373K Algebra I, Homework 4

From Artin

Chapter 2 (pp. 71–74): 5.4, 6.6, 6.7, 6.8, 6.11, 8.5, 8.7, 8.10.

Others:

1. Let G be a group and $S \subset G$ a non-empty subset. Define the *centralizer* of S in G by:

$$C_G(S) = \{g \in G : gs = sg \ \ \forall s \in S\}.$$

(i) Prove that $C_S(G) < G$.

(ii) What is $C_G(G)$?

(iii) What is $C_{S_4}((1,2,3,4))$?

2. Let G be a group and $H < G$. Define the *normalizer* of H in G by:

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

(i) Prove that $N_G(H) < G$ and H is normal in $N_G(H)$.

(ii) Let $H =< (1,2) >< S_3$. What is $N_G(H)$?

3. Let $H < G$, show that if either of the following hold, then H is a normal subgroup of G.

(i) $|H| < \infty$ and H is the only subgroup of G of order $|H|$.

(ii) $[G : H] = m$ and H is the only subgroup of index m in G.

4. Prove that a group G is abelian if and only if the function $f : G \to G$ give by $f(a) = a^{-1}$ is a homomorphism.

5. (i) Suppose that $n \geq 2$. Prove that $\mathrm{SL}(2, \mathbf{Z}/\mathrm{n}\mathbf{Z})$ is a non-abelian group under matrix multiplication.

(ii) Compute the order of $\mathrm{SL}(2, \mathbf{Z}/2\mathbf{Z})$ and identify (via an isomorphism) this group from the groups that we have encountered.

(iii) What is $Z(\mathrm{SL}(2, \mathbf{Z}/\mathrm{n}\mathbf{Z}))$?

(iv) Let $\phi_n : \mathrm{SL}(2, \mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}/\mathrm{n}\mathbf{Z})$ be the map:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} [a] & [b] \\ [c] & [d] \end{pmatrix}$$

(i.e. each entry is sent to its congruence class modulo n). Prove that ϕ_n is a homomorphism.

(v) Prove that for every non-trivial element $g \in \mathrm{SL}(2, \mathbf{Z})$ there exists n so that $\phi_n(g) \neq Id$ (this expresses the important fact that $\mathrm{SL}(2, \mathbf{Z})$ is what is termed *Residually Finite*).