## 373K Algebra I, Homework 6

**From Artin**

Chapter 2 (p. 74–75): 10.2, 10.4, 11.7, 12.2, 12.3.

**Others:**

**1.** Let $G$ be a group, show that $G/Z(G)$ is never isomorphic to $Q$ (the quaternion group of order 8).

**2.** Let $G$ be an abelian group with $|G| = p^n$ for some prime $p$. Show that if $G/H$ is cyclic for every $H \neq 1$, then $G$ is cyclic or $|G| = p^2$ (**Tricky**).

**3.** A group $G$ is called *solvable* if there is a chain of subgroups:

$$1 = G_0 < G_1 < \dots G_k = G$$

such that $G_i$ is normal in $G_{i+1}$ and $G_{i+1}/G_i$ is abelian.

Prove that $\mathrm{PSL}(2, \mathbf{Z}/3\mathbf{Z})$ is solvable.

**4.** Referring to Q3, suppose that $G$ is a group of order $p^n$ were $p$ is a prime. Prove that $G$ is solvable.

**5.** Referring to Q3, prove that a quotient group of solvable group is solvable.

### Sample Midterm 1 Questions

**1.** Let $G$ be a group and for each $g \in G$ define $\phi_g : G \to G$ by $\phi_g(x) = gxg^{-1}$. Define $\mathrm{Inn(G)} = \{\phi_\mathrm{g} : \mathrm{g} \in \mathrm{G}\}$.

(i) Prove that Inn(G) is a normal subgroup of Aut(G).

(ii) Prove that Inn(G) $\cong$ G/Z(G).

**2.** Prove that $S_4$ has no subgroup of order 8. Does it have a normal subgroup of order 3?

**3.** Prove that if $G$ is a group and $H$ a subgroup of index $n$, then $G$ has a normal subgroup $K$ with $[G : K] \leq n!$.

**4.** Exhibit, with a complete explanation, 6 non-isomorphic groups of order 24, at least 4 of which must be non-abelian.