# 373K Midterm 1, Spring 2017

## DO BOTH QUESTIONS IN PART A AND 2 FROM PART B

All questions are worth **25 points**. Indicate which four you want graded. State clearly Theorems you are using.

**Notation:** Throughout, $D_n$ will denote the dihedral group of order $2n$, and $S_n$ the symmetric group on $n$ letters.

## PART A

**1.** Let $G$ denote the set of all rational numbers of the form $2^m 3^n$, with $m, n \in \mathbf{Z}$.

(a) Show that $G$ forms a group under multiplication.

(b) Prove that $G$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$.

**2.** Answer the following *true* or *false*. You must explain your answers to get full credit.

(a) $D_9$ is isomorphic to $D_3 \times \mathbf{Z}/3\mathbf{Z}$?

(b) A group of order 27 is cyclic?

(c) Let $p$ and $q$ be distinct primes and $G$ a group of order $pq$. Every proper subgroup of $G$ is cyclic.

(d) Let $G$ be a group and $H, K$ normal subgroups of $G$. $H \cap K$ is a normal subgroup of $G$.

## PART B

**3.**(a) Let $A$ be an abelian group, prove that all subgroups of $A$ are normal. Does the converse hold? i.e. a group in which all subgroups are normal is abelian?

(b) Let $A$ be an abelian group and let $D$ be the *diagonal subgroup* of $A \times A$; i.e. $D = \{(a, a) : a \in A\}$. Prove that $D$ is a normal subgroup of $A \times A$ and identify the quotient $(A \times A)/D$.

(c) If $A$ is now assumed to be non-abelian, is the digaonal subgroup still normal?

**4.** Let $C =< x >$ be a cyclic group of order 5, and let $G = S_3 \times C$

(a) Compute the order of $G$, and for each element of $G$ compute its order.

(b) Let $H$ be a subgroup of $G$ order 10, show that $H$ is cyclic.

(c) Construct a non-Abelian group $N$ which has the same order as $G$ but is not isomorphic to $G$. You must show your work that the groups are non-isomorphic.

**5.** Recall that by an automorphism $\alpha$ of a group $G$ we mean an isomorphism $\alpha : G \to G$. If $G$ is a group, a subgroup $H$ is called *characteristic* if $\alpha(H) = H$ for all automorphisms of $\alpha$ of $G$.

(a) For each $g \in G$ define $\phi_g : G \to G$ by $\phi_g(x) = gxg^{-1}$. Prove that $\phi_g$ is an automorphism of $G$.

(b) Prove that if $H$ is a characteristic subgroup of $G$, then $H$ is a normal subgroup of $G$.

(c) Let $K$ be a normal subgroup of $G$ and $M$ a characteristic subgroup of $K$. Prove $M$ is normal in $G$.