha y e_2$, so $f(w_2) = Sm^2 + p^2z^2$ and w_2 pairs trivially with w_1. Therefore, the diagonal form $f' = \langle q, Sm^2 + p^2z^2, aS, -a\rangle$ is a subform of f with orthogonal basis $\{w_1, w_2, e_3, e_4\}$. Let $S' = S/p$. Then $f(w_2) = Sm^2 + p^2z^2 = p(S'm^2 + pz^2)$, and, since $S'm^2 + pz^2 \equiv S'm^2 \bmod p$, $f(w_2) \equiv pS' \bmod (\mathbb{Q}_p^*)^2$. Considering f' over $\mathbb{Q}_p$, since $d(f') = -qa^2m^2S(S + p^2z^2)$ and since $\left(\frac{q}{p}\right) = -1$ implies $-q \in (\mathbb{Q}_p^*)^2$ (see Lemma 4.3(a)), we see that $d(f') = -qa^2S(S + p^2z^2) \equiv p^2S'(S' + pz^2) \in (\mathbb{Q}_p^*)^2$. Using Lemma 4.3(c)–(d), $c_p(f')$ simplifies to

$$c_p(f') = -(S' + pz^2, p)_p \cdot (-qS', p)_p \cdot (q, p)_p \cdot (-a, p)_p = -1 = -(-1,-1)_p.$$

Therefore f' is anisotropic over $\mathbb{Q}_p$ and thus also over $\mathbb{Q}$ (see Lemma 4.3(g)). As before, different choices of q yield projectively inequivalent forms, and the conclusion follows as before.

We now prove Claim 1. Since the determinant of g is $-q$ up to squares, by the Hasse principle it suffices to show that g represents S over $k = \mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_q$. By [31, Section IV.2.2, Corollary to Theorem 6], the ternary quadratic form g represents S if $S \neq -q$ in $k^*/(k^*)^2$ or $(-1, q) = c_p(g)$. As $S > 0$ and $-q < 0$, the first case holds over $\mathbb{R}$. As $q \nmid S$, the first case also holds over $\mathbb{Q}_q$. Over $\mathbb{Q}_2$ we have $c_2(g) = (q, -p^2)_2 = (q, -1)_2 = (-1, q)_2$, as required. Therefore g represents S over $\mathbb{Q}$, which proves the claim. □

5 Proof of Theorem 1.2

We begin with a general lemma.

Lemma 5.1 *Let X be an orientable, finite-volume hyperbolic 4–manifold with $\chi(X) = 1$ and containing an embedded, orientable, totally geodesic hyperbolic 3–manifold. Then $b_1(X) > 0$.*

Proof Let $N \hookrightarrow X$ be an embedded, orientable, totally geodesic hyperbolic 3–manifold. Suppose that N separates; then X is decomposed into two finite-volume hyperbolic 4–manifolds with geodesic boundary, whose volumes are proportional to their (integral) Euler characteristic. However, $\chi(X) = 1$, and this is a contradiction. Duality now implies $b_1(X) > 0$. □

Note that the argument in the proof of Lemma 5.1 also proves the following:

Lemma 5.2 *Let X be an orientable, finite-volume hyperbolic 4–manifold with $\chi(X) = 1$ and which contains embedded, orientable, disjoint, totally geodesic hyperbolic 3–manifolds $N_1, N_2, \dots, N_r$. Then $b_1(X) \geq r$.*

Proof The proof of Lemma 5.1 shows that none of the N_i can separate X and, furthermore, also shows that $N_1 \cup \cdots \cup N_r$ cannot separate X. Thus $X \setminus N_1 \cup \cdots \cup N_r$ is connected, and a standard argument now shows that $\pi_1(X)$ surjects a free group of rank r. This proves the lemma. □

Henceforth, throughout this section W is as in the statement of Theorem 1.2 and $M \hookrightarrow W$ is a closed, embedded, orientable, totally geodesic hyperbolic 3–manifold.

Referring to Section 8, since the manifolds labelled 16–22 have first Betti number equal to 0, we can apply Lemma 5.1 to rule out these possibilities for W.

To deal with the remaining 15 possibilities for W, observe that, from Section 8, each of these manifolds admits at least one orientable cross-section.

Lemma 5.3 *M is disjoint from all orientable cross-sections.*

Proof By Lemma 3.1, these cross-sections are all isometric to one of the complements of the links L stated in Lemma 3.1. Suppose that M meets one of the cross-sections; then M must meet $S^3 \setminus L$ in a closed, orientable, embedded, totally geodesic surface. However, this is impossible by Lemma 3.2. □

Since M is disjoint from any orientable cross-section and, from Section 3.1, W is a regular cover of $\mathbb{H}^4/\Gamma(2)$, using the isometries of W induced from the reflections in the coordinate hyperplanes we get at least two disjoint copies of M embedded in W both of which are disjoint from the orientable cross-section, which is itself nonseparating in W (by the proof of Lemma 5.1). Thus we can conclude from Lemma 5.2 that $b_1(W) \geq 3$. Referring to Section 8, we see that this excludes all examples except the first example listed in Section 8. However, in this case, there are three orientable cross-sections, and M is disjoint from all of these by Lemma 5.3. This gives six disjoint embedded copies of M, and so, by Lemma 5.2, we actually have $b_1(W) > 3$. This contradiction completes the proof. □

Remark 5.4 We note here an alternative approach to ruling out the first 15 manifolds of Section 8. Using the three (resp. two) orientable cross-sections in the manifold 1 (resp. manifolds 5 and 6) the six (resp. four) disjoint copies of M embedded in W together with Lemma 5.2 rules out these manifolds. For the manifolds numbered 7–15 of Section 8, where $b_1 = 1$, the two disjoint copies of M embedded in W can be used together with Lemma 5.2 to rule these out. This brings us to the manifolds 2, 3 and 4. In this case, from [27, Table 2], the orientable cross-section is homeomorphic to the complement of 6^3_2 in the case of manifold number 2, and to the complement of 8^3_9 for manifolds 3 and 4. In these cases, that the orientable cross-sections are nonseparating can be seen directly by checking that one of the cusp tori (say T) of the cross-section meets a 3–torus cusp cross-section C of the 4–manifold. The torus T is an embedded nonseparating torus in C and so we can find a dual curve in C that meets T once. It follows that the cross-sections have to be nonseparating.

Remark 5.5 Using the equivalence of the quadratic forms J and $J_7 = x_0^2 + x_1^2 + x_2^2 + 7x_3^2 - 7x_4^2$ given in the proof of Proposition 4.1, one can construct explicit manifolds commensurable with any of the hyperbolic 4–manifolds considered in the proof of Theorem 1.2 containing a closed, embedded, orientable, totally geodesic hyperbolic 3–manifold.

For example, if $\Gamma(49) < \mathrm{O}^+(J, \mathbb{Z})$ denotes the principal congruence subgroup of level 49, then it can be checked that the equivalence described above conjugates $\Gamma(49)$ into a subgroup of the principal congruence subgroup $\Gamma(7) < \mathrm{O}^+(J_7, \mathbb{Z})$. The subform $x_0^2 + x_1^2 + x_2^2 - 7x_4^2$ defines a cocompact subgroup of $\mathrm{O}^+(J_7, \mathbb{Z})$ acting on a hyperbolic 3–space H. Using the reflection in $\mathbb{H}^4$ through H and arguing as in [24], it can be shown that $\mathbb{H}^4/\Gamma(7)$ contains a closed, embedded, orientable, totally geodesic hyperbolic 3–manifold, and hence so does the quotient of $\mathbb{H}^4$ by $\Gamma(49) < \mathrm{O}^+(J, \mathbb{Z})$. The Euler characteristic of $\mathbb{H}^4/\Gamma(49)$ is enormous, exceeding $700\,000$.

6 Volume from tubular neighbourhoods

To prove Theorem 1.3, we will make use of embedded, totally geodesic hyperbolic 3–manifolds in a different way, and, in particular, we will make use of a result of Basmajian [2] which provides disjoint collars about closed, embedded, orientable, totally geodesic hypersurfaces in hyperbolic manifolds. We state this only in the case of interest, namely for hyperbolic 4–manifolds.

Following [2], let $r(x) = \log\coth\left(\frac{1}{2}x\right)$, let $V(r)$ denote the volume of a ball of radius r in $\mathbb{H}^3$. It is noted in [2] that $V(r) = \omega_3 \int_0^r \sinh^2(r)\,dr$, where ω_3 is the area of the unit sphere in $\mathbb{R}^3$ (ie $\omega_3 = 4\pi$).

In [2, pages 213–214], the volume of a tubular neighbourhood of a closed, embedded, orientable, totally geodesic hyperbolic 3–manifold of 3–dimensional hyperbolic volume A in a hyperbolic 4–manifold is given in terms of the 4–dimensional tubular neighbourhood function $c_4(A) = \frac{1}{2}(V \circ r)^{-1}(A)$. Moreover, as noted in [2, Remark 2.1], when the totally geodesic submanifold separates, an improved estimate can be obtained using the tubular neighbourhood function $d_4(A) = \frac{1}{2}(V \circ r)^{-1}\big(\frac{1}{2}A\big)$ and we record this as follows:

Lemma 6.1 *Let X be an orientable, finite-volume hyperbolic 4–manifold containing a closed, embedded, separating, orientable, totally geodesic hyperbolic 3–manifold of 3–dimensional hyperbolic volume A. Then X contains a tubular neighbourhood of M of volume*

$$\mathcal{V}'(A) = 2A \int_0^{d_4(A)} \cosh^3(t)\, dt.$$

Moreover, Basmajian [2] also proves that disjoint embedded, closed, orientable, totally geodesic hyperbolic 3–manifolds in an orientable, finite-volume hyperbolic 4–manifold have disjoint collars, thereby contributing additional volume. For our purposes we summarize what we need in the following:

Corollary 6.2 *Let X be an orientable, finite-volume hyperbolic 4–manifold of Euler characteristic χ containing K disjoint copies of a closed, embedded, orientable, totally geodesic hyperbolic 3–manifold of 3–dimensional hyperbolic volume A. Assume that all of these disjoint copies separate X. Then*

$$\mathrm{Vol}(X) = \tfrac{4}{3}\pi^2\chi \geq K\mathcal{V}'(A).$$

7 Proof of Theorem 1.3

Let W be as in the statement of Theorem 1.3, and suppose that M is a closed, embedded, totally geodesic hyperbolic 3–manifold in W. Since $M \subset W \subset S^4$, M is orientable and separates W. This will allow us to use the formula for the volume of a tubular neighbourhood given in Corollary 6.2.

Let N be the nonorientable manifold 1011 in the census of [28] with $W \to N$ the orientable double cover, and L denote the link 8^3_9. Note that, again by the construction of N in [28], the manifold N is a regular cover of $\mathbb{H}^4/\Gamma(2)$ with covering group K.

Lemma 7.1 *In the case of the manifold N, each of the four cross-sections is isometric to $S^3 \setminus L$.*

Proof In [28, Table 3], the manifold 1011 is given by the code `14FF28`, which represents the side pairing `11114444FFFFFFFF22228888` for the 24 sides of the ideal 24–cell Q^4. In the notation of [28], the four cross-sections have $k_1k_5k_9$ codes

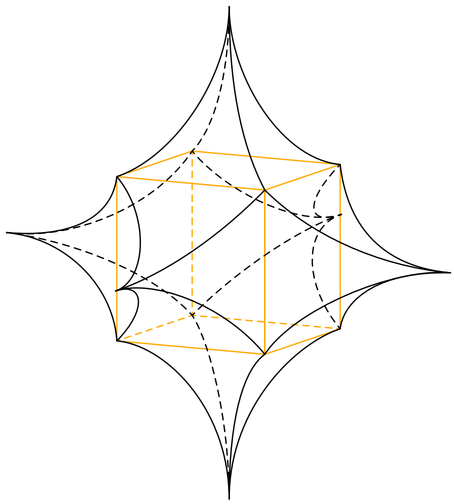

Figure 2: The polytope Q^3.

714, 274, 172 and 147, which correspond to the side pairings $r_i k_i$ for the 12 sides of the polytope Q^3, where r_i is the reflection on side i and $k_1 = k_2 = k_3 = k_4$, $k_5 = k_6 = k_7 = k_8$ and $k_9 = k_{10} = k_{11} = k_{12}$. Since r_i is a reflection, the side pairing $r_i k_i$ is orientation-preserving if and only if the corresponding k_i is orientation-reversing. But this happens only if $k_i \in \{1, 2, 4, 7\}$, since then it corresponds to the diagonal matrices with $1 \leftrightarrow \mathrm{diag}(-1, 1, 1, 1)$, $2 \leftrightarrow \mathrm{diag}(1, -1, 1, 1)$, $4 \leftrightarrow \mathrm{diag}(1, 1, -1, 1)$ and $7 \leftrightarrow \mathrm{diag}(-1, -1, -1, 1)$. Therefore, all four cross-sections of N are orientable.

In [28, Table 1], we see that the code 147 corresponds to the integral congruence two 3–manifold M_2^3 of [28] and which is isometric to the link complement $S^3 \setminus L$ [28, page 108]. The other three codes 714, 274 and 172 do not appear in [28, Table 1]. However, as we briefly describe below, these are equivalent up to symmetries of Q^3 to 147.

First, the symmetries of Q^3 are identified with the symmetries of the cube whose vertices are the actual vertices of Q^3 (see Figure 2). Using this identification, the codes 714 and 274 are equivalent to 147 via a rotation by π along axes between the midpoints of opposite edges of the cube, and the code 172 is equivalent to 147 via a rotation by π along an axis between the centres of two opposite faces. □

Lemma 7.2 (1) *W is a regular cover of $\mathbb{H}^4/\Gamma(2)$.*

(2) *The lift of any cross-section of N to W consists of two embedded totally geodesic copies of $S^3 \setminus L$.*

(3) *M is disjoint from all such lifts.*

Proof For (1) we note that W is the orientable double cover of N; as such, it is a characteristic cover of N. Now N is a regular cover of $\mathbb{H}^4/\Gamma(2)$, hence W is a regular cover of $\mathbb{H}^4/\Gamma(2)$.

For (2), we have from Lemma 7.1 that all cross-sections of N are isometric to the link complement $S^3 \setminus L$. Being orientable, these must lift to two copies in the orientable double cover.

For (3), we argue as in the proof of Lemma 5.3. □

The first part of Theorem 1.3 now follows from Lemma 7.2(2).

For the second part, note that from Lemma 7.2(3), (1), since M is disjoint from all lifts of the cross-sections, and W is a regular cover of $\mathbb{H}^4/\Gamma(2)$, using the isometries of W induced from the reflections in the coordinate hyperplanes, we get 16 disjoint copies of M, all embedded and separating in W. Now the minimal volume of a closed hyperbolic 3–manifold is that of the Weeks manifold and is approximately $0.9427\ldots$ [10]. Using this estimate for $\mathrm{Vol}(M)$ and applying Corollary 6.2, we see that $\mathrm{Vol}(W) \geq 16\mathcal{V}'(0.94)$, which is approximately 28.9. On the other hand, since $\chi(W) = 2$, $\mathrm{Vol}(W) = \frac{8}{3}\pi^2$, which is approximately 26.3, a contradiction. □

Remark 7.3 In [30], an investigation of finite-volume hyperbolic link complements of 2–tori and Klein bottles in other smooth, closed, simply connected 4–manifolds was initiated. Amongst other things, this work provided restrictions on the simply connected manifolds that can admit such link complements; namely they can only be homeomorphic to S^4, $\#_r(S^2 \times S^2)$ or $\#_r(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$ with $r > 0$. Furthermore, using the examples of [14], examples of link complements of 2–tori in $\#_r(S^2 \times S^2)$ for r even were exhibited in [30] (these cover the manifold W above). Other examples of link complements of 2–tori and Klein bottles in closed, simply connected manifolds are also given in [16].

Note that for the example W of [14] considered in Theorem 1.3, it is shown in [15] that the link complement is in S^4 with the standard smooth structure.

Remark 7.4 Every closed, orientable 3–manifold embeds in $\#_r(S^2 \times S^2)$ for some $r > 0$ (see [17, Chapter VII, Theorem 4]). On the other hand, we do not know whether the link complements of 2–tori in $\#_r(S^2 \times S^2)$ that cover W mentioned in Remark 7.3 contain a closed, embedded, totally geodesic hyperbolic 3–manifold.

Motivated by the results of this paper, these remarks and recent work on embedding (arithmetic) hyperbolic manifolds as codimension one, totally geodesic submanifolds (see [19]) we pose the following questions:

Question 7.5 *Is there a cusped, orientable, finite-volume hyperbolic 4–manifold W with $\chi(W) = 1$ (or 2) which contains a closed, embedded, orientable, totally geodesic hyperbolic 3–manifold? If not, what is the minimal Euler characteristic of such a hyperbolic 4–manifold?*

Question 7.6 *Do any of the link complements of 2–tori in $\#_r(S^2 \times S^2)$ that cover W mentioned in Remark 7.3 contain a closed, embedded, orientable, totally geodesic hyperbolic 3–manifold?*

Question 7.7 *Does there exist a finite-volume hyperbolic link complement of 2–tori and Klein bottles in $\#_r(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$ for some $r > 0$?*

8 The orientable integral congruence two, hyperbolic 4–manifolds

Table 1 is composed from data in [27, Table 2; 28, Table 2].

number	b_1	# of orientable cross-sections	number	b_1	# of orientable cross-sections
1	3	3	12	1	1
2	2	1	13	1	1
3	2	1	14	1	1
4	2	1	15	1	1
5	2	2	16	0	0
6	2	2	17	0	0
7	1	1	18	0	0
8	1	1	19	0	0
9	1	1	20	0	0
10	1	1	21	0	0
11	1	1	22	0	0

Table 1: The 22 orientable integral congruence two, hyperbolic 4–manifolds of [28].

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