ARITHMETIC OF DEL PEZZO AND K3 SURFACES: EXERCISES III

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1. Brauer-Manin obstructions

(1) Let X be a nice variety over a global field k. Recall that each class $\mathcal{A} \in \operatorname{Br} X$ gives rise to an evaluation map $\operatorname{ev}_{\mathcal{A}} \colon X(k_v) \to \operatorname{Br} k_v \cong \mathbb{Q}/\mathbb{Z}$ for every place v of k.

  (a) Show that if $\mathbb{Q}/\mathbb{Z}$ is given the discrete topology, then $\operatorname{ev}_{\mathcal{A}}$ is continuous. [Hints: First reduce the problem to showing that $\operatorname{ev}_{\mathcal{A}}^{-1}(0)$ is open. Consider the PGL_n-torsor $f\colon Y \to X$ associated to $\mathcal{A}$ (considered as an Azumaya algebra of rank n^2). Show that $\operatorname{ev}_{\mathcal{A}}^{-1}(0) = f(Y(k_v)) \subset X(k_v)$ and conclude by applying the v-adic implicit function theorem.]

  (b) Conclude that $X(k)^{\mathrm{Br}}$ is closed in $X(\mathbb{A})$.

(2) Let X be a nice variety over a global field k. Verify that if $\mathcal{A} \in \operatorname{Br}_0 X = \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X)$, then the set $X(\mathbb{A})^{\{\mathcal{A}\}}$ is equal to the entire set of local points $X(\mathbb{A})$. Hence, to compute the Brauer-Manin set $X(\mathbb{A})^{\mathrm{Br}}$ it suffices to take the intersection of $X(\mathbb{A})^{\mathcal{A}}$, where $\mathcal{A}$ runs over a set of representative classes for the quotient $\operatorname{Br} X/\operatorname{Br}_0 X$.

(3) Let X be a nice variety over a field k. Assume that either $X(k) \neq \emptyset$ or that k is a global field and $X(\mathbb{A}) \neq \emptyset$.

  (a) Show that the map $\operatorname{Br} k \to \operatorname{Br} X$ coming from the structure morphism is injective.

  (b) Conclude, using the low-degree exact sequence of the Leray spectral sequence
$$E_2^{pq} := \mathrm{H}^p\left(k, \mathrm{H}^q_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{G}_m)\right) \implies H^{p+q}_{\mathrm{et}}(X, \mathbb{G}_m),$$
that the natural map
$$\operatorname{Pic} X \to \operatorname{Pic}\left(X^{\mathrm{s}}\right)^{\operatorname{Gal}(k^{\mathrm{s}}/k)}$$
is an isomorphism under either of the above hypotheses.

(4) Let k be a field. Show that for $n \geq 1$, the group $\operatorname{Br} \mathbb{P}^n_k$ is trivial.