

Obstructions to the Hasse principle and weak approximation on del Pezzo surfaces of low degree

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Recap

Brauer-Manin set I

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dP1s

Fix a global field k , and let Ω_k be the set of places of k .
Let \mathcal{S} be a class of nice (smooth, projective, geometrically
integral) k -varieties.

Definition

We say that \mathcal{S} **satisfies the Hasse principle** if for all $X \in \mathcal{S}$,

$$X(k_v) \neq \emptyset \quad \text{for all } v \in \Omega_k \implies X(k) \neq \emptyset.$$

Definition

We say that a nice k -variety X **satisfies weak approximation**
if the embedding

$$X(k) \hookrightarrow \prod_{v \in \Omega_k} X(k_v)$$

has dense image for the product of the v -adic topologies.

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In the second lecture we sketch the proof of the following theorem.

Theorem

The class of del Pezzo surfaces (over a global field) of degree ≥ 5 satisfies the Hasse principle. These surfaces also satisfy weak approximation.

Del Pezzo surfaces of lower degree need not enjoy these arithmetic properties.

	$d \geq 5$	$d = 4$	$d = 3$	$d = 2$	$d = 1$
HP	✓	[BSD75]	[SD62]	[KT04]	✓
WA	✓	[CTS77]	[SD62]	[KT08]	[VA08]

- (1) Check mark (✓) means: phenomenon holds.
- (2) A reference points to a counterexample in the literature.

Since X is a nice k -variety, we have $\prod_v X(k_v) = X(\mathbf{A}_k)$.

In 1970, Manin used the Brauer group of the variety to construct an intermediate “obstruction set” between $X(k)$ and $X(\mathbf{A}_k)$:

$$X(k) \subseteq X(\mathbf{A}_k)^{\text{Br}} \subseteq X(\mathbf{A}_k). \quad (1)$$

In fact, the set $X(\mathbf{A}_k)^{\text{Br}}$ already contains the closure of $X(k)$ for the adelic topology:

$$\overline{X(k)} \subseteq X(\mathbf{A}_k)^{\text{Br}} \subseteq X(\mathbf{A}_k). \quad (2)$$

This set may be used to explain the failure of the Hasse principle and weak approximation on many kinds of varieties.

Definition

Let X be a nice k -variety, and assume that $X(\mathbf{A}_k) \neq \emptyset$. We say that X is a counter-example to the Hasse principle explained by the **Brauer-Manin obstruction** if

$$X(\mathbf{A}_k)^{\text{Br}} = \emptyset.$$

Definition

Let X be a nice k -variety. We say that X is a counter-example to the weak approximation explained by the **Brauer-Manin obstruction** if

$$X(\mathbf{A}_k) \setminus X(\mathbf{A}_k)^{\text{Br}} \neq \emptyset.$$

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Two definitions for the Brauer group

An **Azumaya algebra** on a scheme X is an \mathcal{O}_X -algebra \mathcal{A} that is coherent and locally free as an \mathcal{O}_X -module, such that the fiber $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathcal{O}_{X,x}} k(x)$ is a central simple algebra over the residue field $k(x)$ for each $x \in X$.

Two Azumaya algebras \mathcal{A} and \mathcal{B} on X are **similar** if there exist locally free coherent \mathcal{O}_X -modules \mathcal{E} and \mathcal{F} such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_X}(\mathcal{E}) \cong \mathcal{B} \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_X}(\mathcal{F}).$$

The **Azumaya Brauer group** $\text{Br}_{\text{Az}} X$ of a scheme X is the set of similarity classes of Azumaya algebras on X , with multiplication induced by tensor product of sheaves.

The **Brauer group** of a scheme X is $\text{Br } X := H_{\text{ét}}^2(X, \mathbb{G}_m)$.

Comparison

If F is a field, then

$$\mathrm{Br}_{\mathrm{Az}}(\mathrm{Spec} F) \cong \mathrm{Br} \mathrm{Spec} F \cong \mathrm{Br} F$$

For any scheme X there is a natural inclusion

$$\mathrm{Br}_{\mathrm{Az}} X \hookrightarrow \mathrm{Br} X.$$

Theorem (Gabber, de Jong)

If X is a scheme endowed with an ample invertible sheaf then the natural map $\mathrm{Br}_{\mathrm{Az}} X \hookrightarrow \mathrm{Br} X$ induces an isomorphism

$$\mathrm{Br}_{\mathrm{Az}} X \xrightarrow{\sim} (\mathrm{Br} X)_{\mathrm{tors}}. \quad \square$$

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If X is an integral, regular and quasi-compact scheme, then the inclusion $\text{Spec } \mathbf{k}(X) \rightarrow X$ gives rise to an injection $\text{Br } X \hookrightarrow \text{Br } \mathbf{k}(X)$.

On the other hand, the group $\text{Br } \mathbf{k}(X)$ is torsion, because it is a Galois cohomology group.

Corollary

Let X be a nice variety over a field. Then

$$\text{Br}_{A_Z} X \cong \text{Br } X. \quad \square$$

Let X be a nice variety over a global field k . For $\mathcal{A} \in \text{Br } X$ and K/k a field extension there is an evaluation map

$$\text{ev}_{\mathcal{A}}: X(K) \rightarrow \text{Br } K, \quad x \mapsto \mathcal{A}_x \otimes_{\theta_{X,x}} K.$$

We put these maps together to construct a pairing

$$\phi: \text{Br } X \times X(\mathbf{A}_k) \rightarrow \mathbb{Q}/\mathbb{Z}, \quad (\mathcal{A}, (x_v)) \mapsto \sum_{v \in \Omega_k} \text{inv}_v(\text{ev}_{\mathcal{A}}(x_v)),$$

where $\text{inv}_v: \text{Br } k_v \rightarrow \mathbb{Q}/\mathbb{Z}$ is the invariant map from LCFT.

For $\mathcal{A} \in \text{Br } X$ we obtain a commutative diagram

$$\begin{array}{ccccccc} X(k) & \longrightarrow & X(\mathbf{A}_k) & & & & \\ \text{ev}_{\mathcal{A}} \downarrow & & \text{ev}_{\mathcal{A}} \downarrow & \searrow \phi(\mathcal{A}, -) & & & \\ 0 \longrightarrow & \text{Br } k & \longrightarrow & \bigoplus_v \text{Br } k_v & \xrightarrow{\sum_v \text{inv}_v} & \mathbb{Q}/\mathbb{Z} & \longrightarrow 0 \end{array}$$

Manin's observation is that an element $\mathcal{A} \in \text{Br } X$ can be used to "carve out" a subset of $X(\mathbf{A}_k)$ that contains $X(k)$:

$$X(\mathbf{A}_k)^{\mathcal{A}} := \{(x_v) \in X(\mathbf{A}_k) : \phi(\mathcal{A}, (x_v)) = 0\}.$$

We call

$$X(\mathbf{A}_k)^{\text{Br}} := \bigcap_{\mathcal{A} \in \text{Br } X} X(\mathbf{A}_k)^{\mathcal{A}}$$

the **Brauer-Manin set** of X .

if \mathbb{Q}/\mathbb{Z} is given the discrete topology, then the map $\phi(\mathcal{A}, -): X(\mathbf{A}_k) \rightarrow \mathbb{Q}/\mathbb{Z}$ is continuous, so $X(\mathbf{A}_k)^{\mathcal{A}}$ is a closed subset of $X(\mathbf{A}_k)$. In particular,

$$\overline{X(k)} \subseteq X(\mathbf{A}_k)^{\text{Br}}.$$

If $\mathcal{A} \in \text{im}(\text{Br } k \rightarrow \text{Br } X) =: \text{Br}_0 X$, then $X(\mathbf{A}_k)^{\mathcal{A}} = X(\mathbf{A}_k)$. This means that to compute $X(\mathbf{A}_k)^{\text{Br}}$, it is enough to consider $X(\mathbf{A}_k)^{\mathcal{A}}$, as \mathcal{A} runs through a set of representatives of the group $\text{Br } X / \text{Br}_0 X$. When $\text{Br } X_{k^{\text{sep}}} = 0$, the Hochschild-Serre spectral sequence in étale cohomology (with \mathbb{G}_m -coefficients) can help us compute this group. The long exact sequence of low degree terms is

$$\begin{aligned} 0 &\rightarrow \text{Pic } X \rightarrow (\text{Pic } X_{k^{\text{sep}}})^{\text{Gal}(k^{\text{sep}}/k)} \rightarrow \text{Br } k \\ &\rightarrow \ker(\text{Br } X \rightarrow \text{Br } X_{k^{\text{sep}}}) \rightarrow H^1(\text{Gal}(k^{\text{sep}}/k), \text{Pic } X_{k^{\text{sep}}}) \\ &\rightarrow H^3(\text{Gal}(k^{\text{sep}}/k), k^{\text{sep}*}). \end{aligned}$$

If k is a global field, then $H^3(\text{Gal}(k^{\text{sep}}/k), k^{\text{sep}*}) = 0$ (Tate).
If $X(\mathbf{A}_k) \neq \emptyset$, then the map $(\text{Pic } X_{k^{\text{sep}}})^{\text{Gal}(k^{\text{sep}}/k)} \rightarrow \text{Br } k$ is
the zero map and hence we have

$$\text{Pic } X \xrightarrow{\sim} (\text{Pic } X_{k^{\text{sep}}})^{\text{Gal}(k^{\text{sep}}/k)}.$$

If X is a geometrically rational surface, then $\text{Br } X_{k^{\text{sep}}} = 0$.
Put this all together and we get

Proposition

Let X be a del Pezzo surface over a global field k . Assume that $X(\mathbf{A}_k) \neq \emptyset$. Then we have

$$\text{Br } X / \text{Br } k \xrightarrow{\sim} H^1(\text{Gal}(k^{\text{sep}}/k), \text{Pic } X_{k^{\text{sep}}}).$$

X : del Pezzo surface over a global field k of degree $d \leq 7$.
Let K be the smallest extension of k in k^{sep} over which all
exceptional curves of X are defined. The group $\text{Pic } X_{K^{\text{sep}}}$ is
generated by the class of exceptional curves, so

$$\text{Pic } X_K \cong \text{Pic } X_{K^{\text{sep}}},$$

and moreover, the inflation map

$$H^1(\text{Gal}(K/k), \text{Pic } X_K) \rightarrow H^1(\text{Gal}(k^{\text{sep}}/k), \text{Pic } X_{K^{\text{sep}}})$$

is an isomorphism (here we assume that $X(\mathbf{A}_k) \neq \emptyset$).

One way of constructing Brauer-Manin obstructions on del
Pezzo surfaces of small degree begins by computing the
group $H^1(\text{Gal}(K/k), \text{Pic } X_K)$ on “reasonable” surfaces.

Many authors have pursued this set of ideas, not just for del Pezzo surfaces: Manin, Swinnerton-Dyer, Colliot-Thélène, Kanevsky, Sansuc, Skorobogatov, Bright, Bruin, Flynn, Logan, Kresch, Tschinkel, Corn, van Luijk, V-A, etc (the list is not meant to be comprehensive).

We will compute an example to weak approximation on a del Pezzo surface of degree 1.

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Del Pezzo surfaces of degree 1: quick review

Anticanonical model of X/k is a smooth sextic hypersurface in $\mathbb{P}_k(1, 1, 2, 3) := \text{Proj}(k[x, y, z, w])$, e.g.,

$$w^2 = z^3 + Ax^6 + By^6, \quad A, B \in k^*.$$

Conversely, any smooth sextic in $\mathbb{P}_k(1, 1, 2, 3)$ is a dP1. $X_{k^{\text{sep}}}$ is isomorphic to the blow-up of $\mathbb{P}_{k^{\text{sep}}}^2$ at 8 points in *general position*. In particular,

$$\text{Pic } X_{k^{\text{sep}}} \cong \mathbb{Z}^9.$$

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Fix a primitive sixth root of unity ζ in $\overline{\mathbb{Q}}$.

Theorem (V-A'08)

Let X be the del Pezzo surface of degree 1 over $k = \mathbb{Q}(\zeta)$
given by

$$w^2 = z^3 + 16x^6 + 16y^6$$

in $\mathbb{P}_k(1, 1, 2, 3)$. Then X is k -minimal and there is a
Brauer-Manin obstruction to weak approximation on X .
Moreover, the obstruction arises from a cyclic algebra class
in $\text{Br } X / \text{Br } k$.

Need the action of $\text{Gal}(k^{\text{sep}}/k)$ on $\text{Pic } X_{k^{\text{sep}}}$ explicitly. Recall that $\text{Pic } X_{k^{\text{sep}}}$ is generated by the exceptional curves of X .

Theorem (V-A'08)

Let X be a del Pezzo surface of degree 1 over a field k , given as a smooth sextic hypersurface $V(f(x, y, z, w))$ in $\mathbb{P}_k(1, 1, 2, 3)$. Let

$$\Gamma = V(z - Q(x, y), w - C(x, y)) \subseteq \mathbb{P}_{k^{\text{sep}}}(1, 1, 2, 3),$$

where $Q(x, y)$ and $C(x, y)$ are homogenous forms of degrees 2 and 3, respectively, in $k^{\text{sep}}[x, y]$. If Γ is a divisor on $X_{k^{\text{sep}}}$, then it is an exceptional curve of X . Conversely, every exceptional curve on X is a divisor of this form.

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Exceptional curves on $w^2 = z^3 + 16x^6 + 16y^6$

Let

$$Q(x, y) = ax^2 + bxy + cy^2,$$

$$C(x, y) = rx^3 + sx^2y + txy^2 + uy^3,$$

Then the identity $C(x, y)^2 = Q(x, y)^3 + 16x^6 + 16y^6$ gives

$$a^3 - r^2 + 16 = 0,$$

$$3a^2b - 2rs = 0,$$

$$3a^2c + 3ab^2 - 2rt - s^2 = 0,$$

$$6abc + b^3 - 2ru - 2st = 0,$$

$$3ac^2 + 3b^2c - 2su - t^2 = 0,$$

$$3bc^2 - 2tu = 0,$$

$$c^3 - u^2 + 16 = 0.$$

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We can use Gröbner bases to solve this system of equations. We get 240 solutions, one for each exceptional curve of the surface. The action of $\text{Gal}(\bar{k}/k)$ can be read off from the coefficients of the equations of the exceptional curves.

Sample exceptional curve: $(s = \sqrt[3]{2}, \zeta = (1 + \sqrt{-3})/2)$

$$z = (-s^2\zeta + s^2 - 2s + 2\zeta)x^2 + (2s^2\zeta - 2s^2 + 3s - 4\zeta)xy \\ + (-s^2\zeta + s^2 - 2s + 2\zeta)y^2,$$

$$w = (2s^2\zeta - 4s^2 + 2s\zeta + 2s - 6\zeta + 3)x^3 \\ + (-5s^2\zeta + 10s^2 - 6s\zeta - 6s + 16\zeta - 8)x^2y \\ + (5s^2\zeta - 10s^2 + 6s\zeta + 6s - 16\zeta + 8)xy^2 \\ + (-2s^2\zeta + 4s^2 - 2s\zeta - 2s + 6\zeta - 3)y^3.$$

The Picard group of X

Let $s = \sqrt[3]{2}$. Consider the exceptional curves on X given by

$$E_1 = V(z + 2sx^2, w - 4y^3),$$

$$E_2 = V(z - (-\zeta_3 + 1)2sx^2, w + 4y^3),$$

$$E_3 = V(z - 2\zeta_3sx^2 + 4y^2, w - 4s(\zeta_3 - 2)x^2y - 4(-2\zeta_3 + 1)y^3),$$

$$E_4 = V(z + 4\zeta_3sx^2 - 2s^2(2\zeta_3 - 1)xy - 4(-\zeta_3 + 1)y^2, \\ w - 12x^3 - 8s(-\zeta_3 - 1)x^2y - 12\zeta_3s^2xy^2 - 4(-2\zeta_3 + 1)y^3),$$

$$E_5 = V(z + 4\zeta_3sx^2 - 2s^2(\zeta_3 - 2)xy - 4\zeta_3y^2 \\ w + 12x^3 - 8s(2\zeta_3 - 1)x^2y - 12s^2xy^2 - 4(-2\zeta_3 + 1)y^3),$$

$$E_6 = V(z - 2s(-s^2\zeta_3 + s^2 - 2s + 2\zeta_3)x^2 - 2s(2s^2\zeta_3 - 2s^2 + 3s - 4\zeta_3)xy - 2s(-s^2\zeta_3 + s^2 - 2s + 2\zeta_3)y^2, \\ w - 4(2s^2\zeta_3 - 4s^2 + 2s\zeta_3 + 2s - 6\zeta_3 + 3)x^3 - 4(-5s^2\zeta_3 + 10s^2 - 6s\zeta_3 - 6s + 16\zeta_3 - 8)x^2y \\ - 4(5s^2\zeta_3 - 10s^2 + 6s\zeta_3 + 6s - 16\zeta_3 + 8)xy^2 - 4(-2s^2\zeta_3 + 4s^2 - 2s\zeta_3 - 2s + 6\zeta_3 - 3)y^3),$$

$$E_7 = V(z - 2s(-s^2 - 2s\zeta_3 + 2s + 2\zeta_3)x^2 - 2s(-2s^2\zeta_3 + 3s + 4\zeta_3 - 4)xy - 2s(-s^2\zeta_3 + s^2 + 2s\zeta_3 - 2)y^2, \\ w - 4(2s^2\zeta_3 + 2s^2 + 2s\zeta_3 - 4s - 6\zeta_3 + 3)x^3 - 4(10s^2\zeta_3 - 5s^2 - 6s\zeta_3 - 6s - 8\zeta_3 + 16)x^2y \\ - 4(5s^2\zeta_3 - 10s^2 - 12s\zeta_3 + 6s + 8\zeta_3 + 8)xy^2 - 4(-2s^2\zeta_3 - 2s^2 - 2s\zeta_3 + 4s + 6\zeta_3 - 3)y^3),$$

$$E_8 = V(z - 2s(s^2\zeta_3 + 2s\zeta_3 + 2\zeta_3)x^2 - 2s(2s^2 + 3s + 4)xy - 2s(-s^2\zeta_3 + s^2 - 2s\zeta_3 + 2s - 2\zeta_3 + 2)y^2, \\ w - 4(-4s^2\zeta_3 + 2s^2 - 4s\zeta_3 + 2s - 6\zeta_3 + 3)x^3 - 4(-5s^2\zeta_3 - 5s^2 - 6s\zeta_3 - 6s - 8\zeta_3 - 8)x^2y \\ - 4(5s^2\zeta_3 - 10s^2 + 6s\zeta_3 - 12s + 8\zeta_3 - 16)xy^2 - 4(4s^2\zeta_3 - 2s^2 + 4s\zeta_3 - 2s + 6\zeta_3 - 3)y^3).$$

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The Picard group of X

...as well as the exceptional curve

$$E_9 = V(z - 2\zeta_3 s^2 xy, w - 4x^3 + 4y^3).$$

Then

$$\text{Pic } X_{\bar{k}} = \text{Pic } X_K \cong \left(\bigoplus_{i=1}^8 \mathbb{Z}[E_i] \right) \oplus \mathbb{Z}[H] = \mathbb{Z}^9,$$

where $H = E_1 + E_2 + E_9$.

The exceptional curves of X are defined over $K := k(\sqrt[3]{2})$.

Let $G := \text{Gal}(K/k) = \langle \rho \rangle$. Note that G is cyclic.

Strategy for inverting

$$\mathrm{Br} X / \mathrm{Br} k \rightarrow H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Pic} X_{k^{\mathrm{sep}}})$$

$$\begin{array}{ccc}
 \mathrm{Br} X / \mathrm{Br} k & \xrightarrow{\sim} & H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Pic} X_{k^{\mathrm{sep}}}) \\
 \downarrow & & \uparrow \sim \\
 \mathrm{Br} \mathbf{k}(X) / \mathrm{Br} k & & H^1(\mathrm{Gal}(K/k), \mathrm{Pic} X_K) \\
 \uparrow & & \downarrow \sim \\
 \mathrm{Br}_{\mathrm{cyc}}(X, K) & \xleftarrow[\sim]{\psi} & \ker \bar{N}_{K/k} / \mathrm{im} \Delta
 \end{array}$$

$$\mathrm{Br}_{\mathrm{cyc}}(X, K) := \left\{ \begin{array}{l} \text{classes } [(K/k, f)] \text{ in the image of the} \\ \text{map } \mathrm{Br} X / \mathrm{Br} k \rightarrow \mathrm{Br} \mathbf{k}(X) / \mathrm{Br} k \end{array} \right\}$$

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The group $\text{Br}_{\text{cyc}}(X, K)$

Explicitly, we have maps

$$\begin{aligned}\bar{N}_{K/k}: \text{Pic } X_K &\rightarrow \text{Pic } X & \Delta: \text{Pic } X_K &\rightarrow \text{Pic } X \\ [D] &\mapsto [D + \rho D + \rho^2 D] & [D] &\mapsto [D - \rho D]\end{aligned}$$

We compute

$$\ker \bar{N}_{K/k} / \text{im } \Delta \cong (\mathbb{Z}/3\mathbb{Z})^4;$$

and the classes

$$\begin{aligned}\mathfrak{h}_1 &= [E_2 + 2E_8 - H], & \mathfrak{h}_2 &= [E_5 + 2E_8 - H], \\ \mathfrak{h}_3 &= [E_7 + 2E_8 - H], & \mathfrak{h}_4 &= [3E_8 - H]\end{aligned}$$

of $\text{Pic } X_K$ give a set of generators for this group.

An Azumaya Algebra

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The group isomorphism

$$\psi: \ker \bar{N}_{K/k} / \text{im } \Delta \rightarrow \text{Br}_{\text{cyc}}(X, K)$$

is given by

$$[D] \mapsto [(K/k, f)],$$

where $f \in k(X)^*$ is any function such that $N_{K/k}(D) = (f)$.

Consider the divisor class $\mathfrak{h}_1 - \mathfrak{h}_2 = [E_2 - E_5] \in \text{Pic } X_K$. It gives rise to a cyclic algebra $\mathcal{A} := (K/k, f) \in \text{Br}_{\text{cyc}}(X, K)$, where $f \in k(X)^*$ is any function such that

$$N_{K/k}(E_2 - E_5) = (f),$$

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To wit, f is a function with zeroes along

$$E_2 + \rho E_2 + \rho^2 E_2$$

and poles along

$$E_5 + \rho E_5 + \rho^2 E_5.$$

Using the explicit equations for E_2 and E_5 we find

$$f := \frac{w + 4y^3}{w + (2\zeta + 2)zy + (-8\zeta + 4)y^3 + 12x^3}$$

does the job.

The Brauer-Manin obstruction

Recall X is given by $w^2 = z^3 + 16x^6 + 16y^6$. Note that

$$P_1 = [1 : 0 : 0 : 4] \quad \text{and} \quad P_2 = [0 : 1 : 0 : 4].$$

are in $X(k)$.

Let \mathfrak{p} be the unique prime above 3 in k . We compute

$$\text{inv}_{\mathfrak{p}}(\mathcal{A}(P_1)) = 0 \quad \text{and} \quad \text{inv}_{\mathfrak{p}}(\mathcal{A}(P_2)) = 1/3.$$

Let $P \in X(\mathbf{A}_k)$ be the point that is equal to P_1 at all places except \mathfrak{p} , and is P_2 at \mathfrak{p} . Then

$$\sum_v \text{inv}_v(\mathcal{A}(P_v)) = 1/3,$$

so $P \in X(\mathbf{A}_k) \setminus X(\mathbf{A}_k)^{\text{Br}}$ and X is a counterexample to weak approximation. □