APPENDIX: A CLOSED 2-GENERATOR NON-ORIENTABLE HYPERBOLIC 3-MANIFOLD WITH NO INTEGRAL LIFT OF w_1

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Abstract. We show that there is a closed non-orientable hyperbolic 3-manifold M whose fundamental group has rank 2 and for which $w_1(M)$ does not admit an integral lift.

1. Integral lifts

In [4], the authors introduce the notion of *immersion equivalence* of closed connected smooth manifolds of the same dimension. Although their focus is mainly in dimension 4, they also discuss the situation in dimensions 2 and 3 and [4, Propositions 2.1 & 2.2] describes what is known in these settings. In particular, in dimension 3, all closed connected orientable 3-manifolds are immersion equivalent to S^3, with the case of closed connected non-orientable 3-manifolds admitting at least two equivalence classes: these are represented by $S^1\tilde{\times}S^2$ (the twisted product of S^1 and S^2) and $\mathbb{RP}^2\times S^1$, and are distinguished by whether $w_1(M)\in H^1(M,\mathbb{Z}/2\mathbb{Z})$ (the first Stiefel-Whitney class of M) has an *integral lift* (i.e. whether there is a class in $H^1(M,\mathbb{Z})$ whose reduction modulo 2 is $w_1(M)$) or not (see Lemma 2.1).

As part of their investigation into immersion equivalence of non-orientable closed connected 3-manifolds, the authors of [4] asked the following question. To state this we recall that a finitely generated group G has *rank* d, if the minimal cardinality of a generating set for G is d. When $G=\pi_1(M)$ where M is a closed connected 3-manifold, then we abuse notation and also say that M has rank d.

Question 1: *Does there exist a closed connected non-orientable hyperbolic 3-manifold M of rank 2 for which $w_1(M)$ does not admit an integral lift?*

Note that $\mathbb{RP}^2\times S^1$ is 2-generator. In this Appendix we answer Question 1, namely we show.

Theorem 1.1. *Question 1 has an affirmative answer.*

More details on the manifold M used to answer Question 1 are given below. For now we simply note that $H_1(M,\mathbb{Z})=\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$, and so from [4, Proposition 2.2], and in the notation of [4], $M\leq\mathbb{RP}^2\times S^1$ but M is not immersion equivalent to $\mathbb{RP}^2\times S^1$. In addition, [4, Proposition 2.2(4)] $S^1\tilde{\times}S^2\leq M$, however using Theorem 1.1 and [4, Proposition 2.2(5)] it follows that M is not immersion equivalent to $S^1\tilde{\times}S^2$. Summarizing we have the following.

Corollary 1.2. *The manifold M constructed above is not immersion equivalent to $S^1\tilde{\times}S^2$ or $\mathbb{RP}^2\times S^1$, and hence represents a distinct immersion equivalence class of closed connected non-orientable 3-manifolds.*

Further motivation for Question 1 is given in §3.

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2. Proof of Theorem 1.1

We begin by proving a lemma that provides an easy way (in principle) to construct closed connected non-orientable 3-manifolds Y for which $w_1(Y)$ does not admit an integral lift (note this applies to $\mathbb{RP}^2 \times S^1$).

Lemma 2.1. *Let Y be a closed non-orientable 3-manifold satisfying the following properties:*

1. *Y is fibered over the circle with fiber a closed non-orientable surface;*
2. *$H_1(Y, \mathbb{Z}) = \mathbb{Z} \times T$, where T is finite.*

Then $w_1(Y)$ does not admit an integral lift.

Proof. Since Y is non-orientable $w_1(Y) \in H^1(Y, \mathbb{Z}/2\mathbb{Z})$ is non-trivial. Furthermore, since Y is fibered over the circle with fiber F, the normal bundle of $F \subset Y$ is trivial and so standard properties of Stiefel-Whitney classes shows that $w_1(Y)|_F = w_1(F) \neq 0$.

On the other hand, $b_1(Y) = 1$ and so the fiber surface F is dual to the unique epimorphism $\phi : \pi_1(Y) \to \mathbb{Z}$ and in particular $\phi(\pi_1(F)) = 0$. Hence the induced epimorphism $\overline{\phi} \in H^1(Y, \mathbb{Z}/2\mathbb{Z})$ satisfies $\overline{\phi}(\pi_1(F)) = 0$. Putting this together with the previous paragraph, we conclude that ϕ (which is unique) cannot be an integral lift of $w_1(Y)$, since $w_1(Y)|_F \neq 0$. □

Theorem 1.1 will follow from Lemma 2.1 once we exhibit a closed non-orientable hyperbolic 3-manifold M of rank 2 that satisfies the hypothesis of Lemma 2.1. There should be many such examples obtained by taking the mapping tori of "suitable" pseudo-Anosov homeomorphisms on closed non-orientable hyperbolic surfaces. However, our approach here is more concrete and computational, and makes use of a closed non-orientable hyperbolic 3-manifold arising in the SnapPy census [2].

The example: *Let M denote the manifold* `m313(1,0)` *of the SnapPy census. Then M satisfies the hypothesis of Lemma 2.1.*

We will make considerable use of SnapPy [2] in our analysis, and using SnapPy we find that $H_1(M, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, and a presentation for $\pi_1(M)$ is given by:

$\pi_1(M) =$ `<a,b|aaaBAAAbbaaababb, aaabAAbbaaaBABAAAB>`,

where X denotes x^{-1}. Note that to find M we used the SnapPy command:

```
for M in NonorientableClosedCensus[:20]:  print(M, M.volume())
```

to list small volume closed non-orientable hyperbolic 3-manifolds. Our manifold M is eighth on this list and has volume approximately $3.17729327860\ldots$.

Remark 2.2. Note that out of the first eight manifolds in the list of small volume closed non-orientable hyperbolic 3-manifolds we generated from SnapPy, six have first homology group $\mathbb{Z}$ and so could not be fibered over the circle with fiber a non-orientable surface, the other example had first homology group $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Using the methods we describe below, we were able to check that this example is fibered with fiber an orientable surface. This is how we were led to M as an appropriate candidate.

That M has the desired properties follows from Claim 1:

Claim 1: *M is fibered with fiber a closed non-orientable surface.*

To prove Claim 1 we first prove:

Claim 2: *The non-orientable cusped hyperbolic 3-manifold $N =$ `m313` has one cusp which has a Klein Bottle cross-section and is fibered with fiber a non-orientable punctured surface.*

Before commencing with the proof of Claim 2, we include some relevant pre-amble. To show that N is fibered we will use a criterion of Brown [1] (see also [3] for a discussion aimed at low-dimensional topologists) which provides an algorithm to decide if a 2-generator, 1-relator group G equipped with an epimorphism $\phi : G \to \mathbb{Z}$ has $\ker(\phi)$ finitely generated (which by a result of Stallings [7] in the 3-manifold setting is equivalent to the manifold being fibered). Below we state a version of Brown's theorem (see [1, Theorem 4.3]) that we shall use. We refer the reader to [1] and [3] for further details.

We begin by fixing some notation. Suppose that $G =< a, b \mid R >$ is a 1-relator group, with R a nontrivial cyclically reduced word in the free group on $\{a, b\}$. Let R_i denote the initial subword consisting of the first i letters of R, and assume that all the initial subwords of R are given by $R_1, R_2, \ldots, R_n$ with $R_n = R$. Following [3, Theorem 5.1] we have:

Theorem 2.3 (Brown)**.** *In the notation above, let $\phi : G \to \mathbb{Z}$ be an epimorphism and assume that $\phi(a)$ and $\phi(b)$ are both non-zero, then $\ker(\phi)$ is finitely generated if and only if the sequence $\phi(R_1), \phi(R_2), \ldots, \phi(R_n)$ has a unique minimum and maximum.*

Proof of Claim 2: Using the SnapPy command `M.cusp_info()` shows that N has a single cusp with cusp cross-section a Klein Bottle (which we denote by $\mathcal{K}$). Also using SnapPy we get $H_1(N, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and a presentation:

$$\pi_1(N) = \texttt{<a,b|aaaBAAAbbaaababb>}.$$

We will apply Brown's algorithm to the unique epimorphism $\phi : \pi_1(N) \to \mathbb{Z}$. One can check that $\phi(a) = -1$ and $\phi(b) = 1$, and so to apply Theorem 2.3 we need to check the values on the initial subwords of the relator $R :=$ `aaaBAAAbbaaababb`.

We identified 16 initial subwords:

$$R_1 = a, R_2 = a^2, R_3 = a^3, R_4 = a^3b^{-1}, \ldots, R_9 = a^3b^{-1}a^{-3}b^2, \ldots, R_{16} = R.$$

The values of $\phi(R_i) \in \{-4, -3, -2, -1, 0, 1\}$ with $\phi(R_4) = -4$ a unique minimum and $\phi(R_9) = 1$ a unique maximum. Hence Theorem 2.3 applies to show that $\ker(\phi)$ is finitely generated, and so we conclude that N is fibered using [7].

To show that the fiber is non-orientable we use the SnapPy commands `G = M.fundamental_group()` together with `G.O31('a')` and `G.O31('b')` to exhibit matrices for a and b whose determinants can then be checked to be (approximately) $+1$ and -1 respectively. In particular the element b is orientation-reversing, as is the element $ab \in \ker(\phi)$ (from above). Hence the fiber is non-orientable and the proof of Claim 2 is complete. □

Completing the proof of Claim 1: Since N is non-orientable with cusp cross-section $\mathcal{K}$, there is a unique way to compactify N by gluing a solid Klein Bottle to $\mathcal{K} \times [0, 1)$, which by definition gives M. Indeed, there is a unique essential simple closed curve (slope) $\alpha \subset \mathcal{K}$ that is the core of the solid Klein Bottle that can be attached to $\mathcal{K} \times [0, 1)$ which compactifes N via Dehn filling and for which α is killed. Now the slope α is the boundary slope of the fiber surface of N (this is made explicit in Remark 2.4 below), and so in particular the fibering of N extends after Dehn filling along α to get M. Hence M is fibered with fiber a closed non-orientable surface as required. □

Remark 2.4. Using SnapPy the filling curve α can be identified as the peripheral element `BAAABaaabAAbbaaaBA` $\in \pi_1(N)$ which can easily be seen to be a re-ordering of the second relation in $\pi_1(M)$. It can also be readily checked that this element of $\pi_1(N)$ maps trivially under the map to $\mathbb{Z}$.

3. Compatibility

One motivation for Question 1 is the following result which needs some additional terminology. Let M and N be closed connected non-orientable 3-manifolds and for which $w_1(M)$ and $w_1(N)$ both do not admit an integral lift. Say that a homomorphism $\theta : \pi_1(M) \to \pi_1(N)$ is w_1-*compatible* if and only the following diagram commutes:

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{\theta} & \pi_1(N) \\ & {\scriptstyle w_1}\searrow \quad \swarrow{\scriptstyle w_1} & \\ & \mathbb{Z}/2\mathbb{Z} & \end{array}$$

Theorem 3.1. *Assume that M and N are closed non-orientable hyperbolic 3-manifolds for which $w_1(M)$ and $w_1(N)$ both do not admit an integral lift. Assume that M has rank 2 and $b_1(N) \geq 3$. Then there is no w_1-compatible homomorphism $\theta : \pi_1(M) \to \pi_1(N)$.*

Proof. The proof given below is essentially that described to the author by M. Freedman. We begin by proving a lemma which is well known in the orientable case (see [5, Theorem VI.4.1]). As in the proof of Theorem 1.1, $\mathcal{K}$ denotes the Klein Bottle.

Lemma 3.2. *Let X be a closed hyperbolic 3-manifold and $H < \pi_1(X)$ a subgroup of rank ≤ 2. Then, either H is free of rank ≤ 2, or H has finite index in $\pi_1(X)$.*

Proof. As stated above, the case when X is orientable follows from [5, Theorem VI.4.1]. We now handle the case when X is non-orientable, and to that end let X^+ denote the orientation double cover of X and $H^+ = H \cap \pi_1(X^+)$.

If H is not free then by [6, Corollary 4], $\chi(H) = 0$. If H does not have finite index in $\pi_1(X)$, then H^+ is an infinite index subgroup of $\pi_1(X^+)$ with $\chi(H^+) = 0$. If C^+ denotes a compact core for $\mathbb{H}^3/H^+$, then a standard argument shows that:

$$\chi(\partial C^+)/2 = \chi(C^+) = \chi(H^+) = 0.$$

Hence ∂C^+ consists of a union of tori ($\mathcal{K}$ is excluded since C^+ is orientable), and since H^+ has infinite index in $\pi_1(X^+)$, and X^+ is a closed orientable hyperbolic 3-manifold, it follows that the only possibility for H is that it is virtually abelian. Since X is hyperbolic, H cannot be free abelian of rank 2, or $\pi_1(\mathcal{K})$, hence $H \cong \mathbb{Z}$ and this completes the proof. □

With this lemma in hand, we complete the proof of Theorem 3.1. To that end assume that there is a w_1-compatible homomorphism $\theta : \pi_1(M) \to \pi_1(N)$, and let $H = \theta(\pi_1(M))$. Compatibility and non-orientability of M imply that $H \neq 1$, and so H is a non-trivial subgroup of $\pi_1(N)$ of rank ≤ 2.

We now consider H in the context of Lemma 3.2. If H is a free group of rank 2, we obtain a sequence of epimorphisms $F_2 \to \pi_1(M) \to H \cong F_2$. However free groups are Hopfian, and so these epimorphisms are all isomorphisms. However, the fundamental group of a closed hyperbolic 3-manifold is not free (since they are irreducible). The case of finite index is ruled out by the hypothesis that $b_1(N) \geq 3$, so that every subgroup G of finite index in $\pi_1(N)$ also has $b_1(G) \geq 3$ (by the transfer homomorphism), which H clearly cannot.

Thus we are reduced to $H = \mathbb{Z}$ and θ factors through a homomorphism onto $\mathbb{Z}$. But this is a contradiction to M not having an integral lift since w_1-compatibility ensures that any orientation-reversing element of $\pi_1(M)$ must map to an odd integer. □

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