Obstructions to the Hasse principle and weak approximation on del Pezzo surfaces of low degree

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Fix a global field k, and let Ω_k be the set of places of k. Let $\mathcal{S}$ be a class of nice (smooth, projective, geometrically integral) k-varieties.

Definition

We say that $\mathcal{S}$ satisfies the Hasse principle if for all $X \in \mathcal{S}$,

$$X(k_v) \neq \emptyset \quad \text{for all} \quad v \in \Omega_k \implies X(k) \neq \emptyset.$$

Definition

We say that a nice k-variety X satisfies weak approximation if the embedding

$$X(k) \hookrightarrow \prod_{v \in \Omega_k} X(k_v)$$

has dense image for the product of the v-adic topologies.

In the second lecture we sketch the proof of the following theorem.

Theorem

The class of del Pezzo surfaces (over a global field) of degree ≥ 5 satisfies the Hasse principle. These surfaces also satisfy weak approximation.

Del Pezzo surfaces of lower degree need not enjoy these arithmetic properties.

	$d \geq 5$	$d = 4$	$d = 3$	$d = 2$	$d = 1$
HP	✓	[BSD75]	[SD62]	[KT04]	✓
WA	✓	[CTS77]	[SD62]	[KT08]	[VA08]

(1) Check mark (✓) means: phenomenon holds.
(2) A reference points to a counterexample in the literature.

Since X is a nice k-variety, we have $\prod_v X(k_v) = X(\mathbf{A}_k)$. In 1970, Manin used the Brauer group of the variety to construct an intermediate "obstruction set" between $X(k)$ and $X(\mathbf{A}_k)$:

$$X(k) \subseteq X(\mathbf{A}_k)^{\mathrm{Br}} \subseteq X(\mathbf{A}_k). \tag{1}$$

In fact, the set $X(\mathbf{A}_k)^{\mathrm{Br}}$ already contains the closure of $X(k)$ for the adelic topology:

$$\overline{X(k)} \subseteq X(\mathbf{A}_k)^{\mathrm{Br}} \subseteq X(\mathbf{A}_k). \tag{2}$$

This set may be used to explain the failure of the Hasse principle and weak approximation on many kinds of varieties.

Definition

Let X be a nice k-variety, and assume that $X(\mathbf{A}_k) \neq \emptyset$. We say that X is a counter-example to the Hasse principle explained by the Brauer-Manin obstruction if

$$X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset.$$

Definition

Let X be a nice k-variety. We say that X is a counter-example to the weak approximation explained by the Brauer-Manin obstruction if

$$X(\mathbf{A}_k) \setminus X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset.$$

Two definitions for the Brauer group

An Azumaya algebra on a scheme X is an $\mathscr{O}_X$-algebra $\mathcal{A}$ that is coherent and locally free as an $\mathscr{O}_X$-module, such that the fiber $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathscr{O}_{X,x}} k(x)$ is a central simple algebra over the residue field $k(x)$ for each $x \in X$.

Two Azumaya algebras $\mathcal{A}$ and $\mathcal{B}$ on X are similar if there exist locally free coherent $\mathscr{O}_X$-modules $\mathcal{E}$ and $\mathcal{F}$ such that

$$\mathcal{A} \otimes_{\mathscr{O}_X} \operatorname{End}_{\mathscr{O}_X}(\mathcal{E}) \cong \mathcal{B} \otimes_{\mathscr{O}_X} \operatorname{End}_{\mathscr{O}_X}(\mathcal{F}).$$

The Azumaya Brauer group $\operatorname{Br}_{\mathrm{Az}} X$ of a scheme X is the set of similarity classes of Azumaya algebras on X, with multiplication induced by tensor product of sheaves.

The Brauer group of a scheme X is $\operatorname{Br} X := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$.

Comparison

If F is a field, then

$$\mathrm{Br}_{\mathrm{Az}}(\operatorname{Spec} F) \cong \operatorname{Br} \operatorname{Spec} F \cong \operatorname{Br} F$$

For any scheme X there is a natural inclusion

$$\mathrm{Br}_{\mathrm{Az}} X \hookrightarrow \operatorname{Br} X.$$

Theorem (Gabber, de Jong)

If X is a scheme endowed with an ample invertible sheaf then the natural map $\mathrm{Br}_{\mathrm{Az}} X \hookrightarrow \operatorname{Br} X$ induces an isomorphism

$$\mathrm{Br}_{\mathrm{Az}} X \xrightarrow{\sim} (\operatorname{Br} X)_{\mathrm{tors}}. \qquad \square$$

If X is an integral, regular and quasi-compact scheme, then the inclusion $\operatorname{Spec} \mathbf{k}(X) \to X$ gives rise to an injection $\operatorname{Br} X \hookrightarrow \operatorname{Br} \mathbf{k}(X)$.

On the other hand, the group $\operatorname{Br} \mathbf{k}(X)$ is torsion, because it is a Galois cohomology group.

Corollary

Let X be a nice variety over a field. Then

$$\operatorname{Br}_{Az} X \cong \operatorname{Br} X. \qquad \square$$

Let X be a nice variety over a global field k. For $\mathcal{A} \in \operatorname{Br} X$ and K/k a field extension there is an evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon X(K) \to \operatorname{Br} K, \quad x \mapsto \mathcal{A}_x \otimes_{\mathscr{O}_{X,x}} K.$$

We put these maps together to construct a pairing

$$\phi : \operatorname{Br} X \times X(\mathbf{A}_k) \to \mathbb{Q}/\mathbb{Z}, \quad (\mathcal{A}, (x_v)) \mapsto \sum_{v \in \Omega_k} \mathrm{inv}_v(\mathrm{ev}_{\mathcal{A}}(x_v)),$$

where $\mathrm{inv}_v \colon \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z}$ is the invariant map from LCFT. For $\mathcal{A} \in \operatorname{Br} X$ we obtain a commutative diagram

$$\begin{tikzcd}
 & X(k) \arrow[r] \arrow[d, "\mathrm{ev}_{\mathcal{A}}"'] & X(\mathbf{A}_k) \arrow[d, "\mathrm{ev}_{\mathcal{A}}"'] \arrow[dr, "\phi(\mathcal{A},-)"] & & \\
0 \arrow[r] & \operatorname{Br} k \arrow[r] & \bigoplus_v \operatorname{Br} k_v \arrow[r, "\sum_v \mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Manin's observation is that an element $\mathcal{A} \in \operatorname{Br} X$ can be used to "carve out" a subset of $X(\mathbf{A}_k)$ that contains $X(k)$:

$$X(\mathbf{A}_k)^{\mathcal{A}} := \{(x_v) \in X(\mathbf{A}_k) : \phi(\mathcal{A}, (x_v)) = 0\}.$$

We call

$$X(\mathbf{A}_k)^{\operatorname{Br}} := \bigcap_{\mathcal{A} \in \operatorname{Br} X} X(\mathbf{A})^{\mathcal{A}}$$

the Brauer-Manin set of X.

if $\mathbb{Q}/\mathbb{Z}$ is given the discrete topology, then the map $\phi(\mathcal{A}, -)\colon X(\mathbf{A}_k) \to \mathbb{Q}/\mathbb{Z}$ is continuous, so $X(\mathbf{A}_k)^{\mathcal{A}}$ is a closed subset of $X(\mathbf{A}_k)$. In particular,

$$\overline{X(k)} \subseteq X(\mathbf{A}_k)^{\operatorname{Br}}.$$

If $\mathcal{A} \in \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X) =: \operatorname{Br}_0 X$, then $X(\mathbf{A}_k)^{\mathcal{A}} = X(\mathbf{A}_k)$. This means that to compute $X(\mathbf{A}_k)^{\operatorname{Br}}$, it is enough to consider $X(\mathbf{A}_k)^{\mathcal{A}}$, as $\mathcal{A}$ runs through a set of representatives of the group $\operatorname{Br} X / \operatorname{Br}_0 X$. When $\operatorname{Br} X_{k^{\text{sep}}} = 0$, the Hochschild-Serre spectral sequence in étale cohomology (with $\mathbb{G}_m$-coefficients) can help us compute this group. The long exact sequence of low degree terms is

$$\begin{aligned} 0 &\to \operatorname{Pic} X \to (\operatorname{Pic} X_{k^{\text{sep}}})^{\operatorname{Gal}(k^{\text{sep}}/k)} \to \operatorname{Br} k \\ &\to \ker(\operatorname{Br} X \to \operatorname{Br} X_{k^{\text{sep}}}) \to \mathrm{H}^1\big(\operatorname{Gal}(k^{\text{sep}}/k), \operatorname{Pic} X_{k^{\text{sep}}}\big) \\ &\to \mathrm{H}^3\big(\operatorname{Gal}(k^{\text{sep}}/k), k^{\text{sep}*}\big). \end{aligned}$$

If k is a global field, then $\mathrm{H}^3\big(\mathrm{Gal}(k^{\mathrm{sep}}/k), k^{\mathrm{sep}*}\big) = 0$ (Tate). If $X(\mathbf{A}_k) \neq \emptyset$, then the map $(\operatorname{Pic} X_{k^{\mathrm{sep}}})^{\mathrm{Gal}(k^{\mathrm{sep}}/k)} \to \operatorname{Br} k$ is the zero map and hence we have

$$\operatorname{Pic} X \xrightarrow{\sim} (\operatorname{Pic} X_{k^{\mathrm{sep}}})^{\mathrm{Gal}(k^{\mathrm{sep}}/k)}.$$

If X is a geometrically rational surface, then $\operatorname{Br} X_{k^{\mathrm{sep}}} = 0$. Put this all together and we get

Proposition

Let X be a del Pezzo surface over a global field k. Assume that $X(\mathbf{A}_k) \neq \emptyset$. Then we have

$$\operatorname{Br} X / \operatorname{Br} k \xrightarrow{\sim} \mathrm{H}^1\big(\mathrm{Gal}(k^{\mathrm{sep}}/k), \operatorname{Pic} X_{k^{\mathrm{sep}}}\big).$$

X: del Pezzo surface over a global field k of degree $d \leq 7$. Let K be the smallest extension of k in k^{sep} over which all exceptional curves of X are defined. The group $\operatorname{Pic} X_{k^{\text{sep}}}$ is generate by the class of exceptional curves, so

$$\operatorname{Pic} X_K \cong \operatorname{Pic} X_{k^{\text{sep}}},$$

and moreover, the inflation map

$$\mathrm{H}^1\big(\operatorname{Gal}(K/k), \operatorname{Pic} X_K\big) \to \mathrm{H}^1\big(\operatorname{Gal}(k^{\text{sep}}/k), \operatorname{Pic} X_{k^{\text{sep}}}\big)$$

is an isomorphism (here we assume that $X(\mathbf{A}_k) \neq \emptyset$). One way of constructing Brauer-Manin obstructions on del Pezzo surfaces of small degree begins by computing the group $\mathrm{H}^1\big(\operatorname{Gal}(K/k), \operatorname{Pic} X_K\big)$ on "reasonable" surfaces.

Many authors have pursued this set of ideas, not just for del Pezzo surfaces: Manin, Swinnerton-Dyer, Colliot-Thélène, Kanevsky, Sansuc, Skorobogatov, Bright, Bruin, Flynn, Logan, Kresch, Tschinkel, Corn, van Luijk, V-A, etc (the list is not meant to be comprehensive).

We will compute an example to weak approximation on a del Pezzo surface of degree 1.

Del Pezzo surfaces of degree 1: quick review

Anticanonical model of X/k is a smooth sextic hypersurface in $\mathbb{P}_k(1,1,2,3) := \mathrm{Proj}(k[x,y,z,w])$, e.g.,

$$w^2 = z^3 + Ax^6 + By^6, \qquad A, B \in k^*.$$

Conversely, any smooth sextic in $\mathbb{P}_k(1,1,2,3)$ is a dP1. $X_{k^{\mathrm{sep}}}$ is isomorphic to the blow-up of $\mathbb{P}^2_{k^{\mathrm{sep}}}$ at 8 points in *general position*. In particular,

$$\operatorname{Pic} X_{k^{\mathrm{sep}}} \cong \mathbb{Z}^9.$$

Fix a primitive sixth root of unity ζ in $\overline{\mathbb{Q}}$.

Theorem (V-A'08)

Let X be the del Pezzo surface of degree 1 over $k = \mathbb{Q}(\zeta)$ given by

$$w^2 = z^3 + 16x^6 + 16y^6$$

in $\mathbb{P}_k(1,1,2,3)$. Then X is k-minimal and there is a Brauer-Manin obstruction to weak approximation on X. Moreover, the obstruction arises from a cyclic algebra class in $\operatorname{Br} X / \operatorname{Br} k$.

Need the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on $\operatorname{Pic} X_{k^{\mathrm{sep}}}$ explicitly. Recall that $\operatorname{Pic} X_{k^{\mathrm{sep}}}$ is generated by the exceptional curves of X.

Theorem (V-A'08)

Let X be a del Pezzo surface of degree 1 over a field k, given as a smooth sextic hypersurface $V(f(x,y,z,w))$ in $\mathbb{P}_k(1,1,2,3)$. Let

$$\Gamma = V(z - Q(x,y), w - C(x,y)) \subseteq \mathbb{P}_{k^{\mathrm{sep}}}(1,1,2,3),$$

where $Q(x,y)$ and $C(x,y)$ are homogenous forms of degrees 2 and 3, respectively, in $k^{\mathrm{sep}}[x,y]$. If Γ is a divisor on $X_{k^{\mathrm{sep}}}$, then it is an exceptional curve of X. Conversely, every exceptional curve on X is a divisor of this form.

Exceptional curves on $w^2 = z^3 + 16x^6 + 16y^6$

Let

$$
\begin{aligned}
Q(x,y) &= ax^2 + bxy + cy^2, \\
C(x,y) &= rx^3 + sx^2y + txy^2 + uy^3,
\end{aligned}
$$

Then the identity $C(x,y)^2 = Q(x,y)^3 + 16x^6 + 16y^6$ gives

$$
\begin{aligned}
a^3 - r^2 + 16 &= 0, \\
3a^2b - 2rs &= 0, \\
3a^2c + 3ab^2 - 2rt - s^2 &= 0, \\
6abc + b^3 - 2ru - 2st &= 0, \\
3ac^2 + 3b^2c - 2su - t^2 &= 0, \\
3bc^2 - 2tu &= 0, \\
c^3 - u^2 + 16 &= 0.
\end{aligned}
$$

We can use Gröbner bases to solve this system of equations. We get 240 solutions, one for each exceptional curve of the surface. The action of $\mathrm{Gal}(\bar{k}/k)$ can be read off from the coefficients of the equations of the exceptional curves. Sample exceptional curve: $(s = \sqrt[3]{2}, \zeta = (1+\sqrt{-3})/2)$

$$\begin{aligned}
z &= (-s^2\zeta + s^2 - 2s + 2\zeta)x^2 + (2s^2\zeta - 2s^2 + 3s - 4\zeta)xy \\
&\quad + (-s^2\zeta + s^2 - 2s + 2\zeta)y^2, \\
w &= (2s^2\zeta - 4s^2 + 2s\zeta + 2s - 6\zeta + 3)x^3 \\
&\quad + (-5s^2\zeta + 10s^2 - 6s\zeta - 6s + 16\zeta - 8)x^2y \\
&\quad + (5s^2\zeta - 10s^2 + 6s\zeta + 6s - 16\zeta + 8)xy^2 \\
&\quad + (-2s^2\zeta + 4s^2 - 2s\zeta - 2s + 6\zeta - 3)y^3.
\end{aligned}$$

The Picard group of X

Let $s = \sqrt[3]{2}$. Consider the exceptional curves on X given by

$$
\begin{aligned}
E_1 &= V(z + 2sx^2,\ w - 4y^3),\\
E_2 &= V(z - (-\zeta_3 + 1)2sx^2,\ w + 4y^3),\\
E_3 &= V(z - 2\zeta_3 sx^2 + 4y^2,\ w - 4s(\zeta_3 - 2)x^2y - 4(-2\zeta_3 + 1)y^3),\\
E_4 &= V(z + 4\zeta_3 sx^2 - 2s^2(2\zeta_3 - 1)xy - 4(-\zeta_3 + 1)y^2,\\
&\qquad w - 12x^3 - 8s(-\zeta_3 - 1)x^2y - 12\zeta_3 s^2xy^2 - 4(-2\zeta_3 + 1)y^3),\\
E_5 &= V(z + 4\zeta_3 sx^2 - 2s^2(\zeta_3 - 2)xy - 4\zeta_3 y^2\\
&\qquad w + 12x^3 - 8s(2\zeta_3 - 1)x^2y - 12s^2xy^2 - 4(-2\zeta_3 + 1)y^3),\\
E_6 &= V(z - 2s(-s^2\zeta_3 + s^2 - 2s + 2\zeta_3)x^2 - 2s(2s^2\zeta_3 - 2s^2 + 3s - 4\zeta_3)xy - 2s(-s^2\zeta_3 + s^2 - 2s + 2\zeta_3)y^2,\\
&\qquad w - 4(2s^2\zeta_3 - 4s^2 + 2s\zeta_3 + 2s - 6\zeta_3 + 3)x^3 - 4(-5s^2\zeta_3 + 10s^2 - 6s\zeta_3 - 6s + 16\zeta_3 - 8)x^2y\\
&\qquad\quad - 4(5s^2\zeta_3 - 10s^2 + 6s\zeta_3 + 6s - 16\zeta_3 + 8)xy^2 - 4(-2s^2\zeta_3 + 4s^2 - 2s\zeta_3 - 2s + 6\zeta_3 - 3)y^3),\\
E_7 &= V(z - 2s(-s^2 - 2s\zeta_3 + 2s + 2\zeta_3)x^2 - 2s(-2s^2\zeta_3 + 3s + 4\zeta_3 - 4)xy - 2s(-s^2\zeta_3 + s^2 + 2s\zeta_3 - 2)y^2,\\
&\qquad w - 4(2s^2\zeta_3 + 2s^2 + 2s\zeta_3 - 4s - 6\zeta_3 + 3)x^3 - 4(10s^2\zeta_3 - 5s^2 - 6s\zeta_3 - 6s - 8\zeta_3 + 16)x^2y\\
&\qquad\quad - 4(5s^2\zeta_3 - 10s^2 - 12s\zeta_3 + 6s + 8\zeta_3 + 8)xy^2 - 4(-2s^2\zeta_3 - 2s^2 - 2s\zeta_3 + 4s + 6\zeta_3 - 3)y^3),\\
E_8 &= V(z - 2s(s^2\zeta_3 + 2s\zeta_3 + 2\zeta_3)x^2 - 2s(2s^2 + 3s + 4)xy - 2s(-s^2\zeta_3 + s^2 - 2s\zeta_3 + 2s - 2\zeta_3 + 2)y^2,\\
&\qquad w - 4(-4s^2\zeta_3 + 2s^2 - 4s\zeta_3 + 2s - 6\zeta_3 + 3)x^3 - 4(-5s^2\zeta_3 - 5s^2 - 6s\zeta_3 - 6s - 8\zeta_3 - 8)x^2y\\
&\qquad\quad - 4(5s^2\zeta_3 - 10s^2 + 6s\zeta_3 - 12s + 8\zeta_3 - 16)xy^2 - 4(4s^2\zeta_3 - 2s^2 + 4s\zeta_3 - 2s + 6\zeta_3 - 3)y^3).
\end{aligned}
$$

The Picard group of X

...as well as the exceptional curve

$$E_9 = V(z - 2\zeta_3 s^2 xy, w - 4x^3 + 4y^3).$$

Then

$$\operatorname{Pic} X_{\bar{k}} = \operatorname{Pic} X_K \cong \left(\bigoplus_{i=1}^{8} \mathbb{Z}[E_i] \right) \oplus \mathbb{Z}[H] = \mathbb{Z}^9,$$

where $H = E_1 + E_2 + E_9$.
The exceptional curves of X are defined over $K := k(\sqrt[3]{2})$.
Let $G := \operatorname{Gal}(K/k) = \langle \rho \rangle$. Note that G is cyclic.

Strategy for inverting $\operatorname{Br} X/\operatorname{Br} k \to \mathrm{H}^1\big(\operatorname{Gal}(k^{\mathrm{sep}}/k), \operatorname{Pic} X_{k^{\mathrm{sep}}}\big)$

$$\begin{tikzcd}
\operatorname{Br} X/\operatorname{Br} k \arrow[rr, "\sim"] \arrow[d, hook] & & \mathrm{H}^1\big(\operatorname{Gal}(k^{\mathrm{sep}}/k), \operatorname{Pic} X_{k^{\mathrm{sep}}}\big) \\
\operatorname{Br} \mathbf{k}(X)/\operatorname{Br} k & & \mathrm{H}^1\big(\operatorname{Gal}(K/k), \operatorname{Pic} X_K\big) \arrow[u, "\mathrm{inf}", "\sim"'] \arrow[d, "\sim"] \\
\operatorname{Br}_{\mathrm{cyc}}(X,K) \arrow[u, hook] & & \ker \overline{N}_{K/k}/\operatorname{im}\Delta \arrow[ll, "\psi", "\sim"']
\end{tikzcd}$$

$$\operatorname{Br}_{\mathrm{cyc}}(X,K) := \left\{ \begin{array}{c} \text{classes } [(K/k, f)] \text{ in the image of the} \\ \text{map } \operatorname{Br} X/\operatorname{Br} k \to \operatorname{Br} \mathbf{k}(X)/\operatorname{Br} k \end{array} \right\}$$

The group $\mathrm{Br}_{\mathrm{cyc}}(X, K)$

Explicitly, we have maps

$$\overline{N}_{K/k}\colon \operatorname{Pic} X_K \to \operatorname{Pic} X \qquad\qquad \Delta\colon \operatorname{Pic} X_K \to \operatorname{Pic} X$$
$$[D] \mapsto [D + {}^{\rho}D + {}^{\rho^2}D] \qquad\qquad [D] \mapsto [D - {}^{\rho}D]$$

We compute

$$\ker \overline{N}_{K/k} / \operatorname{im} \Delta \cong (\mathbb{Z}/3\mathbb{Z})^4;$$

and the classes

$$\mathfrak{h}_1 = [E_2 + 2E_8 - H], \qquad \mathfrak{h}_2 = [E_5 + 2E_8 - H],$$
$$\mathfrak{h}_3 = [E_7 + 2E_8 - H], \qquad \mathfrak{h}_4 = [3E_8 - H]$$

of $\operatorname{Pic} X_K$ give a set of generators for this group.

An Azumaya Algebra

The group isomorphism

$$\psi\colon \ker \overline{N}_{K/k}/\operatorname{im}\Delta \to \mathrm{Br}_{\mathrm{cyc}}(X, K)$$

is given by

$$[D] \mapsto [(K/k, f)],$$

where $f \in k(X)^*$ is any function such that $N_{K/k}(D) = (f)$.

Consider the divisor class $\mathfrak{h}_1 - \mathfrak{h}_2 = [E_2 - E_5] \in \operatorname{Pic} X_K$. It gives rise to a cyclic algebra $\mathscr{A} := (K/k, f) \in \mathrm{Br}_{\mathrm{cyc}}(X, K)$, where $f \in k(X)^*$ is any function such that

$$N_{K/k}(E_2 - E_5) = (f),$$

To wit, f is a function with zeroes along

$$E_2 + {}^{\rho}E_2 + {}^{\rho^2}E_2$$

and poles along

$$E_5 + {}^{\rho}E_5 + {}^{\rho^2}E_5.$$

Using the explicit equations for E_2 and E_5 we find

$$f := \frac{w + 4y^3}{w + (2\zeta + 2)zy + (-8\zeta + 4)y^3 + 12x^3}$$

does the job.

The Brauer-Manin obstruction

Recall X is given by $w^2 = z^3 + 16x^6 + 16y^6$. Note that

$$P_1 = [1 : 0 : 0 : 4] \quad \text{and} \quad P_2 = [0 : 1 : 0 : 4].$$

are in $X(k)$.

Let $\mathfrak{p}$ be the unique prime above 3 in k. We compute

$$\mathrm{inv}_{\mathfrak{p}}(\mathscr{A}(P_1)) = 0 \quad \text{and} \quad \mathrm{inv}_{\mathfrak{p}}(\mathscr{A}(P_2)) = 1/3.$$

Let $P \in X(\mathbf{A}_k)$ be the point that is equal to P_1 at all places except $\mathfrak{p}$, and is P_2 at $\mathfrak{p}$. Then

$$\sum_v \mathrm{inv}_v(\mathscr{A}(P_v)) = 1/3,$$

so $P \in X(\mathbf{A}_k) \setminus X(\mathbf{A}_k)^{\mathrm{Br}}$ and X is a counterexample to weak approximation. $\square$