373K Algebra I, Homework 12

From Artin

Chapter 11 (pages 355–358) 3.8, 3.11, 6.8(a)–(c), 7.3.

Others:

1. Let R be a commutative ring and $\mathcal{F}(R)$ be the set of all functions $R \to R$ with pointwise addition and multiplication.

(a) Show that $\mathcal{F}(R)$ is a commutative ring.

(b) Show that $\mathcal{F}(R)$ is not an integral domain.

(c) How many elements does $\mathcal{F}(\mathbf{F}_2)$ have?

(d) Show that R is isomorphic to the subring of $\mathcal{F}(R)$ consisting of all the constant functions.

2. Let R be a commutative ring. Show that the function $\epsilon : R[x] \to R$ defined by

$$\epsilon(a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0) = a_0$$

is a ring homomorphism. Describe $\ker \epsilon$.

3. Let $f : R \to S$ be a ring homomorphism.

(a) If Q is a prime ideal in S, prove that $f^{-1}(Q)$ is a prime ideal of R.

(b) Give an example to show that if $P \subset R$ is a prime ideal, $f(P)$ need not be prime.

4. Prove that if P is a prime ideal in a commutative ring and $r^n \in P$ for some $r \in R$ and $n \geq 1$, then $r \in P$.

5. Let R be the ring of continuous functions on $[0, 1]$.

(a) Let $M_c = \{f \in R : f(c) = 0\}$. Prove that M_c is a maximal ideal.

(b) Let I be the subset of R consisting of those functions $f(x)$ with $f(1/3) = f(1/2) = 0$. Prove that I is an ideal but it is not a prime ideal.

6. Prove that the ideal $< 2, x > \subset \mathbf{Z}[x]$ is not principal.

7.(a) Prove that $< 2 + i > \subset \mathbf{Z}[i]$ is a prime ideal.

(b) Prove that the ideal $< 3, 2 + \sqrt{-5} > \subset \mathbf{Z}[\sqrt{-5}]$ is not principal.

Sample Midterm 2 Questions

1. Let p be a prime and let P be a group of order p^a.

(a) Prove that $Z(P)$ is non-trivial.

(b) If H is a non-trivial normal subgroup then $H \cap Z(P) \neq 1$.

(c) Deduce that if H is a normal subgroup of order p, then $H < Z(P)$.

2. Let G be a group of order of 315.

(a) Show that the Sylow 7-subgroup is normal.

(b) Assume that a Sylow 3-subgroup is normal. Prove that $Z(G)$ contains a Sylow 3-subgroup and deduce that G is abelian.

3. Answer the following **True** or **False**. You must prove or give counter-examples.

(a) Let G be a group of order $17^6.101^4.97^2$. G contains a subgroup of order 101^4.

(b) There is a non-abelian group of order 19^2.

(c) Let $G = S_3 \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$. A Sylow 3-subgroup is normal.

5. Let R be a ring, and I and J ideals of R. Let

$$I + J = \{a + b : a \in I, b \in J\}$$

$$IJ = \{\Sigma ab : a \in I, b \in J, \quad \text{where all sums are finite}\}.$$

Prove that $I + J$ and IJ are ideals in R.

6. A commutative ring R is called a *local ring*, if it has a unique (proper) maximal ideal.

(a) Prove that the ring of rational numbers whose denominators are odd is a local ring whose unique maximal ideal is the principal ideal generated by 2.

(b) Prove that if R is a local ring with unique maximal ideal M, then every element in $R \setminus M$ is a unit.

(c) Show that if R is a commutative ring with 1 in which the set of non-units forms an ideal M, then R is a local ring with unique maximal ideal M.

7. An integral domain R is called a *a Principal Ideal Domain (P.I.D)* if all ideals are principal.

(a) Give an example of an integral domain that is not a P.I.D.

(b) Prove that the quotient of a P.I.D by a prime ideal is a P.I.D.

(c) Prove that if R is a commutative ring with 1 so that $R[x]$ is a P.I.D. then, R is a field.