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**Groups, Geometry, and Dynamics**


# Pseudo-Anosov homeomorphisms not arising from branched covers

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**Abstract.** In this paper we provide a negative answer to a question of Farb about the relation between the algebraic degree of the stretch factor of a pseudo-Anosov homeomorphism and the genus of the surface on which it is defined.

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## 1. Introduction

Let $S = S_g$ be a closed, orientable surface of genus $g$. A pseudo-Anosov homeomorphism $f: S \to S$ is a *virtual lift* if there exists a branched cover $p: S \to \Sigma$ with degree $\deg(p) > 1$ over a (possibly nonorientable) surface $\Sigma$, and a pseudo-Anosov $\phi: \Sigma \to \Sigma$ so that $\phi$ lifts to a power of $f$ by $p$; that is, there exists $n > 0$ so that $pf^n = \phi p$. We say that $f^n$ *is a lift of* $\phi$ *via* $p$.

Franks and Rykken [6] showed that if $f: S \to S$ is a pseudo-Anosov (with orientable stable/unstable foliations), $g \geq 2$, and if the stretch factor $\lambda(f)$ is a quadratic irrational, then $f$ is a virtual lift—in fact, the branched cover is over a torus $p: S \to \Sigma$ (cf. Gutkin and Judge [7] and Kenyon and Smillie [8]). In 2004, Farb asked (see [19]) if a version of this is true when the degree of the stretch factor was greater than 2. Specifically, he asked if there exists a function $h: \mathbb{N} \to \mathbb{N}$ so that a pseudo-Anosov homeomorphism $f: S_g \to S_g$ is a virtual lift if the degree of $\lambda(f)$ over $\mathbb{Q}$ is at most $d$ and $g \geq h(d)$. Here we prove that the answer is 'no'.

**Main Theorem.** *For any even $d \geq 4$ and all $g \geq \frac{d}{2}+2$, there exist pseudo-Anosov homeomorphisms $f_{g,d}: S_g \to S_g$ with orientable stable/unstable foliations and $\lambda(f_{g,d})$ of degree $d$ over $\mathbb{Q}$, so that $f_{g,d}$ is not a virtual lift.*

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We also mention the related results [2, Lemma 6.2] and [19, Corollary 1.4] that both describe conditions which guarantee that a pseudo-Anosov is not a virtual lift. In the former case no control on the stretch factor is given, and in the latter the stretch factors have degree $6g - 6$ (the maximal possible degree).

We complete the Introduction by briefly describing the idea of the proof of the Main Theorem. The pseudo-Anosov homeomorphisms are constructed as products of high powers of Dehn twists. The twisting curves and powers are chosen in such a way that we can apply Strenner's results from [19] to ensure that the stretch factor has the appropriate degree. To prove that the homeomorphisms are not virtual lifts, we analyze the flat metrics defining the associated Teichmüller axes. Appealing to work of Rafi [18], Minsky [16], and Brock-Canary-Minsky [5], we prove that for carefully chosen twisting curves, there is a bi-infinite collection of simple closed curves that are "characteristic" for the pseudo-Anosov. These characteristic curves are described in terms of Euclidean cylinder neighborhoods with respect to the flat metrics, and if a pseudo-Anosov homeomorphism is a virtual lift, we prove that they must project to the quotient surface in a very specific way. The proof is completed by choosing the twisting curves so that the associated bi-infinite sequence of curves cannot project to any nontrivial quotient surface in that way.

**Remark 1.1.** In fact, no pseudo-Anosov element of the Veech group containing the pseudo-Anosov mapping class from the Main Theorem will be a virtual lift; see §5.1. However, we do not know whether there are other elements in the Veech group, so we have not made it a point to emphasize this fact. There is a simpler proof for the special case of $d = 4$, where we can find more elements of the Veech group that are not virtual lifts, and this is Theorem 5.3, whose proof also appears in §5.1. We have made this last section mostly independent from the rest of the paper, so one can find a negative answer to Farb's question in these few pages, at least in the special case of $d = 4$.

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## 2. Surfaces, curves, and annular projections

Suppose $S$ is any orientable hyperbolic surface of finite topological type. When convenient, we view $S$ as a Riemann surface in which punctures are filled in and

treated as marked points. Here we collect a few facts about curve complexes and subsurface projections. See [13] and [14] for more details.

The *curve graph of* $S$, $\mathcal{C}(S)$, is the simplicial complex whose vertex set $\mathcal{C}^{(0)}(S)$ is the set of isotopy classes of essential simple closed curves on $S$. A pair of isotopy classes determine an edge if and only if they can be realized disjointly on $S$--equivalently, the geodesic representatives with respect to the hyperbolic metric are disjoint. We make $\mathcal{C}(S)$ into a geodesic metric space by declaring each edge to have length 1. According to [13], $\mathcal{C}(S)$ is $\delta$-hyperbolic.

If $Y$ is an annulus, we define the *curve graph of* $Y$, $\mathcal{C}(Y)$, in a similar fashion: the vertex set consists of isotopy classes of essential arcs in $Y$, where isotopies are required to fix the boundary pointwise. Edges connect isotopy classes when there are representatives with disjoint interiors, and we similarly make $\mathcal{C}(Y)$ into a geodesic metric space.

The curve graphs of annuli arise from annular subsurfaces of $S$ as follows. Given an essential annulus $Y \subset S$, there is a corresponding covering space $\widetilde{Y} \to S$. The ideal boundary of the universal covering $\mathbb{H}^2 \to S$ determines an ideal boundary of $\widetilde{Y}$, and we let $\overline{Y}$ denote $\widetilde{Y}$ together with its ideal boundary, making $\overline{Y}$ into a compact surface with boundary. Given a vertex $\alpha$ of $\mathcal{C}(S)$, representing $\alpha$ by its hyperbolic geodesic representative, we let $\tilde{\alpha}$ denote the union of the arcs in the preimage of $\alpha$ in $\overline{Y}$. We define $\pi_Y(\alpha)$ to be the union of the components of $\tilde{\alpha}$ which are essential in $\overline{Y}$ (together with their ideal endpoints); that is, the components with an endpoint on each boundary component of $\overline{Y}$. We view $\pi_Y(\alpha)$ as a subset of $\mathcal{C}(Y)$. If $\mu$ is a measured foliation on $S$, we can similarly define $\pi_Y(\mu)$ to be the set of lifts of non-singular leaves with endpoints on distinct boundary components. Note that if $\alpha$ is either a curve or a measured foliation, $\pi_Y(\alpha) \subset \mathcal{C}(Y)$ has diameter 1 (any two components are disjoint). Given two curves or measured foliations $\alpha, \beta$, if $\pi_Y(\alpha)$ and $\pi_Y(\beta)$ are both nonempty, we define the projection distance

$$d_Y(\alpha, \beta) = \operatorname{diam}(\pi_Y(\alpha) \cup \pi_Y(\beta)).$$

One also has $d_Y(\alpha, \beta) = \max i(\alpha_0, \beta_0) + 1$, where the maximum is taken over $\alpha_0 \in \pi_Y(\alpha)$, $\beta_0 \in \pi_Y(\beta)$, and $i$ denotes the geometric intersection number of the isotopy classes of arcs $\alpha_0, \beta_0$ (the number of intersection points of the interiors, minimized over representative of the relative isotopy classes, also equal to the absolute value of the algebraic intersection number). With these definition, $d_Y$ satisfies a triangle inequality whenever the projections involved are nonempty. See [14], especially §2.4, for more on these (and other) subsurface projections.

The core curve of $Y$ is an essential simple closed curve $\gamma$ in $S$ and every essential simple closed curve is the core curve of an essential annulus. We sometimes write $\mathcal{C}(\gamma)$, $\pi_\gamma$, and $d_\gamma$ instead of $\mathcal{C}(Y)$, $\pi_Y$, and $d_Y$, respectively. We have $\pi_\gamma(\alpha) \neq \emptyset$ if and only if the geometric intersection number, $i(\alpha, \gamma) \neq 0$.

One of the key features of subsurface projections is the following *Bounded Geodesic Image Theorem* (see [14]) in the case of annuli.

**Proposition 2.1.** *There exists a constant $M > 0$ with the following property. If $\alpha, \beta$ are two curves in $\mathcal{C}(S)$ and $d_\gamma(\alpha, \beta) > M$, then the geodesic from $\alpha$ to $\beta$ contains a vertex $\delta$ so that $i(\delta, \gamma) = 0$, and hence $\delta$ is adjacent to $\gamma$ in $\mathcal{C}(S)$.*

The following is a special case of the Behrstock Inequality [4] for annuli that we will need.

**Proposition 2.2.** *Suppose $\alpha, \beta, \gamma$ are three simple closed curves on $S$ that pairwise intersect. If $d_\gamma(\alpha, \beta) \geq 10$ then $d_\alpha(\gamma, \beta) \leq 3$.*

This version with explicit constants is proved by Mangahas in [11, 12].

## 3. Teichmuller geodesics and Euclidean cone metrics

A pseudo-Anosov homeomorphism $f: S \to S$ preserves a Teichmüller geodesic axis defined by a unit area Euclidean cone metric $q_0$ with cone angles greater than $2\pi$ at non-marked points, for which the stable and unstable foliations $\mu^\pm$ are orthogonal, geodesic foliations. Furthermore, in preferred coordinates $\mu^\pm$ are horizontal and vertical, respectively, and the transverse measures are given by horizontal and vertical variation, respectively. The different points along the axis are conformal structures of Euclidean cone metrics $q_t$ in which the stable and unstable foliations have their transverse measures scaled as $e^t\mu^+, e^{-t}\mu^-$ (maintaining unit area for the Euclidean cone metrics). We call the family of Euclidean cone metrics $Q = \{q_t\}_{t\in\mathbb{R}}$ the *associated flat metics*. Note that any two metrics in the family differ by an affine diffeomorphism (away from the cone points). We write $\ell_{q_t}(\gamma)$ for the $q_t$-length of a curve $\gamma$.

If $f^n$ is a lift of $\phi: \Sigma \to \Sigma$ via a branched cover $p: S \to \Sigma$, then the associated flat metrics $\Xi = \{\xi_t\}$ for $\phi$ can be chosen so that $q_t = p^*(\xi_t)/\sqrt{\deg(p)}$ (this scaling is necessary since $q_t$ and $\xi_t$ have unit area). In this case, we say that $Q = \{q_t\}$ and $\Xi = \{\xi_t\}$ are *compatible*.

If $Q = \{q_t\}$ are the flat metrics associated to a pseudo-Anosov on $S$ as above, a *$Q$-cylinder* or *flat cylinder for $Q$* (or just flat cylinder, if $Q$ is understood) is an annulus $Y \subset S$ so that the path metric on $Y$ coming from some $q_t \in Q$ makes $Y$ into a Euclidean product $I \times S^1$, where $I$ is an interval (we allow the possibility that $Y$ is only embedded on its interior, but still write $Y \subset S$). Note that if the metric on $Y$ is a Euclidean product for some $q_t \in Q$, then it is for all $q_t \in Q$ (and any two such metrics differ by affine diffeomorphism). The $q_t$-modulus of a flat cylinder $Y \subset S$, denoted $M(Y, q_t)$, is the ratio of the height to circumference, and $M(Y, Q) = \max\{M(Y, q_t) \mid t \in \mathbb{R}\}$ is the maximum modulus. If $\gamma \subset S$ is a two-sided simple closed curve, there is a maximal flat cylinder $Y_\gamma \subset S$ whose core curves are isotopic to $\gamma$, and we set $M(\gamma, q_t) = M(Y_\gamma, q_t)$ and $M(\gamma, Q) = M(Y_\gamma, Q)$. We are allowing the possibility of a degenerate cylinder,

that is, one with width zero. In this case, the cylinder consists of the unique geodesic representative (which is a concatenation of saddle connections), and we have $M(\gamma, q_t) = 0$ for all $t$.

We say that $\gamma$ is a *$Q$-cylinder curve* if $M(\gamma, Q) > 0$. There is a unique $t_\gamma \in \mathbb{R}$, called the *balance time* of $\gamma$, so that the vertical and horizontal variations of $\gamma$ agree (see e.g. [13, 18]), and hence also the time when the core geodesics of the cylinder make angle $\pm\frac{\pi}{4}$ with these foliations. Since the $q_t$-length $\ell_{q_t}(\gamma)$ is the square root of the sum of the squares of these variations, this length is also minimized at $t_\gamma$, and we can write

$$\ell_{q_t}(\gamma) = \ell_{q_{t_\gamma}}(\gamma)\cosh^{\frac{1}{2}}(2(t - t_\gamma)).$$

Because the modulus is the ratio of the area of the cylinder (which is constant in $t$) and the square of the length, it follows that $M(\gamma, q_{t_\gamma}) = M(\gamma, Q)$.

The following is an easy consequence of work of Rafi (see Lemma 3.8, Corollary 5.3, and Theorem 5.6 of [18]). Since this exact statement doesn't appear in-adjustable [18], we give a proof here for completeness.

**Proposition 3.1.** *Suppose $f : S \to S$ is a pseudo-Anosov homeomorphism, $Q$ is the associated family of flat metrics, and $\mu^\pm$ are the stable and unstable foliations. If $d_\gamma(\mu^+, \mu^-) \geq 4$, then $\gamma$ is a $Q$-cylinder curve. In general, if $\gamma$ is a $Q$-cylinder curve, then*

$$\left| M(\gamma, Q) - \frac{d_\gamma(\mu^+, \mu^-)}{2} \right| \leq 2$$

*Proof.* Suppose $t = t_\gamma$, the balance time of $\gamma$ and suppose $S$ is endowed with the metric $q_t$. Choose lifts of nonsingular leaves $\delta^+$ of $\mu^+$ and $\delta^-$ of $\mu^-$ to the annular cover $\widetilde{Y}_\gamma$ of $S$ so that

$$d_\gamma(\mu^+, \mu^-) = i(\delta^+, \delta^-) + 1.$$

Since $\delta^+, \delta^-$ are $q_t$-geodesics, these realize the minimal intersection number in their relative isotopy classes, and so intersect in at least 3 points.

Now observe that any three consecutive points of intersection along $\delta^+$ determines two consecutive, compact arcs in $\delta^+$, as well as three consecutive points of intersection along $\delta^-$ and two compact arcs of $\delta^-$. These four arcs determine a quadrilateral in $\widetilde{Y}_\gamma$. Since the geodesics intersect in right angles, the Gauss–Bonnet formula implies that there are no singular points inside the quadrilateral, and hence this is the image of an isometrically immersed rectangle, which is an embedding except at one pair of vertices. Furthermore, the diagonal of the rectangle connecting the identified vertices is a geodesic representative of $\gamma$, and since $t = t_\gamma$ the balance time, the rectangle is actually a Euclidean square; see Figure 1. In fact, since the diagonal of square has length $\ell_{q_t}(\gamma)$, the sides have length $\ell_{q_t}(\gamma)/\sqrt{2}$.

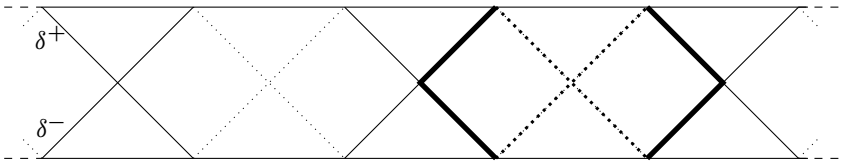


Figure 1. Nonsingular lifts $\delta^+$ and $\delta^-$ in the annulus $\widetilde{Y}_\gamma$, with five intersection points shown; three in "front" and two in "back." Squares are formed from arcs along any three consecutive intersection points. One such square is highlighted by thicker lines.

Next, observe that the geodesic which is a diagonal of a square from three consecutive intersection points contains no cone points, and hence there is a nondegenerate flat cylinder containing for $\gamma$. Consequently, $\gamma$ is a cylinder curve. For any four consecutive intersection points there are two squares in $\widetilde{Y}_\gamma$ that have two sides in common. The geodesics from the diagonals of these two "consecutive" squares form a flat cylinder of circumference $\ell_{q_t}(\gamma)$ and height $\ell_{q_t}(\gamma)/2$ (half a diagonal). There are $i(\delta^+, \delta^-) - 3$ such cylinders in $\widetilde{Y}$ glued end-to-end, and so

$$\begin{aligned} M(\gamma, Q) &= M(\gamma, q_t) \\ &\geq \frac{1}{2}(i(\delta^+, \delta^-) - 3) \\ &= \frac{1}{2}(d_\gamma(\mu^+, \mu^-) - 4) \\ &= \frac{d_\gamma(\mu^+, \mu^-)}{2} - 2. \end{aligned}$$

On the other hand, consider the maximal flat cylinder in $\widetilde{Y}_\gamma$, and choose $\delta_0^+, \delta_0^-$ to be a pair of lifts of leaves of $\mu^+, \mu^-$, respectively, with an intersection point on one boundary component of this cylinder. Considering the squares in the cylinders from triples of consecutive intersection points as above we find that there are at least $\lfloor 2M(\gamma, Q) \rfloor + 1$ intersection points of $\delta_0^+, \delta_0^-$ inside the flat cylinder. From the Gauss-Bonnet argument, it follows that there can be at most one more intersection point of $\delta_0^+, \delta_0^-$ outside the maximal cylinder, and hence

$$i(\delta_0^+, \delta_0^-) \geq \lfloor 2M(\gamma, Q) \rfloor + 2 \geq 2M(\gamma, Q) + 1.$$

Since $\delta_0^+, \delta_0^-$ are arbitrary leaves, we have $d_\gamma(\mu^+, \mu^-) \geq i(\delta_0^+, \delta_0^-) - 1$ and hence

$$M(\gamma, Q) \leq \frac{d_\gamma(\mu^+, \mu^-)}{2}.$$

Combining this with the inequality above completes the proof. □

The proof of our main theorem will rely on understanding how $Q$-cylinders in $S$ are mapped down to $\Sigma$. The images need not be cylinders, but with some additional mild assumptions, they are very well behaved. A *Euclidean half-pillowcase* is the quotient of a Euclidean cylinder $S^1 \times [-T, T]$ by the group generated by the involution $\tau(e^{i\theta}, t) = (e^{-i\theta}, -t)$. Considering a fundamental domain for this action, we can equivalently describe this as the Euclidean orbifold obtained by gluing a component of the boundary of a Euclidean cylinder $S^1 \times [0, T]$ to itself by the map $(e^{i\theta}, 0) \sim (e^{-i\theta}, 0)$. Topologically, a half-pillow case is a disk with two marked points. The two marked points are cone points with cone angle $\pi$ and there is a geodesic segment, the *core segment*, connecting those points whose complement is itself a half-open Euclidean cylinder. We will refer to the modulus of the complementary Euclidean cylinder as the *modulus of the half-pillowcase*.

**Lemma 3.2.** *Suppose $\Sigma$ is an orientable surface and $\phi\colon \Sigma \to \Sigma$ a pseudo-Anosov homeomorphism with associated flat metrics $\Xi = \{\xi_t\}$. Assume that the only marked points of $\Sigma$ are cone points of $\xi_t$ with cone angle $\pi$ and that $\Sigma$ is not a torus or a sphere with four marked points. Let $h\colon Y \to \Sigma$ denote a map of an open Euclidean cylinder into $\Sigma$ which for some $\xi_t \in \Xi$, is a local isometry away from a finite number of branched points. Then either $h(Y)$ is a Euclidean cylinder in $\Sigma$ and $h$ is a covering map onto its image or else $h(Y)$ is a Euclidean half-pillowcase. In either case, $M(h(Y), \xi_t) \geq \frac{M(Y)}{2}$.*

*Proof.* First suppose that there are no branch points in $Y$. In this case, each core geodesic of $Y$ maps to a geodesic. Since the holonomy of $\xi_t$ is $\{\pm I\}$, it follows that these geodesics are simple. We wish to show that no two core geodesics map to the same geodesic. Suppose on the contrary that $\alpha, \beta$ are two distinct core geodesics in $Y$ that map to the same geodesic. Since $\Sigma$ is orientable, the sub-cylinder between $\alpha$ and $\beta$ provides an isotopy from one to the other. Orient both $\alpha$ and $\beta$ in the same direction coming from the annulus (so the isotopy between them is orientation preserving). Again, because $\Sigma$ is orientable, $\alpha$ and $\beta$ must map to the same oriented curve. Since the sub-cylinder between $\alpha$ and $\beta$ lies on different sides of these two curves (each are two-sided curves), it follows that image of the cylinder lies on both sides of the image. Thus, we can identify $\alpha$ and $\beta$ in the sub-cylinder producing a torus which maps locally isometrically to $\Sigma$. Therefore $\Sigma$ is a flat torus, which is a contradiction. Thus, no two core geodesics of $Y$ are sent to the same curve, and it follows that $h(Y)$ is a cylinder, foliated by the images of the core geodesics. Since $h$ restricts to a covering map from each core geodesic onto its image, it follows that $h$ restricts to a covering map from $Y$ onto its image.

Now suppose $h$ is nontrivially branched at some point $\zeta \in Y$. Note that $h(\zeta)$ must be a cone point of angle $\pi$. Let $\alpha$ be a core geodesic through $\zeta$. We first want to show that $h$ can only be branched at points on $\alpha$. For this, observe that $\alpha$ must project to a geodesic segment between a pair of cone points with angle $\pi$. In particular, there is an antipodal point $\zeta'$ on $\alpha$ that projects to the other cone

point (there may be several points that project to $\zeta'$, but one must be antipodal). Geodesics sufficiently close to $\alpha$ project to geodesics surrounding $h(\alpha)$, and hence a cylinder neighborhood of $\alpha$ maps down to a Euclidean half-pillowcase. We need to show that no other core geodesic contains a point where $h$ is branched. So, suppose there were another such geodesic $\beta \neq \alpha$ of $Y$ that also contains a branch point, and choose one that is closest to $\alpha$. Observe that the Euclidean cylinder between $\alpha$ and $\beta$ contains no points where $h$ branches, and so the boundary components can be glued together ("folded" at antipodal points one each boundary component where $h$ branches) to produce a sphere with four cone points that maps locally isometrically (away from the preimage of the cone points) onto $\Sigma$ (this is similar to the case of no branch points where we showed that $\Sigma$ was the image of a flat torus). The only orientable Euclidean cone surfaces with holonomy $\{\pm I\}$ which is the image of a locally isometric map of the sphere with four cone points is the sphere with four cone points, and so $\Sigma$ is a sphere with four cone points, a contradiction. Thus, there is only one geodesic $\alpha$ which contains branch points.

The sub-cylinders on either side of $\alpha$ map to $\Sigma$ without branched points, so by the previous paragraph, these cover cylinder. Thus $h(Y)$ is a Euclidean half-pillowcase, namely the union of the half-pillowcase neighborhood of the image of $\alpha$, together with these two cylinders (which share some core geodesics).

If $h: Y \to h(Y)$ is a covering map, then the modulus of $h(Y)$ is the modulus of $Y$ times the degree of this covering. In the two-fold quotient from a Euclidean cylinder to a half-pillowcase, the modulus is reduced by half. The lower bound on modulus now follows. This completes the proof. □

**Remark 3.3.** We note that when $h(Y)$ is a Euclidean half-pillowcase, the map $h$ is not necessarily a (branched) covering map from $Y$ to $h(Y)$: the two distances from the core geodesic $\alpha$ to the two boundary components might be different.

**Lemma 3.4.** *Suppose $\Sigma$ is a nonorientable surface and $\phi: \Sigma \to \Sigma$ a pseudo-Anosov homeomorphism with associated flat metrics $\Xi = \{\xi_t\}$. Let $h: Y \to \Sigma$ denote a map of an open Euclidean cylinder into $\Sigma$ which for some $\xi_t \in \Xi$, is a local isometry away from a finite number of branched points. Further assume that the modulus of $Y$ is strictly greater than 2. Then $h(Y)$ is either a Euclidean cylinder or a Euclidean half-pillowcase and $M(h(Y), \xi_t) \geq \frac{M(Y)}{2}$.*

*Proof.* Letting $g : \Sigma' \to \Sigma$ denote the orientation double cover, we claim that $h$ lifts to $h': Y \to \Sigma'$. To see this, let $\Sigma_0 \subset \Sigma$ and $Y_0 \subset Y$ denote the complements of the branched points and their preimages, respectively, so that $h|_{Y_0}$ is a local diffeomorphism. Since the orientation double cover of $\Sigma_0$ is the orientation bundle (that is, it is the bundle $\mathbb{P}\Lambda^2 T(\Sigma_0)$), a choice of orientation on $Y_0$ defines a lift of $h|_{Y_0}$ to the orientation double cover. Since $\Sigma$ is orientable in a disk neighborhood of the cone points, this lift extends to all of $Y$. A pseudo-Anosov homeomorphism on a torus or sphere with four marked points cannot be a lift of

a pseudo-Anosov homeomorphism of a nonorientable surface: this follows from [20, Proposition 2.3], for example, where it is shown that lifts of pseudo-Anosov homeomorphisms from a nonorientable surface have stretch factors that are *not* Galois conjugates, while stretch factors of pseudo-Anosov homeomorphisms of the torus and sphere with four marked points are quadratic irrational algebraic integers, and hence their inverses are their Galois conjugates. Therefore, by Lemma 3.2, $h'(Y) \subset \Sigma'$ is either a Euclidean cylinder or half-pillowcase with the required lower bound on modulus.

Since $g$ is a two-fold covering, there is another lift $h'': Y \to \Sigma'$. We claim that $h'(Y)$ and $h''(Y)$ are disjoint, and hence the restriction of $g$ to $h'(Y)$ is a homeomorphism onto $h(Y)$, which by Lemma 3.2 will complete the proof. Therefore we suppose $h'(Y) \cap h''(Y) \neq \emptyset$ and obtain a contradiction. The map $h''$ differs from $h'$ by composing with the order two covering transformation $\tau: \Sigma' \to \Sigma'$, which is orientation reversing. Thus, there is a point $z$ of $h'(Y)$ for which $\tau(z) \in h'(Y)$.

If $h'(Y)$ is a cylinder, we denote it $A = h'(Y)$. If $h'(Y)$ is a half-pillowcase, then since $h'(Y) \cap h''(Y)$ is an open set, we can assume that $z$ and $\tau(z)$ lie in the Euclidean cylinder surrounding the core segment between the cone points. By our assumption, this cylinder has modulus strictly greater than 1, and we denote it $A$. In either case, $A$ is a Euclidean cylinder of modulus greater than 1 containing $z$ and $\tau(z)$.

Choose an oriented orthonormal basis $e_1, e_2$ on $A$ so that $e_1$ is tangent to the core curves of $A$. The derivative $d\tau_z: T_z(A) \to T_{\tau(z)}(A)$ is orientation reversing, hence a reflection. Since the stable/unstable foliations are preserved by $\tau$, the line of reflection must be tangent to one of these foliations. Since these foliations are orthogonal, and neither has closed leaves, we see that the lines of reflection are not spanned by either $e_1$ or $e_2$. It follows that $\tau$ must send the core geodesic of $A$ through $z$ transverse to the core geodesic through $\tau(z)$. Since the modulus of $A$ is greater than 1, the core geodesic is shorter than the distance between the boundary components, which is a contradiction. Therefore, $h'(Y)$ and $h''(Y)$ are disjoint, completing the proof. □

We also need to understand what the preimage of cylinders look like under a branched cover $p: S \to \Sigma$.

**Lemma 3.5.** *Given $S$ and $d > 0$ there exists $B = B(S, d) > 0$ with the following property. Suppose that $p: S \to \Sigma$ is a branched covering of degree at most $d$, $f: S \to S$ a lift of the pseudo-Anosov $\phi: \Sigma \to \Sigma$, $Q = \{q_t\}$ and $\Xi = \{\xi_t\}$ are the associated, compatible flat metrics, and $Y \subset \Sigma$ is a maximal open $Q$-cylinder with maximal modulus $M(Y, \Xi)$. Then there is a sub-cylinder $Y_0 \subset Y$ so that $p^{-1}(Y_0)$ is a union of Euclidean cylinders in $S$, each with maximal modulus at least $BM(Y, \Xi)$.*

*Proof.* Fix the metrics $\xi_t$ and $q_t$ at the balance time $t$ of the core curve of $Y$. By the Riemann-Hurwitz Theorem, there is a bound $b > 0$ on the number of branched points of $p$, in terms of $d$ and $\chi(S)$, and we set $B = \frac{1}{d(b+1)}$. Since $Y$ contains at most $b$ branch points, there are at least $b+1$ open Euclidean sub-cylinders in $Y$ disjoint from the branch points so that the boundaries of the closures in $\Sigma$ are either in the boundary of the closure of $Y$ or else contain a branched point. The sum of the moduli of these is precisely the modulus of $Y$, and consequently one of them, call it $Y_0$, has modulus at least $\frac{M(Y,\Xi)}{b+1} = \frac{M(Y,\xi_t)}{b+1}$. The preimage $p^{-1}(Y_0)$ is a Euclidean cylinder and for any component $\widetilde{Y}_0 \subset p^{-1}(Y_0)$ the restriction of $p$,

$$p|_{\widetilde{Y}_0} : \widetilde{Y}_0 \longrightarrow Y_0,$$

is a covering map of degree at most $d$. Therefore, $M(\widetilde{Y}_0, Q) \geq \frac{M(Y,\Xi)}{d(b+1)} = BM(Y, \Xi)$, as required. □

## 4. Pseudo-Anosovs from Dehn twists.

Suppose $c_1, c_2, \ldots, c_n$ are curves that fill a surface $S = S_g$ with $g \geq 2$ so that $i(c_i, c_{i+1}) \neq 0$ for all $1 \leq i \leq n$ and with $1 \leq i+1 \leq n$ taken modulo $n$. Let $k_1, k_2, \ldots, k_n \in \mathbb{Z}$. Our construction involves analyzing the mapping class

$$f = T_{c_1}^{k_1} T_{c_2}^{k_2} \cdots T_{c_n}^{k_n}.$$

We first extend the finite sets of curves and integers to infinite sequences $\{c_j\}_{j=1}^{\infty}$ and $\{k_j\}_{j=1}^{\infty}$ by setting

$$c_j = c_{j'} \text{ and } k_j = k_{j'}$$

where $1 \leq j' \leq n$ and $j \equiv j'$ modulo $n$. Then for all $j \geq 1$ set

$$f_j = T_{c_1}^{k_1} T_{c_2}^{k_2} \cdots T_{c_j}^{k_j}.$$

Observe that for all $m \geq 0$, and $j \geq 0$ we have

$$f_{nm+j} = f^m f_j. \tag{1}$$

Now construct a new infinite sequence of curves $\{\gamma_j\}_{j=1}^{\infty}$ by $\gamma_j = f_j(c_j)$. For all $j \geq 1$, since $c_j = c_{j+n}$, (1) implies

$$f(\gamma_j) = f f_j(c_j) = f_{j+n}(c_{j+n}) = \gamma_{j+n}. \tag{2}$$

Thus, $f$ acts as the $n^{\text{th}}$ power of the shift on the sequence $\{\gamma_j\}_{j=1}^{\infty}$. Therefore, we can extend the infinite sequence of curves to a bi-infinite sequence $\{\gamma_j\}_{j\in\mathbb{Z}}$ so that (2) holds for all $j \in \mathbb{Z}$.

**Lemma 4.1.** *Given curves $c_1, \dots, c_n$ as above, there exists $R > 0$ and $K > 0$ so that if $|k_j| \geq K$ for all $j \geq 1$, then*

(i) $i(\gamma_i, \gamma_j) \neq 0$ *for all* $i, j \in \mathbb{Z}$, $i \neq j$,

(ii) $|d_{\gamma_\ell}(\gamma_i, \gamma_j) - |k_\ell|| \leq R$ *for all* $i, j, \ell \in \mathbb{Z}$ *with* $i < \ell < j$.

(iii) $\{\gamma_i\}$ *is an $f$-invariant, uniform quasi-geodesic in the curve complex.*

*From* (iii), *it follows that $f$ is pseudo-Anosov, and $\{\gamma_j\}_{j\in\mathbb{Z}}$ is a quasi-geodesic axis. Moreover, if we let $\mu^\pm$ denote the stable/unstable foliations of $f$, then*

(iv) $|d_{\gamma_j}(\mu^+, \mu^-) - |k_j|| \leq R + 2$

*for all* $j \in \mathbb{Z}$.

The meaning of (iii) is that there exists constants $A, B > 0$, depending only on $c_1, \dots, c_n$, so that

$$\frac{1}{A}|i - j| - B \leq d(\gamma_i, \gamma_j) \leq A|i - j| + B.$$

We have avoided cluttering the already lengthy statement by excluding explicit mention of these constants.

*Proof.* We have already established the $f$-invariance of $\{\gamma_j\}$. In particular, it suffices to prove the statements (i)–(iii) for positive indices.

First consider a triple of any three consecutive curves $(\gamma_{j-1}, \gamma_j, \gamma_{j+1})$. We want to describe this triple of curves up to homeomorphism. By applying a sufficiently high positive power of $f$, we can assume that $j > 1$. Then applying $f_{j-1}^{-1}$ to this triple we get

$$\begin{aligned} f_{j-1}^{-1}(\gamma_{j-1}, \gamma_j, \gamma_{j+1}) &= f_{j-1}^{-1}(f_{j-1}(c_{j-1}), f_j(c_j), f_{j+1}(c_{j+1})) \\ &= (c_{j-1}, T_{c_j}^{k_j}(c_j), T_{c_j}^{k_j} T_{c_{j+1}}^{k_{j+1}}(c_{j+1})) \\ &= (c_{j-1}, c_j, T_{c_j}^{k_j}(c_{j+1})) \end{aligned}$$

Since the sequences $\{c_j\}$ and $\{k_j\}$ are $n$-periodic, we see that up to homeomorphism, any consecutive triple looks like

$$c_{j-1}, c_j, T_{c_j}^{k_j}(c_{j+1}),$$

for $1 \leq j \leq n$ and the other two indices $1 \leq j-1, j+1 \leq n$ taken modulo $n$. Since consecutive curves intersect nontrivially, we can apply the triangle inequality for projection distances to obtain

$$|d_{c_j}(c_{j-1}, T_{c_j}^{k_j}(c_{j+1})) - d_{c_j}(c_{j+1}, T_{c_j}^{k_j}(c_{j+1}))| \leq d_{c_j}(c_{j-1}, c_{j+1}).$$

The right hand side is uniformly bounded by $n$-periodicity, and we claim that

$$|d_{c_j}(c_{j+1}, T_{c_j}^{k_j}(c_{j+1})) - |k_j|| \leq 3.$$

This follows from the triangle inequality, the fact that both $\pi_{c_j}(c_{j+1})$ and $\pi_{c_j}(T_{c_j}^{k_j}(c_{j+1}))$ have diameter at most 1, and the fact that the $k^{\text{th}}$ power of a Dehn twist translates any arc by at most $k+1$ on the curve graph of the annulus (note that some arcs are translated more than $k$ because there is more than one lift of the twisting curve). Therefore, taking $R_0 > 0$ to be at least three more than that uniform bound implies

$$|d_{c_j}(c_{j-1}, T_{c_j}^{k_j}(c_{j+1})) - |k_j|| \leq R_0.$$

Applying the homeomorphism $f_{j-1}$ to all curves in this inequality, we obtain

$$|d_{\gamma_j}(\gamma_{j-1}, \gamma_{j+1}) - |k_j|| \leq R_0. \tag{3}$$

For now, assume $K \geq R_0 + 16$ (later we will increase the lower bound on $K$). If $|k_j| \geq K$, it then follows that we also have

$$d_{\gamma_j}(\gamma_{j-1}, \gamma_{j+1}) \geq 16.$$

Consequently, $i(\gamma_{j-1}, \gamma_{j+1}) \neq 0$ (and hence, $\gamma_{j-1}, \gamma_j, \gamma_{j+1}$ pairwise intersect).

**Claim.** If $i < j$, then $i(\gamma_i, \gamma_j) \neq 0$ and for all $i < \ell < j$, we have $d_{\gamma_i}(\gamma_\ell, \gamma_j) \leq 3$ and $d_{\gamma_j}(\gamma_i, \gamma_\ell) \leq 3$.

*Proof.* We prove the claim by induction on $j - i$. For $j - i = 1$ there is no such $\ell$, and the nonzero intersection number statement is a consequence of the description of triples. If $j - i = 2$, then the triples description implies $i(\gamma_i, \gamma_j) \neq 0$, and by Proposition 2.2, it follows that $d_{\gamma_i}(\gamma_\ell, \gamma_j) \leq 3$ and $d_{\gamma_j}(\gamma_i, \gamma_\ell) \leq 3$. These serve as the base cases.

Now suppose the statement is true whenever the difference in indices is at most $m$, and suppose $j - i = m + 1$. Without loss of generality, we may assume that $m + 1 \geq 3$. Let $i < \ell < j$ be any index. Suppose first that

$$i < \ell - 1 < \ell < \ell + 1 < j.$$

Then by induction $\gamma_\ell, \gamma_{\ell+1}, \gamma_j$ pairwise intersect, $\gamma_i, \gamma_{\ell-1}, \gamma_\ell$ pairwise intersect, and

$$d_{\gamma_\ell}(\gamma_{\ell+1}, \gamma_j) \leq 3 \quad \text{and} \quad d_{\gamma_\ell}(\gamma_i, \gamma_{\ell-1}) \leq 3.$$

By the triangle inequality, we have

$$d_{\gamma_\ell}(\gamma_i, \gamma_j) \geq d_{\gamma_\ell}(\gamma_{\ell-1}, \gamma_{\ell+1}) - d_{\gamma_\ell}(\gamma_{\ell-1}, \gamma_i) - d_{\gamma_\ell}(\gamma_{\ell+1}, \gamma_j) \geq 16 - 3 - 3 = 10.$$

In particular, $\gamma_i$ and $\gamma_j$ nontrivially intersect. Furthermore, by Proposition 2.2, we have

$$d_{\gamma_i}(\gamma_\ell, \gamma_j) \leq 3 \text{ and } d_{\gamma_j}(\gamma_i, \gamma_\ell) \leq 3,$$

as required.

If we do not have $i < \ell - 1 < \ell < \ell + 1 < j$, then it must be that either $\ell + 1 = j$ or $\ell - 1 = i$, and we can argue similarly. For example, if $i = \ell - 1$, then $\ell + 1 < j$ and by induction

$$d_{\gamma_\ell}(\gamma_{\ell+1}, \gamma_j) \leq 3 \quad \text{and} \quad d_{\gamma_\ell}(\gamma_i, \gamma_{\ell+1}) \geq 16.$$

So $d_{\gamma_\ell}(\gamma_i, \gamma_j) \geq 13$, thus $i(\gamma_i, \gamma_j) \neq 0$, and applying Proposition 2.2 we have

$$d_{\gamma_i}(\gamma_\ell, \gamma_j) \leq 3 \quad \text{and} \quad d_{\gamma_j}(\gamma_i, \gamma_\ell) \leq 3$$

as required. The case $\ell + 1 = j$ is similar. This completes the induction, and hence proves the claim. △

Observe that part (i) follows from the first part of the claim. For part (ii), let $i < \ell < j$. Then by the claim and the triangle inequality we have

$$|d_{\gamma_\ell}(\gamma_i, \gamma_j) - d_{\gamma_\ell}(\gamma_{\ell-1}, \gamma_{\ell+1})| \leq d_{\gamma_\ell}(\gamma_{\ell-1}, \gamma_i) + d_{\gamma_\ell}(\gamma_{\ell+1}, \gamma_j) \leq 6.$$

So, setting $R = R_0 + 6$, part (ii) of the lemma follows from Inequality (3).

To prove part (iii), we first prove

**Claim.** For any $j \in \mathbb{Z}$, the curves $\gamma_{j+1}, \gamma_{j+2}, \ldots, \gamma_{j+n}$ fill $S$.

*Proof.* By applying an appropriate power of $f$, and cyclically permuting the original indices $1, 2, \ldots, n$, it suffices to prove that $\gamma_1, \ldots, \gamma_n$ fill $S$. For this, we show that for any $1 \leq j \leq n$, the subsurface $X_j$ filled by $\gamma_1, \ldots, \gamma_j$ is the same as the subsurface $Z_j$ filled by $c_1, \ldots, c_j$. We do this by induction on $j$.

The base case is $j = 1$, and then $\gamma_1 = c_1$, so $X_1 = Z_1$ is the annular neighborhood. Now suppose that $X_{j-1} = Z_{j-1}$ for some $j \geq 2$ and we prove $X_j = Z_j$. First observe that

$$f_{j-1} = T_{c_1}^{k_1} \cdots T_{c_{j-1}}^{k_{j-1}}$$

is supported on $Z_{j-1} = X_{j-1}$ since $c_1, \ldots, c_{j-1}$ are contained in $Z_{j-1}$. If $c_j \subset Z_{j-1}$, then $Z_j = Z_{j-1}$, while on the other hand

$$\gamma_j = f_{j-1} T_{c_j}^{k_j}(c_j) = f_{j-1}(c_j) \subset Z_{j-1} = X_{j-1}$$

and hence $X_j = X_{j-1} = Z_{j-1} = Z_j$. Thus if $c_j \subset Z_{j-1}$, we are done. So, suppose $c_j \not\subset Z_{j-1}$. Then $Z_j$ is determined by $Z_{j-1}$ and the isotopy classes of arcs of $c_j - Z_{j-1}$ in $S - Z_{j-1}$. We will be done if we can show that these isotopy classes of arcs are the same as those of $\gamma_j - X_{j-1}$ in $S - X_{j-1} = S - Z_{j-1}$. For this, observe that as above $\gamma_j = f_{j-1}(c_j)$, and since $f_{j-1}$ is supported on $X_{j-1} = Z_{j-1}$, $f_{j-1}$ cannot change the isotopy classes of arcs of $c_j - Z_{j-1}$. Hence $\gamma_j - X_{j-1} = \gamma_j - Z_{j-1}$ is isotopic to $c_j - Z_{j-1}$, as required. This proves the claim. △

Now observe that by $f$-invariance, if $|j-i| \le n$, then $d(\gamma_i, \gamma_j) \le A'$ for some constant $A'$. In particular, $d(\gamma_i, \gamma_j) \le A'|j-i|$ for $0 < |j-i| \le n$. By the triangle inequality, $d(\gamma_i, \gamma_j) \le A'|j-i|$ for all $i, j$.

At this point we further assume that $K \ge R_0 + 16 + M$, where $M$ is the constant from Proposition 2.1. Consider any geodesic $\sigma$ in $\mathcal{C}(S)$ from $\gamma_i$ to $\gamma_j$ and list the vertices consecutively as $\gamma_i = \alpha_0, \alpha_1, \ldots, \alpha_r = \gamma_j$ from $\gamma_i$ to $\gamma_j$. The bound on $K$ implies $d_{\gamma_\ell}(\gamma_i, \gamma_j) > M$, for all $i < \ell < j$. So by Proposition 2.1 there is a vertex $\alpha_s$ of $\sigma$ which is disjoint from $\gamma_\ell$. There may be more than one, but there can be at most 3 since $\sigma$ is a geodesic (if there were more than three, two would be distance at least 3 apart, which is impossible since they are distance 1 from $\gamma_\ell$). For each such $\ell$, let $\alpha_{s(\ell)}$ be the vertex closest to $\gamma_j$ which is disjoint from $\gamma_\ell$. As in [3, Lemma 4.4], $s(\ell) \le s(\ell')$ if $\ell \le \ell'$. On the other hand, since every $n$ consecutive curves fill, we have $s(\ell) < s(\ell + n)$. Consequently, the number of vertices in $\sigma$ between $\gamma_i$ and $\gamma_j$ is at least $\frac{j-1}{n}$ and hence the distance is at least

$$d(\gamma_i, \gamma_j) \ge \frac{j-i}{n} - 1.$$

This provides the desired lower bound, and hence $\{\gamma_j\}$ is a uniform quasi-geodesic.

Finally, for part (iv), we note that since $\{\gamma_j\}_{j\in\mathbb{Z}}$ is a quasi-geodesic, and is $f$-invariant, $f$ must be pseudo-Anosov, and we have

$$\lim_{j\to\pm\infty} \gamma_j = \mu^{\pm},$$

in the Hausdorff topology on $S$, after throwing away any isolated leaves of the limit. Therefore, for every $\ell \in \mathbb{Z}$, every arc of $\pi_{\gamma_\ell}(\mu^+) \cup \pi_{\gamma_\ell}(\mu^-)$ is a limit of arcs in $\pi_{\gamma_\ell}(\gamma_j) \cup \pi_{\gamma_\ell}(\gamma_{-j})$, as $j$ tends to infinity. Since some limits of arcs in the latter set can disappear (since isolated leaves of the Hausdorfff limits are discarded), the difference in diameters between the former and latter sets (for $j$ sufficiently large) is at most 2. Part (iv) now follows from part (ii). □

Now suppose $c_1, \ldots, c_n$ are as above, $\kappa_1, \ldots, \kappa_n \in \{\pm 1\}$, and $m \ge K$, with $K$ as in Lemma 4.1. Let $k_j(m) = \kappa_j m$ for $1 \le j \le n$, and extend this to $\{k_j(m)\}_{j\in\mathbb{Z}}$ as above. Construct a sequence of homeomorphisms $\{f_m : S \to S\}_{m=1}^{\infty}$ by

$$f_m = T_{c_1}^{k_1(m)} T_{c_2}^{k_2(m)} \cdots T_{c_n}^{k_n(m)}. \tag{4}$$

**Proposition 4.2.** *Let $\{f_m : S \to S\}_{m=K}^{\infty}$ be a sequence of pseudo-Anosov homeomorphisms defined as in equation (4), $Q(m) = \{q_t(m)\}$ the associated flat metrics, and $\{\gamma_j(m)\}_{j\in\mathbb{Z}}$ the associated $f_m$-invariant collection of curves, for each $m$. Then for all $j$,*

$$M(\gamma_j(m), Q(m)) \ge \frac{m - R - 6}{2},$$

*where $R$ is the constant from Lemma 4.1. Furthermore, there is a constant $D > 0$ so that for any $m$ and curve $\gamma \notin \{\gamma_j(m)\}_{j\in\mathbb{Z}}$,*

$$M(\gamma, Q(m)) \le D.$$

*Proof.* Let $\mu^{\pm}(m)$ denote the stable/unstable foliations of $f_m$. Since $|k_j(m)| = m \geq K$, $\{\gamma_j(m)\}_{j\in\mathbb{Z}}$ satisfies the conclusion of the Lemma 4.1. Combining this with Proposition 3.1 we have

$$M(\gamma_j(m), Q(m)) \geq \frac{d_{\gamma_j(m)}(\mu^+(m), \mu^-(m))}{2} - 3 \geq \frac{m - R - 6}{2}.$$

This proves the first statement.

Let $X_{f_m}$ denote the mapping torus of $f_m$ equipped with its hyperbolic metric, and $\widetilde{X}_{f_m}$ the cover of $X_{f_m}$ corresponding to the fiber subgroup $\pi_1(S)$. Appealing to the *Short Curve Theorem* of Minsky [16] (see also the *Length Bound Theorem* from Brock-Canary-Minsky's [5]), the curves $\gamma_j(m)$ all have length in $\widetilde{X}_{f_m}$ tending to zero as $m$ tends to infinity. Being $f_m$-invariant, they push down to $n$ closed geodesics in $X_{f_m}$.

The geometric limit of the sequence of hyperbolic 3-manifolds $X_{f_m}$ is the cusped hyperbolic 3-manifold $X_\infty$ obtained by drilling out the $n$ curves, realized on $n$ different fibers of $X_{f_m}$ (see [21]) and $X_{f_m}$ is obtained from $X_\infty$ by $(1, k_j(m))$-Dehn filling on $X_\infty$ for all $m > 0$ as in [10]. The geometric convergence ensures that there is a uniform lower bound to the length of any curve in $X_{f_m}$ which is not one of the $n$ curves, and hence there is a uniform lower bound (independent of $m$) to the length of any curve $\gamma$ in $\widetilde{X}_{f_m}$ which is not in $\{\gamma_j(m)\}_{j\in\mathbb{Z}}$. By the Short Curve Theorem again, it follows that $d_\gamma(\mu^+(m), \mu^-(m))$ is uniformly bounded, independent of $m$ and $\gamma$. By Propostion 3.1, the modulus $M_t(\gamma)$ of any $q_t(m)$-Euclidean cylinder with core curve isotopic to $\gamma$ is uniformly bounded, independent of $m$ and $\gamma$, as required. □

The following provides a useful mechanism for deciding when a pseudo-Anosov $f: S \to S$ constructed as above is not a virtual lift.

**Theorem 4.3.** *Suppose $\{f_m: S \to S\}_m$ is a sequence of pseudo-Anosov homeomorphisms as in equation (4), $\{\{\gamma_j(m)\}_{j\in\mathbb{Z}}\}_m$ are the associated sequences of curves, and that the stretch factors $\lambda(f_m)$ have degree greater than 2 over $\mathbb{Q}$. Then there exists a positive integer $N \geq K$, so that if $m \geq N$ and $f_m$ is a virtual lift of some $\phi_m: \Sigma_m \to \Sigma_m$ via a branched covering $p_m: S \to \Sigma_m$, then there are representatives of the curves $\gamma_j(m)$ so that $p_m^{-1}(p_m(\gamma_j(m))) = \gamma_j(m)$ for all $j$.*

The choice of representative $\gamma_j(m)$ is a convenience for the statement: for an arbitrary representative, all components of the preimage of the image will be seen to be isotopic.

*Proof.* To begin, assume $N$ is large enough so that if $m \geq N$, then Proposition 4.2 ensures that for all $j$, $M(\gamma_j(m), Q(m)) > 0$ and hence $\gamma_j(m)$ is a cylinder curve.

Suppose that $p_m: S \to \Sigma_m$ is a branched covering and $\phi_m$ a map that lifts to a power of $f_m$. Since $\lambda(f_m)$ is not quadratic irrational, $\Sigma_m$ is not a sphere with

four marked points or a torus. Let $\Xi(m) = \{\xi_t(m)\}$ and $Q(m) = \{q_t(m)\}$ be the associated compatible family of flat metrics. By Lemmas 3.2 and 3.4, for each $j$, we can choose a representative of $\gamma_j(m)$ so that $p_m(\gamma_j(m))$ is a cylinder curve with

$$M(p_m(\gamma_j(m)), \Xi(m)) \geq \frac{M(\gamma_j(m), Q(m))}{2}.$$

On the other hand, by the Riemann–Hurwitz Theorem, there is a bound $d$ on the degree of $p_m$. Let $B = B(S, d)$ be the constant from Lemma 3.5. Then there is a sub-cylinder $Y_j(m)$ of the cylinder about $p_m(\gamma_j(m))$ so that each component of $\widetilde{Y}_j(m) = p_m^{-1}(Y_j(m))$ is a Euclidean cylinder and has maximal modulus at least

$$BM(p_m(\gamma_j(m)), \Xi(m)) \geq \frac{BM(\gamma_j(m), Q(m))}{2}.$$

One component of $\widetilde{Y}_j(m)$ is contained in the original cylinder with core curve $\gamma_j(m)$. Without loss of generality, we may choose $\gamma_j(m)$ so that $p_m(\gamma_j(m)) \subset Y_j(m)$, and hence $\gamma_j(m) \subset p_m^{-1}(p_m(\gamma_j(m))) \subset \widetilde{Y}_j(m)$.

Let $D > 0$ be the constant from Proposition 4.2. We choose $N > K$ so that if $m \geq N$, then for all $j$

$$\frac{B(m - R - 6)}{4} > D.$$

Then if $\gamma_j(m)'$ is any component of $p_m^{-1}(p_m(\gamma_j(m)))$, and $\widetilde{Y}_j'(m) \subset \widetilde{Y}_j(m)$ is the component containing it, then the bound above on the maximal modulus of $\widetilde{Y}_j'(m)$ combined with Proposition 4.2 implies

$$M(\gamma_j(m)', Q(m)) \geq \frac{BM(\gamma_j(m), Q(m))}{2} \geq \frac{B(m - R - 6)}{4} > D.$$

Consequently, $\gamma_j(m)'$ must be one of the curves $\gamma_{j'}(m)$. However, the direction of $\gamma_j(m)$ and $\gamma_j(m)'$ in the Euclidean cone metric are the same, while if $j' \neq j$, the curves $\gamma_{j'}(m)$ and $\gamma_j(m)$ intersect nontrivially by Lemma 4.1. Therefore, $\gamma_j(m)'$ and $\gamma_j(m)$ must either be equal or isotopic.

Thus, all the components of $p_m^{-1}(p_m(\gamma_j(m)))$ are isotopic to $\gamma_j(m)$, and are hence contained in a single cylinder. By Lemma 3.2, either $p_m$ restricted to this cylinder is a covering map—in which case, $p_m^{-1}(p_m(\gamma_j(m))) = \gamma_j(m)$, and we are done—or the image of the cylinder is a half-pillowcase. If the latter happens, then we take $\gamma_j(m)$ to be the unique core curve in the cylinder that projects to the core geodesic segment of the half-pillowcase, we get $p_m^{-1}(p_m(\gamma_j(m))) = \gamma_j(m)$, as required. □

**Corollary 4.4.** *In addition to the assumptions from Theorem 4.3, suppose that $i(c_j, c_{j+1}) = 1$ for some $j$. If $m \geq N$, and $f_m$ is a virtual lift of some $\phi_m\colon \Sigma_m \to \Sigma_m$ via a branched covering $p_m\colon S \to \Sigma_m$, then $\Sigma_m$ is the quotient by an orientation preserving involution preserving the isotopy classes of $c_1, \ldots, c_m$.*

*Proof.* Choose representatives $\gamma_i(m)$ for the isotopy classes, for all $i$, as in Theorem 4.3. Note that $i(\gamma_j(m), \gamma_{j+1}(m)) = i(c_j, c_{j+1}) = 1$. Since $p_m^{-1}(p_m(\gamma_i(m))) = \gamma_i(m)$ for all $i$, it follows that if $x = \gamma_j(m) \cap \gamma_{j+1}(m)$, then $p_m^{-1}(p_m(x)) = \{x\}$. Since the image of the cylinders about $\gamma_j(m)$ are either cylinders or half-pillowcases, the local degree of $p_m$ near $x$ must be 1 or 2. The degree cannot be 1 since the definition of virtual lift requires a branched cover of degree greater than 1. Therefore, the degree is 2 and hence the branched covering is regular (since index 2 subgroups are always normal). The covering group is thus generated by an orientation preserving involution $\tau$.

Since $p_m^{-1}(p_m(\gamma_i(m))) = \gamma_i(m)$ for all $i$, it follows that $\tau(\gamma_i(m)) = \gamma_i(m)$. We now show that $\tau(c_i) = c_i$ for each $i = 1, \ldots, n$. For $i = 1$, note that $c_1 = \gamma_1(m)$. We use this as the base case for induction. Assuming $\tau(c_i) = c_i$ for all $1 \le i \le \ell < n$, we prove that $\tau(c_{\ell+1}) = c_{\ell+1}$. For this, observe that

$$\gamma_{\ell+1}(m) = T_{c_1}^{k_1(m)} \cdots T_{c_\ell}^{k_\ell(m)} T_{c_{\ell+1}}^{k_{\ell+1}(m)}(c_{\ell+1}) = T_{c_1}^{k_1(m)} \cdots T_{c_\ell}^{k_\ell(m)}(c_{\ell+1})$$

since $T_{c_{\ell+1}}$ fixes $c_{\ell+1}$. Therefore, we have

$$T_{c_\ell}^{-k_\ell(m)} \cdots T_{c_1}^{-k_1(m)}(\gamma_{\ell+1}(m)) = c_{\ell+1}.$$

Since $\tau$ preserves each of $c_1, \ldots, c_\ell$, it commutes with $T = T_{c_\ell}^{-k_\ell(m)} \cdots T_{c_1}^{-k_1(m)}$, thus the equations $T(\gamma_{\ell+1}(m)) = c_{\ell+1}$ and $\tau(\gamma_{\ell+1}(m)) = \gamma_{\ell+1}(m)$ imply

$$\tau(c_{\ell+1}) = \tau T(\gamma_{\ell+1}(m)) = \tau T \tau^{-1} \tau(\gamma_{\ell+1}(m)) = T(\gamma_{\ell+1}(m)) = c_{\ell+1}.$$

This completes the proof. □

**4.1. Strenner's construction.** The key to obtaining the required degree for the dilatation is the following special case of a result of Strenner [19, Theorem 5.3], building on a theorem of Penner [17].

**Theorem 4.5** (Strenner). *Suppose $A = a_1 \cup \ldots \cup a_n$ and $B = b_1 \cup \cdots \cup b_n$ are multicurves that fill the surface $S$, and let $N = (i(a_i, b_j))_{ij}$ be the matrix of intersection numbers and $G$ the associated (bipartite) adjacency graph (with a vertex for every $a_i$ and every $b_j$ and an edge between $a_i$ and $b_j$ if $i(a_i, b_j) \neq 0$). Suppose*

(1) *$rk(N) = r > 0$,*

(2) *$a_{i_1} b_{i_1} a_{i_2} b_{i_2} \cdots a_{i_d} b_{i_d} a_{i_1}$ are the vertices of a closed, contractible loop in $G$ visiting every vertex.*

*Then for all $m > 0$ sufficiently large, the mapping classes*

$$f_m = T_{a_{i_1}}^m T_{b_{i_1}}^{-m} \cdots T_{a_{i_d}}^m T_{b_{i_d}}^{-m}$$

*are pseudo-Anosov and $\lambda(f_m)$ has degree $2r$.*

We note that Strenner's theorem from [19] states that the degree of the stretch factor should be the rank of the intersection matrix, whereas Theorem 4.5 states that the degree should be twice the rank. The discrepancy comes from the fact that in [19], Strenner considers the matrix of intersection numbers of all pairs of curves. Since $i(a_i, a_j) = i(b_i, b_j) = 0$ for all $i, j$, Strenner's intersection matrix is given by

$$\begin{pmatrix} \mathbf{0} & N \\ N^T & \mathbf{0} \end{pmatrix},$$

where the $\mathbf{0}$'s are appropriately sized zero-matrices. Strenner's matrix thus has rank exactly twice the rank of $N$.

## 5. Proof of the main theorem

We will apply the results of the preceding section to a particular pair of multicurves. For this, we start with a particular pair of simple closed curves $a, b$ that fill a genus 3 surface $X$ with one boundary component and intersect in exactly 5 points with the same sign (after orienting them appropriately). This pair is described in Figure 2.

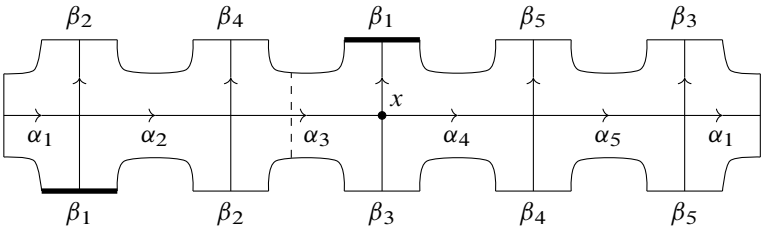


Figure 2. The curves $a$ and $b$ are cut into arcs $a = \alpha_1 \cup \cdots \cup \alpha_5$ and $b = \beta_1 \cup \cdots \cup \beta_5$ at the points of intersection $a \cap b$. The surface $X$ of genus 3 with one boundary component is shown, cut open along essential arcs meeting each of the arcs $\beta_1, \ldots, \beta_5$ and $\alpha_1$ as labelled. The point $x$ is the fixed point of an involution $\tau$ of $X$ leaving each of $a$ and $b$ invariant. The thick line represents an essential arc $\delta$ meeting $b$ in the arc $\beta_1$.

**Lemma 5.1.** *Up to isotopy, the surface $X$ admits exactly one orientation preserving involution $\tau$ leaving both $a$ and $b$ invariant.*

*Proof.* Let $\tau\colon X \to X$ denote the "obvious" involution of $X$, evident in Figure 2, given by rotation about the point $x$—it is straightforward to check that the rotation extends over the gluing of the arcs in the reconstruction of $X$. To see that $\tau$ is the only orientation preserving involution preserving $a$ and $b$, we note that such an involution would define a graph automorphism of $a \cup b$, viewed as a four-valent

graph with 5 vertices, and would preserve the cyclic ordering around each vertex. Any such nontrivial graph automorphism would necessarily fix one of the vertices, and would be determined by which vertex it fixed. It is now easy to show that the only such nontrivial graph automorphism is $\tau$. □

We now prove the following theorem, which implies the Main Theorem in the introduction.

**Theorem 5.2.** *For each integer $r > 1$ and closed orientable surface $S = S_g$ with $g \geq r + 2$, there exists a pseudo-Anosov homeomorphism $f: S \to S$ with stretch factor $\lambda(f)$ of degree $2r$ and orientable foliations that is not a virtual lift.*

*Proof.* Embed the surface $X$ of genus 3 with one boundary component as an essential subsurface of $S_g$. The complement $Z$ is a surface of genus $g - 3$ with one boundary component. Let $a_1 = a$ and $b_1 = b$ as constructed above. The arc $\delta$ from Figure 2 can be connected to a nonseparating arc $\delta'$ in $Z$ to construct an essential simple closed curve we denote $a_2$, that has intersection number 1 with $b_1$ and 0 with $a_1$.

If $r = 2$, then we choose any essential simple closed curve $b_2$ in $Z$ which fills with $\delta'$ so that all $k$ intersection points have the same sign, which is possible since $\delta'$ is a nonseparating arc in $Z$. The intersection matrix is

$$(i(a_i, b_j)) = \begin{pmatrix} 5 & 1 \\ 0 & k \end{pmatrix}.$$

This has rank 2. Now consider the sequence of mapping classes defined by:

$$f_m = T_{a_1}^m T_{b_1}^{-m} T_{a_2}^m T_{b_2}^{-m} T_{a_2}^m T_{b_1}^{-m}.$$

On the one hand, the sequence $\{f_m\}_{m=1}^{\infty}$ satisfies the hypothesis of Theorem 4.5, and so for $m$ sufficiently large, the $f_m$ are pseudo-Anosov, and have stretch factors $\lambda(f_m)$ having degree 4 over $\mathbb{Q}$. On the other hand, consecutive curves in the sequence $a_1, b_1, a_2, b_2, a_2, b_1$ intersect nontrivially (cyclically), and $i(b_1, a_2) = 1$. Consequently, the sequence $\{f_m\}$ also satisfies Corollary 4.4, and so by taking $m$ larger if necessary, it follows that if $f_m$ is a virtual lift via a branched covering $p_m: S \to \Sigma_m$, then $p_m$ has degree two, and $\Sigma_m$ is the quotient by an orientation preserving involution $\tau$ preserving $a_1, b_1, a_2, b_2$. The involution $\tau$ must restrict to $\tau$ on $X$ (up to isotopy) by Lemma 5.1. However, $\tau$ does not preserve the isotopy class of $\delta$ in $X$, and so $\tau$ cannot preserve $b_1$, a contradiction. Therefore, there is no such involution $\tau$, and hence $f$ is not a virtual lift.

From Penner's construction, the invariant foliations are carried by bigon tracks obtained by smoothing the points of intersection. In our construction the curves can be oriented so the intersection points have all the same sign, and so these tracks are orientable. Therefore, the invariant foliations are orientable.

If $r > 2$, we proceed in a similar fashion, choosing a curve $b_2$ that intersects $a_2$ once and is disjoint from all other curves. Note that this is possible since $\delta'$ was a nonseparating arc in $Z$. We continue, choosing $a_3$ intersecting $b_2$ once and disjoint from all other curves, $b_3$ intersecting $a_3$ once and disjoint from all other curves, etc., until we obtain a set of curves

$$a_1, \quad b_1, \quad a_2, \quad b_2, \quad \ldots, \quad a_{r-1}, \quad b_{r-1}, \quad a_r.$$

That this is possible follows from an Euler characteristic computation, and the classification of surfaces. We finally choose $b_r$ so that the union of all the curves fills $S$ and so that $b_r$ is disjoint from all curves except $a_r$, which it intersects in $k$ points, for some $k > 0$, all with the same sign. The $r \times r$ intersection matrix now has the form

$$(i(a_i, b_j)) = \begin{pmatrix} 5 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & k \end{pmatrix}$$

This has determinant $5k$, and so has rank $r$. As above, we can now use Theorem 4.5 and Corollary 4.4 to construct a sequence of pseudo-Anosov homeomorphisms

$$f_m = T_{a_1}^m T_{b_1}^{-m} \cdots T_{a_r}^m T_{b_r}^{-m} T_{a_r}^m \cdots T_{a_2}^m T_{b_1}^{-m},$$

and arguing exactly as in the case $r = 2$ to deduce that for $m$ sufficiently large $\lambda(f_m)$ has degree $2r$ over $\mathbb{Q}$, and that $f_m$ is not a virtual lift. □

**5.1. Veech groups and degree 4.** Recall that the Veech group of a flat metric $q$ defined by a quadratic differential is the group of all mapping classes represented by affine homeomorphisms with respect to $q$; see e.g. [8]. Since the proof of the Main Theorem involves showing that there is no branched covering $S \to \Sigma$ so that the flat metric on $S$ pulls back from one on $\Sigma$, it actually shows that no pseudo-Anosov in the Veech group for that flat metric is a virtual lift. We do not know, however, if there are any elements in the Veech group other than the pseudo-Anosov homeomorphism (and its powers) which we constructed.

It turns out that to prove the theorem in the special case of $d = 2r = 4$, one may bypass much of the technical machinery from Section 4 by appealing to the *Thurston construction*; see [22, 9]. This has the added benefit that the Veech group is nonelementary (and hence contains a nonabelian free, purely pseudo-Anosov subgroup), no pseudo-Anosov element of which is a virtual lift.

**Theorem 5.3.** *For any genus $g \geq 4$, there exists a nonelementary Veech group $G$ in* $\mathrm{Mod}(S_g)$ *so that no pseudo-Anosov element of $G$ is a virtual lift.*

*Proof.* To begin, for every $g \geq 4$, we will construct a pair of multicurves $A = a_1 \cup a_2$ and $B = b_1 \cup b_2$ with $i(a_1, b_1) = 5$, $i(a_1, b_2) = 1 = i(a_2, b_1)$, and $i(a_2, b_2) = k$, where

$$k = \begin{cases} 0 & \text{for } g = 4, \\ 3 & \text{for } g = 5, \\ 2g - 8 & \text{for } g \geq 6. \end{cases}$$

To do this, we again embed the genus 3 surface with one boundary component $X$ in $S_g$, and take $a_1 = a$ and $b_1 = b$, so that $i(a_1, b_1) = 5$. Next, we describe how to construct the curves $a_2$ and $b_2$. The intersection of these curves with $X$ will be the thick horizontal arc and the dashed vertical arc from Figure 2, respectively. In particular, $i(a_1, b_2) = 1 = i(a_2, b_1)$. From the figure we see that the endpoints of these two arcs alternate around the boundary of $X$ (i.e. the endpoints of one arc link with the endpoints of the other). In the complement, $S_g \setminus X$, we construct a pair of arcs whose endpoints also alternate around the common boundary $\partial X$ and fill $S_g \setminus X$ as follows. First, on a closed surface of genus $g - 3$, take a minimally intersecting, filling pair of curves; see e.g. [1]. If $g - 3 = 1$, then this is a pair of curves intersecting once; when $g - 3 = 2$ this is a pair of curves intersecting 4 times; and on a genus $g - 3 \geq 3$ surface, this is a pair of curves intersecting $2(g - 3) - 1 = 2g - 7$ times. Now at one of the points of intersection removing a small disk produces a pair of arcs on a surface of genus $g - 3$ that intersect $k$ times, with $k$ as above. We identify this surface with $S_g \setminus X$ so that the arcs glue up and produce the curves $a_2, b_2$. Thus $i(a_2, b_2) = k$, as required.

Next, we recall that from $A$ and $B$, the Thurston construction produces a flat metric $q$ from a quadratic differential so that the multitwists $T_A = T_{a_1} T_{a_2}$ and $T_B = T_{b_1} T_{b_2}$ have affine representatives with respect to $q$. The derivatives in preferred local coordinates for $q$, which are well-defined up to sign, are given by

$$DT_A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad DT_B = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix},$$

where $\mu^2$ is the spectral radius of the matrix $NN^T$, and $N$ is the intersection matrix

$$N = \begin{pmatrix} 5 & 1 \\ 1 & k \end{pmatrix}.$$

Since $N$ is symmetric, $\mu$ is the spectral radius of $N$, and so computing we find $\mu = \frac{1}{2}(5 + k + \sqrt{(5 - k)^2 + 4})$.

Letting $G$ denote the Veech group of $q$, $\langle T_A, T_B \rangle$ is a nonelementary subgroup of $G$. By the chain rule the derivative of elements in $G$ in preferred local coordinates defines a homomorphism $D\colon G \to \mathrm{PSL}(2, \mathbb{R})$, and an element $f \in G$

is pseudo-Anosov if and only $|\mathrm{Trace}(Df)| > 2$. In this case, $\lambda(f)$ is equal to the spectral radius of $Df$, and the associated 1-parameter family of flat metrics associated to $f$ are all affine deformations of $q$. For example, for the element $f_0 = T_A T_B^{-1}$, we can compute $\mathrm{Trace}(Df_0) = 2 + \mu^2 > 2$. Thus $f_0$ is pseudo-Anosov, and $\lambda(f_0)$ is the largest root of $x^2 - (2 + \mu^2)x + 1$.

Next we claim that the number $\mu$ (and hence $\mu^2$) is quadratic irrational. For this, it suffices to prove that $(5-k)^2+4$ is not a perfect square, $z^2$, for an integer $z$. If it were, then $z^2 - (5-k)^2 = 4$. But the only pair of squares that differ by 4 are 0 and 4. However, from the description of $k$ above (depending on $g$), we see that $k$ is never equal to 5, so $(k-5)^2 + 4$ is never a square. Therefore, $\mu$ and $\mu^2$ are quadratic irrational numbers, and hence so is $\mathrm{Trace}(Df_0) = 2 + \mu^2$.

According to [8, Theorem 28] (see also [15, Corollary 9.6]) for any other pseudo-Anosov element $f \in G$, $\mathrm{Trace}(Df)$ will also be a quadratic irrational and generate the same extension $\mathbb{Q}(\mu^2) = \mathbb{Q}(\mu)$ over $\mathbb{Q}$. Since (up to sign) the stretch factor $\lambda = \lambda(f)$ and its inverse $\lambda^{-1}$ are roots of the polynomial $x^2 - \mathrm{Trace}(Df)x + 1$, it follows that either $\lambda$ and $\lambda^{-1}$ are Galois conjugates and have degree 2 over $\mathbb{Q}(\mu)$ and degree 4 over $\mathbb{Q}$, or else they are not Galois conjugates, they lie in $\mathbb{Q}(\mu)$ and so have degree 2 over $\mathbb{Q}$. We claim that the latter case cannot happen. To see this, note that $\lambda$ is a unit in the ring of integers, and hence its minimal polynomial has the form $x^2 - kx \pm 1$ for some $k \in \mathbb{Z}$. Since $\lambda$ and $\lambda^{-1}$ are not Galois conjugates, it follows that $\lambda$ and $-\lambda^{-1}$ are Galois conjugates, and hence so are $\lambda^{-1}$ and $-\lambda$. However, considering the branched cover orienting the foliations, the lifted pseudo-Anosov is orientation preserving and hence both $\lambda$ and $\lambda^{-1}$ or both $-\lambda$ and $-\lambda^{-1}$ are roots of the characteristic polynomial of the action of the lift on homology. Since the minimal polynomial divides this polynomial, we see that all four numbers $\pm\lambda$ and $\pm\lambda^{-1}$ are roots of this polynomial. On the other hand exactly one of $\lambda$ or $-\lambda$ is the unique root of maximal modulus for this polynomial [15]. Since $|\lambda| = |-\lambda|$, this is a contradiction. Therefore, $\lambda$ and $\lambda^{-1}$ have degree 4 and are Galois conjugates. According to [20, Proposition 2.3] no pseudo-Anosov $f \in G$ can be a lift of a pseudo-Anosov homeomorphism on a nonorientable surface. Therefore, we need only consider branched covers of $S$ over orientable surfaces.

**Claim 5.4.** *There is no nontrivial branched covering $p: S \to \Sigma$ where $\Sigma$ is orientable and admits a flat metric $\xi$ from a quadratic differential $\xi$ so that $p^*\xi = q$.*

Note that if we prove this, then no pseudo-Anosov element in $G$ can be a virtual lift since the associated 1-parameter of flat metrics on $S$ are affine deformations of $q$, and so those on $\Sigma$ would have to be affine deformations of a metric $\xi$ with $p^*\xi = q$.

Before we get to the proof of the claim, we first recall that the Thurston construction produces the metric $q$ so as to have horizontal and vertical foliations defining complete cylinder decompositions with core curves representing the isotopy classes of $a_1, a_2$ and $b_1, b_2$, respectively. Furthermore, by symmetry, the heights of the cylinders can be chosen to be $V_1$ for both $a_1$ and $b_1$ and $V_2$ for $a_2$ and $b_2$, where $(V_1, V_2)$ is an eigenvector for the eigenvalue $\mu$ of $N$. Since $\mu$ is quadratic irrational, one can see that $V_2/V_1$ is irrational. Every time $a_i$ crosses $b_j$, it picks up length $V_j$ (and $b_j$ picks up length $V_i$), and hence the $q$-length of $a_1$ and $b_1$ is $5V_1+V_2$, while the $q$-lengths of $a_2$ and $b_2$ are $V_1+kV_2$. So, the moduli of $a_1$ and $b_1$ is $\frac{V_1}{5V_1+V_2}$ and the moduli of $a_2$ and $b_2$ is $\frac{V_2}{V_1+kV_2}$. From Thurston's construction, the ratio of these moduli is rational, but since $\frac{V_1}{V_2}$ is irrational. Therefore, the ratio $r = \frac{5V_1+V_2}{V_1+kV_2}$ of the $q$-lengths of $a_1$ and $a_2$ is irrational (as is the ratio of the $q$-lengths of $b_1$ and $b_2$).

Now, to prove the claim, suppose on the contrary that we have a nontrivial branched covering $p\colon S \to \Sigma$ as in the claim with metric $\xi$ on $\Sigma$. Since the horizontal and vertical foliations of $q$ define a complete cylinder decomposition, so must the horizontal and vertical foliations of $\xi$. In fact, away from the finite set of branch points, the core curves of the $q$-cylinders must push down to core curves for the $\xi$-cylinders. Furthermore, because the ratio of the lengths of the $a_1$ and $a_2$ curves is irrational, the $a_1$ and $a_2$ cylinders must each (branched) cover a cylinder or half-pillow case whose interiors are disjoint (and likewise for the $b_1$ and $b_2$). Thus, for all four $q$-cylinders $C$ (the two vertical and two horizontal), we have $p^{-1}(p(\mathrm{int}(C))) = \mathrm{int}(C)$ (c.f. §3). Since $i(a_1, b_2) = 1$, arguing as in Corollary 4.4, the degree of $p$ must be 2, and there must be an orientation preserving involution of $S$ leaving each of $a_1$, $a_2$, $b_1$, and $b_2$ invariant. We now complete the argument as in the proof of the Main Theorem: up to isotopy the involution must send $X$ to itself and the arc of $a_2$ intersecting $X$ to itself. Since there is a unique involution of $X$ leaving $a$ and $b$ invariant by Lemma 5.1, and since this involution does not preserve the arc of intersection of $a_2$ with $X$, we have a contradiction.

Therefore, no pseudo-Anosov element of $G$ is a virtual lift. □

**Remark 5.5.** The referee has pointed out that the pseudo-Anosov mapping classes $f$ from Theorem 4.5 have the property that $\lambda = \lambda(f)$ and $\lambda^{-1}$ are Galois conjugates. Therefore, such $f$ cannot be a virtual lift from a non-orientable surface, as was argued in this last proof, appealing to [20, Proposition 2.3]. This means that one could avoid Lemma 3.4 in the proof of the Main Theorem, shortening the argument. Since that lemma is elementary and may be of independent interest, we have opted to keep the original proof.

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