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Frattini and Related Subgroups of Mapping Class Groups

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To V.P. Platonov on the occasion of his 75th birthday

Abstract—Let $\Gamma_{g,b}$ denote the orientation-preserving mapping class group of a closed orientable surface of genus g with b punctures. For a group G let $\Phi_f(G)$ denote the intersection of all maximal subgroups of finite index in G. Motivated by a question of Ivanov as to whether $\Phi_f(G)$ is nilpotent when G is a finitely generated subgroup of $\Gamma_{g,b}$, in this paper we compute $\Phi_f(G)$ for certain subgroups of $\Gamma_{g,b}$. In particular, we answer Ivanov's question in the affirmative for these subgroups of $\Gamma_{g,b}$.

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1. INTRODUCTION

We fix the following notation throughout this paper: Let $\Gamma_{g,b}$ denote the orientation-preserving mapping class group of a closed orientable surface of genus g with b punctures. When $b = 0$, we simply write Γ_g. In addition, when $b > 0$, we let $\mathrm{P}\Gamma_{g,b}$ denote the *pure mapping class group*, i.e. the subgroup of $\Gamma_{g,b}$ consisting of those elements that fix the punctures pointwise. The Torelli group $\mathcal{I}_g$ is the subgroup of Γ_g arising as the kernel of the homomorphism $\Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z})$ coming from the action of Γ_g on $H_1(\Sigma_g, \mathbb{Z})$. As usual, $\mathrm{Out}(F_n)$ will denote the outer automorphism group of a free group of rank $n \geq 2$.

For a group G, the *Frattini subgroup* $\Phi(G)$ of G is defined to be the intersection of all maximal subgroups of G (if they exist); otherwise it is defined to be the group G itself. (Here a maximal subgroup is a strict subgroup which is maximal with respect to inclusion.) In addition we define $\Phi_f(G)$ to be the intersection of all maximal subgroups of finite index in G. Note that $\Phi(G) < \Phi_f(G)$.[1]

Frattini's original theorem is that if G is finite, then $\Phi(G) = \Phi_f(G)$ is a nilpotent group (see, for example, [29, Theorem 11.5]). For infinite groups this is not the case: there are examples of finitely generated infinite groups G with $\Phi(G)$ not nilpotent [15, p. 328]. On the other hand, in [25] Platonov showed that if G is any finitely generated linear group, then $\Phi(G)$ and $\Phi_f(G)$ are nilpotent.

Motivated by the question as to whether Γ_g is a linear group, in [21], Long proved that $\Phi(\Gamma_g) = 1$ for $g \geq 3$, and $\Phi(\Gamma_2) = \mathbb{Z}/2\mathbb{Z}$. This was extended by Ivanov [18], who showed that (as in the linear case) $\Phi(G)$ is nilpotent for any finitely generated subgroup $G < \Gamma_{g,b}$. Regarding $\Phi_f(G)$, in [18] (and then again in [19]), Ivanov asks whether the same is true for Φ_f:

Question (Ivanov [18, 19]). Is $\Phi_f(G)$ nilpotent for every finitely generated subgroup G of $\Gamma_{g,b}$?

The aim of this note is to prove some results in the direction of answering Ivanov's question. In particular, the following theorem answers Ivanov's question in the affirmative for Γ_g and some of its subgroups in the case where $g \geq 3$.

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[1] We use the notation $G_1 < G_2$ to indicate that G_1 is a subgroup of G_2 (including the case where $G_1 = G_2$).

Theorem 1.1. *Suppose that $g \geq 3$, and that G is either*

(i) *the mapping class group Γ_g, or*

(ii) *a normal subgroup of Γ_g (for example, the Torelli group $\mathcal{I}_g$, the Johnson kernel $\mathcal{K}_g$, or any higher term in the Johnson filtration of Γ_g), or*

(iii) *a subgroup of Γ_g which contains a finite index subgroup of the Torelli group $\mathcal{I}_g$.*

Then $\Phi_f(G) = 1$.

Remarks. 1. Since $\Phi(G) < \Phi_f(G)$, our methods also give a different proof of Long's result that $\Phi(\Gamma_g) = 1$ for $g \geq 3$. As for the case $g \leq 2$, note that Γ_1 and Γ_2 are linear (see [6] for $g = 2$) and so Platonov's result [25] applies to answer Ivanov's question in the affirmative in these cases for all finitely generated subgroups. On the other hand, for $g \geq 3$ the mapping class group is not known to be linear and no other technique for answering Ivanov's question was known. In fact, as pointed out by Ivanov, neither of the methods of [21] or [18] apply to Φ_f, and so even the case of Φ_f of the mapping class group itself was not known.

2. Note that in Platonov's and Ivanov's theorems and in Ivanov's question, the Frattini subgroup and its variant Φ_f are considered for finitely generated subgroups. In reference to Theorem 1.1 above, it remains an open question as to whether the Johnson kernel $\mathcal{K}_g$, or any higher term in the Johnson filtration of Γ_g, is finitely generated or not.

Perhaps the most interesting feature of the proof of Theorem 1.1 is that it is another application of the projective unitary representations arising in Topological Quantum Field Theory (TQFT) first constructed by Reshetikhin and Turaev [26] (although as in [23], the perspective here is that of the skein-theoretical approach of [7]).

We are also able to prove

Theorem 1.2. *Assume that $b > 0$; then $\Phi_f(\mathrm{P}\Gamma_{g,b})$ is either trivial or $\mathbb{Z}/2\mathbb{Z}$. Indeed, $\Phi_f(\mathrm{P}\Gamma_{g,b}) = 1$ unless $(g, b) \in \{(1, b), (2, b)\}$.*

The reason for separating out the case when $b > 0$ is that the proof does not directly use the TQFT framework, but rather makes use of Theorem 1.1(i) in conjunction with the Birman exact sequence and a general group-theoretic lemma (see Section 6). We expect that the methods of this paper will also answer Ivanov's question for $\Gamma_{g,b}$, but at present we are unable to do so. We comment further on this at the end of Section 6.

In addition our methods can also be used to give a straightforward proof of the following.

Theorem 1.3. *Suppose that $n \geq 3$; then $\Phi(\mathrm{Out}(F_n)) = \Phi_f(\mathrm{Out}(F_n)) = 1$.*

Note that it was shown in [17] that $\Phi(\mathrm{Out}(F_n))$ is finite.

As remarked upon above, this note was largely motivated by the questions of Ivanov. To that end, we discuss a possible approach to answering Ivanov's question in general using the aforementioned projective unitary representations arising from TQFT, coupled with Platonov's work [25]. Another motivation for this work arose from attempts to understand the nature of the Frattini subgroup and the center of the profinite completion of Γ_g and $\mathcal{I}_g$. We discuss these further in Section 8.

2. PROVING TRIVIALITY OF Φ_f

Before stating and proving an elementary but useful technical result, we introduce some notation. Let Γ be a finitely generated group, and let $\mathcal{S} = \{G_n\}$ be a collection of finite groups together with epimorphisms $\phi_n \colon \Gamma \to G_n$. We say that Γ is *residually-$\mathcal{S}$* if, given any non-trivial element $\gamma \in \Gamma$, there is some group $G_n \in \mathcal{S}$ and an epimorphism ϕ_n for which $\phi_n(\gamma) \neq 1$. Note that, as usual, this is equivalent to the statement $\bigcap \ker \phi_n = 1$.

Proposition 2.1. *Let* Γ *and* $\mathcal{S}$ *be as above with* Γ *being residually-*$\mathcal{S}$*. Assume further that* $\Phi(G_n) = 1$ *for every* $G_n \in \mathcal{S}$*. Then* $\Phi(\Gamma) = \Phi_f(\Gamma) = 1$*.*

Before commencing with the proof of this proposition, we recall the following property.

Lemma 2.2. *Let* Γ *and* G *be groups and* $\alpha\colon \Gamma \to G$ *an epimorphism. Then* $\alpha(\Phi(\Gamma)) \subset \Phi(G)$ *and* $\alpha(\Phi_f(\Gamma)) \subset \Phi_f(G)$*.*

Proof. We prove the last statement. Let M be a maximal subgroup of G of finite index. Then $\alpha^{-1}(M)$ is a maximal subgroup of Γ of finite index in Γ, and hence $\Phi_f(\Gamma) \subset \alpha^{-1}(M)$. Thus $\alpha(\Phi_f(\Gamma)) \subset M$ for all maximal subgroups M of finite index in G and the result follows. $\square$

Remark. As pointed out in Section 1, for a finite group G, $\Phi(G) = \Phi_f(G)$.

Proof of Proposition 2.1. We give the argument for $\Phi_f(\Gamma)$; the argument for $\Phi(\Gamma)$ is exactly the same. Thus suppose that $g \in \Phi_f(\Gamma)$ is a non-trivial element. Since Γ is residually-$\mathcal{S}$, there exists some n so that $\phi_n(g) \in G_n$ is non-trivial. However, by Lemma 2.2 (and the remark following it) we have

$$\phi_n(g) \in \phi_n(\Phi_f(\Gamma)) < \Phi_f(G_n) = \Phi(G_n),$$

and in particular $\Phi(G_n) \neq 1$, a contradiction. $\square$

3. QUANTUM REPRESENTATIONS AND FINITE QUOTIENTS

We briefly recall some of [23] (which uses [7] and [14]). As in [23], we only consider the case of p a prime satisfying $p \equiv 3 \pmod 4$.

Let Σ be a closed orientable surface of genus $g \geq 3$. The integral SO(3)-TQFT constructed in [14] provides a representation of a central extension $\widetilde{\Gamma}_g$ of Γ_g,

$$\rho_p\colon\ \widetilde{\Gamma}_g \to \mathrm{GL}(N_g(p), \mathbb{Z}[\zeta_p]),$$

where ζ_p is a primitive pth root of unity, $\mathbb{Z}[\zeta_p]$ is the ring of cyclotomic integers and $N_g(p)$ is the dimension of a vector space $V_p(\Sigma)$ on which the representation acts. It is known that $N_g(p)$ is given by a Verlinde-type formula and goes to infinity as $p \to \infty$. For convenience we simply set $N = N_g(p)$.

As in [23], the image group $\rho_p(\widetilde{\Gamma}_g)$ will be denoted by Δ_g. As indicated in [23], $\Delta_g < \mathrm{SL}(N, \mathbb{Z}[\zeta_p])$, and moreover, Δ_g is actually contained in a special unitary group $\mathrm{SU}(V_p, H_p; \mathbb{Z}[\zeta_p])$, where H_p is a Hermitian form defined over the real field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$.

Furthermore, the homomorphism ρ_p descends to a projective representation of Γ_g (which we denote by $\overline{\rho}_p$):

$$\overline{\rho}_p\colon\ \Gamma_g \to \mathrm{PSU}(V_p, H_p; \mathbb{Z}[\zeta_p]).$$

What we need from [23] is the following. We can find infinitely many rational primes q which split completely in $\mathbb{Z}[\zeta_p]$, and for every such prime $\widetilde{q}$ of $\mathbb{Z}[\zeta_p]$ lying over such a q, we can consider the group

$$\pi_{\widetilde{q}}(\Delta_g) \subset \mathrm{SL}(N, q),$$

where $\pi_{\widetilde{q}}$ is the reduction homomorphism from $\mathrm{SL}(N, \mathbb{Z}[\zeta_p])$ to $\mathrm{SL}(N, q)$ induced by the isomorphism $\mathbb{Z}[\zeta_p]/\widetilde{q} \simeq \mathbb{F}_q$. As shown in [23] (see also [12]), we obtain epimorphisms $\Delta_g \twoheadrightarrow \mathrm{SL}(N, q)$ for all but finitely many of these primes $\widetilde{q}$, and it then follows easily that we obtain epimorphisms $\Gamma_g \twoheadrightarrow \mathrm{PSL}(N, q)$. We denote these epimorphisms by $\rho_{p,\widetilde{q}}$. These should be thought of as reducing the images of $\overline{\rho}_p$ modulo $\widetilde{q}$. That one obtains finite simple groups of the form PSL rather than PSU when q is a split prime is discussed in [23, Sect. 2.2].

Lemma 3.1. *For each* $g \geq 3$, $\bigcap \ker \rho_{p,\widetilde{q}} = 1$.

Proof. Fix $g \geq 3$ and suppose that there exists a non-trivial element $\gamma \in \bigcap \ker \rho_{p,\widetilde{q}}$. Now it follows from asymptotic faithfulness [2, 11] that $\bigcap \ker \overline{\rho}_p = 1$. Thus for some p there exists $\overline{\rho}_p$ such that $\overline{\rho}_p(\gamma) \neq 1$. Now $\rho_{p,\widetilde{q}}(\gamma)$ is obtained by reducing $\overline{\rho}_p(\gamma)$ modulo $\widetilde{q}$, and so there clearly exists $\widetilde{q}$ so that $\rho_{p,\widetilde{q}}(\gamma) \neq 1$, a contradiction. □

4. PROOFS OF THEOREMS 1.1 AND 1.3

The proof of Theorem 1.1 for $G = \Gamma_g$ follows easily as a special case of our next result. To state this, we introduce some notation: If $H < \Gamma_g$, we denote by $\widetilde{H}$ the inverse image of H under the projection $\widetilde{\Gamma}_g \to \Gamma_g$.

Proposition 4.1. *Let $g \geq 3$, and assume that H is a finitely generated subgroup of Γ_g for which $\rho_p(\widetilde{H})$ has the same Zariski closure and adjoint trace field as Δ_g. Then $\Phi(H) = \Phi_f(H) = 1$.*

Proof. We begin with a remark. That the homomorphisms $\rho_{p,\widetilde{q}}$ of Section 3 are surjective is proved using strong approximation. The main ingredients of this are the Zariski density of Δ_g in the algebraic group $\mathrm{SU}(V_p, H_p)$ and the fact that the adjoint trace field of Δ_g is the field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ over which the group $\mathrm{SU}(V_p, H_p)$ is defined (see [23] for more details). In particular, the proof establishes surjectivity of $\rho_{p,\widetilde{q}}$ when restricted to any subgroup $H < \Gamma_g$ satisfying the hypothesis of the proposition.

To complete the proof, the groups $\mathrm{PSL}(N, q)$ are finite simple groups (since the dimensions N are all very large) so their Frattini subgroup is trivial. This follows from Frattini's theorem, or, more simply, from the fact that the Frattini subgroup of a finite group is a normal subgroup which is moreover a strict subgroup (since finite groups do have maximal subgroups). Hence the result follows from Lemma 3.1, Proposition 2.1 and the remark at the start of the proof. □

In particular, Γ_g satisfies the hypothesis of Proposition 4.1, and so $\Phi_f(\Gamma_g) = 1$. This also recovers the result of Long [21] proving triviality of the Frattini subgroup.

The proof of Theorem 1.1 in case (ii), that is, when G is a normal subgroup of Γ_g, follows from this and the following general fact:

Proposition 4.2. *If N is a normal subgroup of a group Γ, then $\Phi_f(N) < \Phi_f(\Gamma)$.*

This fact is known for Frattini subgroups of finite groups, and the proof can be adapted to our situation. We defer the details to Section 5.

In the remaining case (iii) of Theorem 1.1, G is a subgroup of Γ_g which contains a finite index subgroup of the Torelli group $\mathcal{I}_g$. We shall show that G satisfies the hypothesis of Proposition 4.1, and deduce $\Phi_f(G) = 1$ as before.

Consider first the case where G is the Torelli group $\mathcal{I}_g$ itself. Recall the short exact sequence

$$1 \to \mathcal{I}_g \to \Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1.$$

We now use the following well-known facts:

- Γ_g is generated by Dehn twists, which map to transvections in $\mathrm{Sp}(2g, \mathbb{Z})$.
- The central extension $\widetilde{\Gamma}_g$ of Γ_g is generated by certain lifts of Dehn twists, and ρ_p of every such lift is a matrix of order p.
- The quotient of $\mathrm{Sp}(2g, \mathbb{Z})$ by the normal subgroup generated by pth powers of transvections is the finite group $\mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$ (see [5], for example).

It follows that the finite group $\mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$ admits a surjection onto the quotient group

$$\Delta_g / \rho_p(\widetilde{\mathcal{I}}_g)$$

(recall that $\Delta_g = \rho_p(\widetilde{\Gamma}_g)$) and hence the group $\rho_p(\widetilde{\mathcal{I}}_g)$ has finite index in Δ_g. But the Zariski closure of Δ_g is the connected, simple, algebraic group $\mathrm{SU}(V_p, H_p)$. Thus $\rho_p(\widetilde{\mathcal{I}}_g)$ and Δ_g have the same

Zariski closure. Again using the fact that $\mathrm{SU}(V_p, H_p)$ is a simple algebraic group, we also deduce that $\rho_p(\widetilde{\mathcal{I}_g})$ has the same adjoint trace field as Δ_g (this follows from [8, Proposition 12.2.1], for example). This shows that $\mathcal{I}_g$ indeed satisfies the hypothesis of Proposition 4.1, and so once again $\Phi_f(\mathcal{I}_g) = 1$. The same arguments work when G has finite index in $\mathcal{I}_g$, and also when G is any subgroup of Γ_g which contains a finite index subgroup of $\mathcal{I}_g$. This completes the proof of Theorem 1.1. □

We now turn to the proof of Theorem 1.3. To deal with the case of $\mathrm{Out}(F_n)$, we recall that R. Gilman [13] showed that for $n \geq 3$, $\mathrm{Out}(F_n)$ is residually alternating; i.e., in the notation of Section 2, the collection $\mathcal{S}$ consists of alternating groups.

Proof of Theorem 1.3. For $n \geq 3$, the abelianization of $\mathrm{Out}(F_n)$ is $\mathbb{Z}/2\mathbb{Z}$ (as can be seen directly from Nielsen's presentation of $\mathrm{Out}(F_n)$, see [30, Sect. 2.1]). Hence, $\mathrm{Out}(F_n)$ does not admit a surjection onto A_3 or A_4. Thus all the alternating quotients described by Gilman's result above have trivial Frattini subgroups (as in the proof of Proposition 4.1). The proof is completed using the residual alternating property and Proposition 2.1. □

5. PROOF OF PROPOSITION 4.2

Let Γ be a group and N a normal subgroup of Γ. We wish to show that $\Phi_f(N) < \Phi_f(\Gamma)$. We proceed as follows.

First a preliminary observation. Let $K = \Phi_f(N)$. It is easy to see that K is characteristic in N (i.e., fixed by every automorphism of N). Since N is normal in Γ, it follows that K is normal in Γ. This implies that for every subgroup M of Γ, the set

$$KM = \{km \mid k \in K,\ m \in M\}$$

is a subgroup of Γ. Moreover, since $K < N$, we have

$$KM \cap N = KM_1 \tag{5.1}$$

where $M_1 = M \cap N$. To see the inclusion $KM \cap N \subset KM_1$, write an element of $KM \cap N$ as $km = n$ and observe that $m \in N$ since $K < N$. Thus $m \in M_1$. The reverse inclusion is immediate.

Now suppose for a contradiction that $K = \Phi_f(N)$ is not contained in $\Phi_f(\Gamma)$. Then there exists a maximal subgroup $M < \Gamma$ of finite index such that K is not contained in M. Write

$$M_1 = M \cap N$$

as above. Then M_1 is a finite index subgroup of N. If $M_1 = N$, then N is contained in M, and hence so is K, which is a contradiction. Thus M_1 is a strict subgroup of N, and since its index in N is finite, M_1 is contained in a maximal subgroup H, say, of N.

The proof is now concluded as follows. By definition, $K = \Phi_f(N)$ is also contained in H. Hence the group KM_1 is contained in H and is therefore strictly smaller than N. On the other hand, by the maximality of M in Γ, we have $KM = \Gamma$, and hence, using (5.1), we have

$$KM_1 = KM \cap N = \Gamma \cap N = N.$$

This contradiction completes the proof. □

Remark. If we consider the original Frattini group Φ in place of Φ_f, one can show similarly that $\Phi(N) < \Phi(G)$, provided that every subgroup of N is contained in a maximal subgroup of N; e.g., when N is finitely generated.

6. PROOF OF THEOREM 1.2

We begin by recalling the Birman exact sequence. Let $\Sigma_{g,b}$ denote the closed orientable surface of genus g with b punctures. If $b = 0$, we abbreviate to Σ_g. There is a short exact sequence (the *Birman exact sequence*)

$$1 \to \pi_1(\Sigma_{g,(b-1)}) \to \mathrm{P}\Gamma_{g,b} \to \mathrm{P}\Gamma_{g,(b-1)} \to 1,$$

where the map $\mathrm{P}\Gamma_{g,b} \to \mathrm{P}\Gamma_{g,(b-1)}$ is the forgetful map, and the map $\pi_1(\Sigma_{g,(b-1)}) \to \mathrm{P}\Gamma_{g,b}$ is the point pushing map (see [10, Ch. 4.2] for details). Also in the case when $b = 1$, the symbol $\mathrm{P}\Gamma_{g,0}$ simply denotes the mapping class group Γ_g.

It will be useful to recall that an alternative description of $\mathrm{P}\Gamma_{g,b}$ is as the kernel of an epimorphism $\Gamma_{g,b} \to S_b$ (the symmetric group on b letters).

The proof will proceed by induction, using Theorem 1.1(i) to get started, together with the following (which is an adaptation of Lemma 3.5 of [1] to the case of Φ_f). The proof is included in Section 7 below.

We introduce the following notation. Recalling Section 2, let G be a group, say, G is *residually simple* if the collection $\mathcal{S} = \{G_n\}$ (as in Section 2) consists of finite non-abelian simple groups.

Lemma 6.1. *Let N be a finitely generated normal subgroup of the group G and assume that N is residually simple. Then $N \cap \Phi_f(G) = 1$. In particular, if $\Phi_f(G/N) = 1$, then $\Phi_f(G) = 1$.*

Given this we now complete the proof. In the cases of $(0,1)$, $(0,2)$ and $(0,3)$, it is easily seen that the subgroup Φ_f is trivial.

Thus we now assume that we are not in those cases. As is well-known, $\pi_1(\Sigma_{g,b})$ is residually simple for those surface groups under consideration, except the case of $\pi_1(\Sigma_1)$, which we deal with separately below. For example, this follows by uniformizing the surface by a Fuchsian group with algebraic matrix entries and then using strong approximation.

Assume first that $g \geq 3$; then Theorem 1.1(i), Lemma 6.1 and the Birman exact sequence immediately prove that the statement holds for $\mathrm{P}\Gamma_{g,1}$. The remarks above, Lemma 6.1 and induction then prove the result for $\mathrm{P}\Gamma_{g,b}$ whenever $g \geq 3$ and $b > 0$.

Now assume that $g = 0$. The base case of the induction here is $\mathrm{P}\Gamma_{0,4}$. From the above, it is easy to see that $\mathrm{P}\Gamma_{0,3}$ is trivial, and so $\mathrm{P}\Gamma_{0,4}$ is a free group of rank 2. As such, it follows that $\Phi_f(\mathrm{P}\Gamma_{0,4}) = 1$. The remarks above, Lemma 6.1 and induction then prove the result for $\mathrm{P}\Gamma_{0,b}$ whenever $b > 0$.

When $g = 1$, $\Gamma_1 \cong \Gamma_{1,1} \cong \mathrm{SL}(2,\mathbb{Z})$ and it is easy to check that $\Phi_f(\mathrm{SL}(2,\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$ (coinciding with the center of $\mathrm{SL}(2,\mathbb{Z})$). Now $\mathrm{P}\Gamma_{1,1} = \Gamma_{1,1}$ and so these facts together with Lemma 6.1 and induction then prove the result (i.e., that $\Phi_f(\mathrm{P}\Gamma_{1,b})$ is either trivial or $\mathbb{Z}/2\mathbb{Z}$).

In the case of $g = 2$, by [6] Γ_2 is linear, and so [25] also proves that $\Phi_f(\Gamma_2)$ is nilpotent. We claim that this forces $\Phi_f(\Gamma_2) = \mathbb{Z}/2\mathbb{Z}$. To see this, we argue as follows.

If $\Phi_f(\Gamma_2)$ is finite, it is central by [21, Lemma 2.2]. Since $\Phi_f(\Gamma_2)$ contains $\Phi(\Gamma_2)$, which is equal to the center $\mathbb{Z}/2\mathbb{Z}$ of Γ_2 by [21, Theorem 3.2], it follows that $\Phi_f(\Gamma_2) = \mathbb{Z}/2\mathbb{Z}$.

Thus it is enough to show that $\Phi_f(\Gamma_2)$ is finite. Assume that it is not. Then by [21, Lemma 2.5], $\Phi_f(\Gamma_2)$ contains a pseudo-Anosov element. Indeed, [21, Lemma 2.6] shows that the set of invariant laminations of pseudo-Anosov elements in $\Phi_f(\Gamma_2)$ is dense in the projective measured lamination space. This contradicts $\Phi_f(\Gamma_2)$ being nilpotent (e.g., the argument of [21, p. 86] constructs a free subgroup).

As before, using Lemma 6.1 and by induction via the Birman exact sequence, we can now handle the cases of $\Gamma_{2,b}$ with $b > 0$. $\square$

Remark 1. Recall that the *hyperelliptic mapping class group* (which we denote by Γ_g^h) is defined to be the subgroup of Γ_g consisting of those elements that commute with a fixed hyperelliptic involution. It is pointed out in [6, p. 706] that the arguments used in [6] prove that Γ_g^h is linear. Hence once again $\Phi_f(G)$ is nilpotent for every finitely generated subgroup G of Γ_g^h.

Remark 2. We make some comments on the case of $\Gamma_{g,b}$ with $b > 0$. First, since $\mathrm{P}\Gamma_{g,b} = \ker\{\Gamma_{g,b} \to S_b\}$ and $\Phi_f(S_b) = 1$, if $\mathrm{P}\Gamma_{g,b}$ were known to be residually simple, then the argument in the proof of Theorem 1.2 could be used to show that $\Phi_f(\Gamma_{g,b}) = 1$. Hence we raise here:

Question. Is $\mathrm{P}\Gamma_{g,b}$ residually simple?

Another approach to showing that $\Phi_f(\Gamma_{g,b}) = 1$ is to directly use the representations arising from TQFT. In this case the result of Larsen and Wang [20] that allows us to prove Zariski density in [23] needs to be established. Given this, the proof (for most (g, b)) would then follow as above.

7. PROOF OF LEMMA 6.1

As already mentioned, in what follows we adapt the proof of Lemma 3.5 of [1] to the case of Φ_f.

We argue by contradiction and assume that there exists a non-trivial element $x \in N \cap \Phi_f(G)$. By the residually simple assumption, we can find a non-abelian finite simple group S_0 and an epimorphism $f\colon N \to S_0$ for which $f(x) \neq 1$. Set $K_0 = \ker f$ and let $K_0, K_1 \dots, K_n$ be the distinct copies of K_0 which arise on mapping K_0 under the automorphism group of N (this set being finite since N is finitely generated). Set $K = \bigcap K_i$, a characteristic subgroup of finite index in N. As in [1], it follows from standard finite group theory that N/K is isomorphic to a direct product of finite simple groups (all of which are isomorphic to $S_0 = N/K_0$).

Now K being characteristic in N implies that K is a normal subgroup of G. Put $G_1 = G/K$ and let $f_1\colon G \to G_1$ denote the canonical homomorphism. Also write N_1 for $N/K = f_1(N)$. Now $f_1(x) \in N_1$ and $f_1(x) \in f_1(\Phi_f(G))$, which by Lemma 2.2 implies that $f_1(x) \in \Phi_f(G_1)$. Hence

$$f_1(x) \in N_1 \cap \Phi_f(G_1).$$

Following [1], let C denote the centralizer of N_1 in G_1, and as in [1], we can deduce various properties about the groups N_1 and C. Namely,

(i) since N_1 is a finite group, its centralizer C in G_1 is of finite index in G_1;

(ii) since N_1 is a product of non-abelian finite simple groups, it has trivial center, and so $C \cap N_1 = 1$;

(iii) since N_1 is normal in G_1, C is normal in G_1.

Using (iii), put $G_2 = G_1/C$ and let $f_2\colon G_1 \to G_2$ denote the canonical homomorphism. Also write N_2 for $f_2(N_1)$. Arguing as before (again invoking Lemma 2.2), we have

$$f_2(f_1(x)) \in N_2 \cap \Phi_f(G_2).$$

Moreover, since $f_1(x) \in N_1$ and $f_1(x) \neq 1$ by construction, we have from (ii) that $f_2 f_1(x) \neq 1$. Thus the intersection

$$H := N_2 \cap \Phi_f(G_2)$$

is a non-trivial group.

As in [1], we will now get a contradiction by showing that H is both a nilpotent group and a direct product of non-abelian finite simple groups, which is possible only if H is trivial. Here is the argument. From (i) above we deduce that G_2 is a finite group; hence $\Phi_f(G_2) = \Phi(G_2)$ is nilpotent by Frattini's theorem. Thus $H < \Phi_f(G_2)$ is nilpotent. On the other hand, N_2 is a quotient of N_1 and hence a direct product of non-abelian finite simple groups. But H is normal in N_2 (since $\Phi_f(G_2)$ is normal in G_2). Thus H is a direct product of non-abelian finite simple groups.

This contradiction shows that $N \cap \Phi_f(G) = 1$, which was the first assertion of the lemma. The second assertion of the lemma now follows from Lemma 2.2. This completes the proof. □

8. FINAL COMMENTS

8.1. An approach to Ivanov's question. We will now discuss an approach to answering Ivanov's question (i.e., the nilpotency of $\Phi_f(G)$ for finitely generated subgroups G of Γ_g) using the projective unitary representations described in Section 3. In the remainder of this section G is an infinite, finitely generated subgroup of Γ_g.

The following conjecture is the starting point to this approach. Recall that a subgroup G of Γ_g is *reducible* if there is a collection of essential simple closed curves C on the surface Σ_g, such that for any $\beta \in G$ there is a diffeomorphism $\overline{\beta}\colon \Sigma_g \to \Sigma_g$ in the isotopy class of β so that $\overline{\beta}(C) = C$. Otherwise G is called *irreducible*. As shown in [18, Theorem 2], an irreducible subgroup G is either virtually an infinite cyclic subgroup generated by a pseudo-Anosov element or contains a free subgroup of rank 2 generated by two pseudo-Anosov elements.

Conjecture. *If G is a finitely generated irreducible non-virtually cyclic subgroup of Γ_g, then $\Phi_f(G) = 1$.*

The motivation for this conjecture is that the irreducible (non-virtually cyclic) hypothesis should be enough to guarantee that the image group $\rho_p(\widetilde{G}) < \Delta_g$ is Zariski dense (with the same adjoint trace field). Roughly speaking, the irreducibility hypothesis should ensure that there is no reason for Zariski density to fail (i.e., the image is sufficiently complicated). Indeed, in this regard, we note that an emerging theme in linear groups is that random subgroups of linear groups are Zariski dense (see [4] and [28], for example). Below we discuss a possible approach to proving the conjecture.

The idea now is to follow Ivanov's proof in [18] that the Frattini subgroup is nilpotent. Very briefly, if the subgroup is reducible, then we first identify Φ_f on the pieces and then build up to identify $\Phi_f(G)$. In Ivanov's argument, this involves passing to certain subgroups of G ("pure subgroups"), understanding the Frattini subgroup of these pure subgroups when restricted to the connected components of $S \setminus C$, and then building $\Phi(G)$ from this information. This uses several statements about the Frattini subgroup, at least one of which (part (iv) of Lemma 10.2 of [18]) does not seem to easily extend to Φ_f.

Remark. As a cautionary note to the previous discussion, at present, it still remains conjectural that the image of a fixed pseudo-Anosov element of Γ_g under the representations $\overline{\rho}_p$ is of infinite order for big enough p (which was raised in [3]).

An approach to the conjecture. We begin by recalling that in [25] Platonov also proves that $\Phi_f(H)$ is nilpotent for every finitely generated linear group H. Note that if G is irreducible and virtually infinite cyclic, then G is a linear group, and so [25] implies that $\Phi_f(G)$ is nilpotent.

Thus we now assume that G is irreducible as in the conjecture. Consider $\rho_p(\widetilde{\Phi_f(G)})$: by Lemma 2.2 above we deduce that $\rho_p(\widetilde{\Phi_f(G)})$ is a nilpotent normal subgroup of $\rho_p(\widetilde{G})$. Now $\overline{\rho}_p(\Gamma_g) < \mathrm{PSU}(V_p, H_p; \mathbb{Z}[\zeta_p])$ and it follows from this that (in the notation of Section 3) $\Delta_g < \Lambda_p = \mathrm{SU}(V_p, H_p; \mathbb{Z}[\zeta_p])$. As discussed in [23], Λ_p is a cocompact arithmetic lattice in the algebraic group $\mathrm{SU}(V_p, H_p)$. Thus $\rho_p(\widetilde{\Phi_f(G)}) < \rho_p(\widetilde{G}) < \Lambda_p$. It follows from general properties of cocompact lattices acting on symmetric spaces (see, e.g., [9, Proposition 10.3.7]) that $\rho_p(\widetilde{\Phi_f(G)})$ contains a maximal normal abelian subgroup of finite index. Now there is a general bound on the index of this abelian subgroup that is a function of the dimension $N_g(p)$. However, in our setting, if the index can be bounded by some fixed constant R independent of $N_g(p)$, then we claim that $\Phi_f(G)$ can at least be shown to be finite. To see this, we argue as follows.

Assume that $\Phi_f(G)$ is infinite. Since G is an irreducible subgroup containing a free subgroup generated by a pair of pseudo-Anosov elements, the same holds for the infinite normal subgroup $\Phi_f(G)$ (by standard dynamical properties of pseudo-Anosov elements; see, for example, [21, pp. 83–84]).

Thus we can find $x, y \in \Phi_f(G)$ a pair of non-commuting pseudo-Anosov elements. Also note that $[x^t, y^t] \neq 1$ for all non-zero integers t. From Lemma 2.2 we have that $\rho_p(\widetilde{\Phi_f(G)}) < \Phi_f(\rho_p(\widetilde{G}))$ and from the assumption above it therefore follows that $\rho_p(\widetilde{\Phi_f(G)})$ contains a maximal normal abelian subgroup A_p of index bounded by R (independent of p). Thus, setting $R_1 = R!$, we have $[\overline{\rho}_p(x^{R_1}), \overline{\rho}_p(y^{R_1})] = 1$ for all p. However, as noted above, $[x^{R_1}, y^{R_1}]$ is a non-trivial element of G, and by asymptotic faithfulness this cannot be mapped trivially for all p. This is a contradiction.

8.2. The profinite completion of Γ_g. We remind the reader that the profinite completion $\widehat{\Gamma}$ of a group Γ is the inverse limit of the finite quotients Γ/N of Γ. (The maps in the inverse system are the obvious ones: if $N_1 < N_2$, then $\Gamma/N_1 \to \Gamma/N_2$.) The Frattini subgroup $\Phi(G)$ of a profinite group G is defined to be the intersection of all maximal open subgroups of G. Open subgroups are of finite index, and if G is finitely generated as a profinite group, then Nikolov and Segal [24] show that finite index subgroups are always open. Hence we can simply take $\Phi(G)$ to be the intersection of all maximal finite index subgroups of G.

Now if Γ is a finitely generated residually finite discrete group, the correspondence theorem between finite index subgroups of Γ and its profinite completion (see [27, Proposition 3.2.2]) shows that $\overline{\Phi_f(\Gamma)} < \Phi(\widehat{\Gamma})$.

There is a well-known connection between the center of a group G, denoted $Z(G)$ (profinite or otherwise), and $\Phi(G)$. We include a proof for completeness. Note that for a profinite group $\Phi(G)$ is a closed subgroup of G, $Z(G)$ is a closed subgroup and by [24] the commutator subgroup $[G,G]$ is a closed subgroup.

Lemma 8.1. *Let G be a finitely generated profinite group. Then $\Phi(G) > Z(G) \cap [G,G]$.*

Proof. Let U be a maximal finite index subgroup of G, and assume that $Z(G)$ is not contained in U. Then $\langle Z(G), U\rangle = G$ by maximality. It also easily follows that U is a normal subgroup of G. But then $G/U = Z(G)U/U \cong Z(G)/(U \cap Z(G))$, which is abelian, and so $[G,G] < U$. This being true for every maximal finite index subgroup U, we deduce that $\Phi(G) > Z(G) \cap [G,G]$ as required. $\square$

We now turn to the following questions which were also part of the motivation of this note.

Question 1. For $g \geq 3$, is $Z(\widehat{\Gamma}_g) = 1$?

Question 2. For $g \geq 3$, is $Z(\widehat{\mathcal{I}}_g) = 1$?

Regarding Question 1, it is shown in [16] that the completion of Γ_g arising from the congruence topology on Γ_g has trivial center. Regarding Question 2, if $Z(\widehat{\mathcal{I}}_g) = 1$, then the profinite topology on Γ_g will induce the full profinite topology on $\mathcal{I}_g$ (see [22, Lemma 2.6]). Motivated by this and Lemma 8.1, we can also ask

Question 1′. For $g \geq 3$, is $\Phi(\widehat{\Gamma}_g) = 1$?

Question 2′. For $g \geq 3$, is $\Phi(\widehat{\mathcal{I}}_g) = 1$?

Although the results in this paper do not impact directly on Questions 1, 1′, 2 and 2′, we note that since Γ_g is finitely generated and perfect for $g \geq 3$, it follows that $\widehat{\Gamma}_g$ is also perfect and hence

$$Z(\widehat{\Gamma}_g) < \Phi(\widehat{\Gamma}_g)$$

by Lemma 8.1. As remarked above, the correspondence theorem gives

$$\overline{\Phi_f(\Gamma_g)} < \Phi(\widehat{\Gamma}_g).$$

Thus our result that $\Phi_f(\Gamma_g) = 1$ for $g \geq 3$ (which implies $\overline{\Phi_f(\Gamma_g)} = 1$) is consistent with triviality of $Z(\widehat{\Gamma}_g)$ (and similarly for $Z(\widehat{\mathcal{I}}_g)$).

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