373K Algebra I, Homework 2

From Artin

Chapter 2 (pp. 69–71): 2.1, 2.4, 4.1, 4.3, 4.4, 4.7, 6.1, 6.3.

Others:

1. Prove that if G is a group and $|G| < 6$ then G is abelian.

2. How many elements of order 2 in S_5, and in S_6. Any guesses for how many in S_n for arbitrary n?

3. If G is a finite group prove that there exists a positive integer N such that $a^N = 1$ for all $a \in G$.

4. Prove that a finite group of even order contains an element of order 2.

5. Show that any group G in which every non-trivial element has order 2 is abelian.

6. Let G be a group and $H < G$. Define a relation $\sim$ on G by $a \sim b$ if $ab^{-1} \in H$. Prove that $\sim$ is an equivalence relation on G.

7. Let $(G_1, \cdot_1)$ and $(G_2, \cdot_2)$ be groups. Prove that $G_1 \times G_2$ is a group under the following operation:

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 \cdot_1 b_1, a_2 \cdot_2 b_2)$$

$\forall a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$.

8. Prove that if G is a group and $H, K < G$ then $H \cap K$ is a subgroup of G.

9. If G is a group define the *center* of G to be:

$$\{x \in G : xg = gx \text{ for all } g \in G\}.$$

(i) Prove that the center of G is a subgroup of G (typically denoted $Z(G)$).

(ii) What is the center of an abelian group?

(iii) What is the center of S_4?

Additional Question(tricky) If G is a group in which $(ab)^i = a^i b^i$ for 3 consecutive integers i and for all $a, b \in G$, show that G is abelian.