Arithmetic of del Pezzo surfaces of degree 1

by

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Abstract

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We study the density of rational points on del Pezzo surfaces of degree 1 for the Zariski topology and the adèlic topology. For a large class of these surfaces over $\mathbb{Q}$, we show that the set of rational points is dense for the Zariski topology. We achieve our results by carefully studying variations of root numbers among the fibers of elliptic surfaces associated to del Pezzo surfaces of degree 1. Our results in this direction are conditional on the finiteness of Tate-Shafarevich groups for elliptic curves over $\mathbb{Q}$.

We also explicitly study the Galois action on the geometric Picard group of del Pezzo surfaces of degree 1 of the form

$$w^2 = z^3 + Ax^6 + By^6$$

in the weighted projective space $\mathbb{P}_k(1,1,2,3)$, where k is a global field of characteristic not 2 or 3 and $A, B \in k^*$. Over a number field, we exhibit an infinite family of minimal surfaces for which the rational points are not dense for the adèlic topology; i.e., minimal surfaces that fail to satisfy weak approximation. These counterexamples are explained by a Brauer-Manin obstruction.

Professor Bjorn Poonen
Dissertation Committee Chair

To my father

Contents

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Chapter 1

Motivation and main results

1.1 Guiding questions in diophantine geometry

Let k be a global field, i.e., a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$ for some prime p, let $\mathbb{A}_k$ denote its ring of adèles, and let X be a smooth projective geometrically integral variety over k. Generally speaking, diophantine geometers seek to "describe" the set $X(k)$ of k-rational points of X. For example, we are interested in determining whether $X(k)$ is empty or not. If $X(k) \neq \emptyset$, then we may further want to know something about the qualitative nature of $X(k)$: is it dense for the Zariski topology of X? Is the image of the natural embedding $X(k) \hookrightarrow X(\mathbb{A}_k)$ dense for the adèlic topology? If not, can we account for the paucity of k-rational points? We may also pursue a more quantitative study of $X(k)$. For instance, we might try to prove asymptotic formulas for the number of k-points of bounded height on some special Zariski-open subset of X.

On the other hand, if $X(k) = \emptyset$, then we might try to account for the absence of k-rational points. For example, the existence of embeddings $X(k) \hookrightarrow X(k_v)$ for every completion k_v of k shows that a necessary condition for X to have a k-rational point is

$$X(k_v) \neq \emptyset \text{ for all completions } k_v \text{ of } k. \tag{1.1}$$

To illustrate this, note that the projective plane conic $x^2+y^2 = 3z^2$ over $\mathbb{Q}$ has no $\mathbb{Q}_3$-points, and hence it contains no $\mathbb{Q}$-points.

We say that X is locally soluble whenever (1.1) is satisfied, and we note that local solubility makes sense for any variety over a global field. Whenever checking (1.1) suffices to show that $X(k) \neq \emptyset$, we say that X satisfies the Hasse principle. Many classes of varieties, such as plane quadrics, satisfy the Hasse principle.

Perhaps the first known counterexample to the Hasse principle is due to Lind and Reichardt, who show that the genus 1 plane curve over $\mathbb{Q}$ with affine model given by $2y^2 = x^4 - 17$ is locally soluble, but lacks $\mathbb{Q}$-rational points; see [Lin40, Rei42]. Failures of the Hasse principle are often explained by the presence of cohomologically flavored obstructions, such as the Brauer-Manin obstruction. These kinds of obstructions may also produce examples of varieties X as above, with $X(k) \neq \emptyset$, for which the embedding $X(k) \hookrightarrow X(\mathbb{A}_k)$ is not dense.

In this thesis, we study the above circle of questions for the class of del Pezzo surfaces of degree 1. We think of these surfaces as smooth sextics in the weighted projective space $\mathbb{P}_k(1,1,2,3)$. Among other things, we show that many such surfaces over $\mathbb{Q}$ have a Zariski dense set of rational points, provided that Tate-Shafarevich groups of elliptic curves are finite. By systematically studying the Galois action on the set of exceptional curves on these surfaces, we also produce the first explicit (minimal) examples for which the embedding $X(k) \hookrightarrow X(\mathbb{A}_k)$ is not dense. For detailed statements of our principal results, see §1.5.

To appreciate how our results fit in the literature, we explain in §1.2 how the answers to the guiding questions we have outlined depend only on the birational class of a variety. We then use a birational classification theorem of Iskovskikh to focus our efforts del Pezzo surfaces and rational conic bundles (§1.3), and we present a synopsis of known answers to our guiding questions in §1.4. Our knowledge gaps on the the arithmetic of del Pezzo surfaces of degree 1 will become transparent. To the author's knowledge, the results in this thesis represent the first progress on the arithmetic of del Pezzo surfaces of degree 1 since [Man74].

Notation. The following notation will remain in force throughout this thesis. First, k denotes a field, $\bar{k}$ is a fixed algebraic closure of k, and $k^s \subseteq \bar{k}$ is the separable closure of k in $\bar{k}$. If k is a global field then we write $\mathbb{A}_k$ for the adèle ring of k, Ω_k for the set of places of k, and k_v for the completion of k at $v \in \Omega_k$. By a k-variety X we mean a separated scheme of finite type over k (we will omit the reference to k when it can cause no confusion). If X and Y are S-schemes then we write $X_Y := X \times_S Y$. However, if $Y = \operatorname{Spec} A$ then we write X_A instead of $X_{\operatorname{Spec} A}$. A k-variety X is said to be nice if it is smooth, projective and geometrically integral. If T is a k-scheme, then we write $X(T)$ for the set of T-valued points of X. If, however, $T = \operatorname{Spec} A$ is affine, then we write $X(A)$ instead of $X(\operatorname{Spec} A)$.

1.2 Birational invariance and a theorem of Iskovskikh

Let X be a nice k-variety. Many properties of $X(k)$, such as "being nonempty," depend only on X up to birational equivalence, as follows.

Existence of a smooth k-point. The Lang-Nishimura lemma guarantees that if $X' \dashrightarrow X$ is a birational map between proper integral k-varieties then X' has a smooth k-point if and only if X has a smooth k-point; see [Lan54, Nis55].

Zariski density of k-rational points. If X, X' are two nice birationally equivalent k-varieties, then $X(k)$ is Zariski dense in X if and only if $X'(k)$ is Zariski dense in X': the key point to keep in mind is that any two nonempty open sets in the Zariski topology have nonempty intersection.

Weak approximation. Let X be a geometrically integral variety over a global field k. We say that X satisfies weak approximation if the diagonal embedding

$$X(k) \hookrightarrow \prod_{v \in \Omega_k} X(k_v)$$

is dense for the product of the v-adic topologies. If X is a nice k-variety then $X(\mathbb{A}_k) = \prod_v X(k_v)$, the latter considered with the product topology of the v-adic topologies; see [Sko01, pp. 98–99]. In this case X satisfies weak approximation if the image of the natural map $X(k) \hookrightarrow X(\mathbb{A}_k)$ is dense for the adèlic topology. Note also that if X does not satisfy the Hasse principle, then automatically X does not satisfy weak approximation.

Lemma 1.2.1. *If X and X' are smooth, geometrically integral and birationally equivalent varieties over a global field k, then X' satisfies weak approximation if and only if X satisfies weak approximation.*

Sketch of proof. It is enough to prove the lemma in the case $X' = X \setminus W$, where W is a proper closed subvariety of X, i.e., X' is a dense open subset of X. Then, if X satisfies weak approximation, then clearly so does X'. On the other hand, by the v-adic implicit function theorem, the set $X'(k_v)$ is dense in $X(k_v)$; see [CTCS80, Lemme 3.1.2]. Suppose that X' satisfies weak approximation and let $(x_v) \in \prod_v X(k_v)$ be given. Choose $(y_v) \in$

$\prod_v X'(k_v) \subseteq \prod_v X(k_v)$ as close as desired to (x_v) for the product topology. By hypothesis, there is a rational point $y \in X'(k)$ whose image in $\prod_v X'(k_v)$ is arbitrarily close to (y_v); then y is also close to (x_v), and X satisfies weak approximation. □

Remark 1.2.2. There is a useful variant of weak approximation, as follows. Let X be a geometrically integral variety over a global field k. We say X satisfies weak-weak approximation if there exists a finite set $T \subseteq \Omega_k$ such that for every other finite set $S \subseteq \Omega_k$ with $S \cap T = \emptyset$, the image of the embedding

$$X(k) \hookrightarrow \prod_{v \in S} X(k_v)$$

is dense for the product topology of the v-adic topologies. Note that X satisfies weak approximation if we can take $T = \emptyset$. If X is smooth then weak-weak approximation depends only on a birational model of X.

It is thus natural to ask the qualitative questions of §1.1 in the context of a *fixed* birational class for X. In particular, we will fix the dimension of X. In this thesis, we will consider these questions only for nice *surfaces*. In addition, we require that X be geometrically rational, i.e., $X \times_k \bar{k}$ is birational to $\mathbb{P}^2_{\bar{k}}$. The reason for this last restriction is the existence of the following beautiful classification theorem due to Iskovskikh, which describes the possible birational classes for X.

Theorem 1.2.3 ([Isk79, Theorem 1])**.** *Let k be a field, and let X be a smooth projective geometrically rational surface over k. Then X is k-birational to either a del Pezzo surface of degree $1 \leq d \leq 9$ or a rational conic bundle.* □

1.3 Del Pezzo surfaces and rational conic bundles

In light of Theorem 1.2.3, we take a moment to review the definition and some basic properties of del Pezzo surfaces and rational conic bundles. The reader is referred to Chapter 2 for further particulars on del Pezzo surfaces. In this section, we work over an arbitrary field k.

We begin by recalling some basic facts and setting some notation. If X is a nice surface, then there is an intersection pairing on the Picard group $(\,\cdot\,,\cdot\,)_X \colon \operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$; see [Kle05, Appendix B] We omit the subscript on the pairing if no confusion can arise. For such an X, we identify $\operatorname{Pic}(X)$ with the Weil divisor class group (see [Har77, Corollary

II.6.16]); in particular, we will use additive notation for the group law on $\operatorname{Pic} X$. If X is a nice k-variety, then we write K_X for the class of the canonical sheaf ω_X in $\operatorname{Pic} X$; the anticanonical sheaf of X is $\omega_X^{\otimes -1}$. An exceptional curve on a smooth projective k-surface X is an irreducible curve $C \subseteq X_{\bar{k}}$ such that $(C, C) = (K_X, C) = -1$. By the adjunction formula, an exceptional curve on X has arithmetic genus 0, and hence it is isomorphic to $\mathbb{P}^1_{\bar{k}}$; see [Ser88, IV.8, Proposition 5].

Definition 1.3.1. A del Pezzo surface X is a nice k-surface with ample anticanonical sheaf. The degree of X is the intersection number $d := (K_X, K_X)$.

If X is a del Pezzo surface then the Riemann-Roch theorem for surfaces and Castelnuovo's rationality criterion show that X is geometrically rational. Moreover, X_{k^s} is isomorphic to either $\mathbb{P}^1_{k^s} \times \mathbb{P}^1_{k^s}$ (in which case $d = 8$), or the blow-up of $\mathbb{P}^2_{k^s}$ at $r \leq 8$ distinct closed points (in which case $d = 9 - r$); this is the content of Theorem 2.1.1 below. In the latter case, the points must be in general position: this means no 3 of them on a line, no 6 of them on a conic and no 8 of them on a cubic with a singularity at one of the points. General position of the blown-up points is equivalent to ampleness of the anticanonical class on the blown-up surface; see [Dem80, Théorème 1, p. 27].

Definition 1.3.2. We say a nice surface X over a field k is k-minimal (or just minimal) if there is no nonempty $\operatorname{Gal}(k^s/k)$-stable set S of pairwise nonintersecting exceptional curves.

When a nice surface X is not k-minimal, there is a $\operatorname{Gal}(k^s/k)$-stable set S of pairwise nonintersecting exceptional curves which can be simultaneously 'blown-down'. This process can be iterated on the 'blown-down' surface until there are no more $\operatorname{Gal}(k^s/k)$-stable sets of pairwise nonintersecting exceptional curves. This is a finite process since the Picard number of the surface decreases at each stage; the final surface is k-minimal. In fact, when k is perfect, X is minimal if and only if any birational k-morphism to a nice surface Y is an isomorphism; see [Has09, Theorem 3.2].

Definition 1.3.3. A rational conic bundle X over a field k is a minimal smooth projective geometrically rational surface together with a dominant k-morphism $\pi\colon X \to C$ for which the base curve and the generic fiber are smooth curves of genus 0. The degree of X is the intersection number $d := (K_X, K_X)$.

If $f\colon X \to C$ is a rational conic bundle, then each smooth fiber of f is a geometrically reduced plane conic split by a quadratic extension of k. Moreover, the non-smooth fibers of $f_{k^s}\colon X_{k^s} \to C_{k^s}$ consist of pairs of exceptional curves intersecting transversely at one point; see [Has09, Theorem 3.6].

Remark 1.3.4. It is possible for X as in Theorem 1.2.3 to be k-birational to *both* a del Pezzo surface and a rational conic bundle. More precisely, a rational conic bundle is birational to a *minimal* del Pezzo surface if and only if $d = 1, 2$ or 4 and there are two distinct representations of X as a rational conic bundle; see [Isk79, Theorems 4 and 5].

Examples of rational conic bundles are certain smooth projective models of affine surfaces defined by an equation of the form

$$y^2 - az^2 = P(x), \tag{1.2}$$

where $a \in k^*$, and $P(x)$ is a nonzero polynomial. We may assume (by making suitable rational changes of variables) that $P(x)$ is a separable polynomial of even degree. For an explicit construction of the smooth projective model of these surfaces, see [Poo08, §4].

The geometry of rational conic bundles has been extensively studied; see [MT86, §2.2] and [Has09, §3.2] for a survey of geometric results.

1.4 Survey of arithmetic results

We survey known answers to the questions we raised in §1.1 for smooth projective geometrically rational surfaces over a field k, in light of Theorem 1.2.3.

Existence of a smooth k-point. Del Pezzo surfaces of degrees 1, 5 and 7 are known to carry k-rational points. If X is such a surface of degree 1, then the linear system $|-K_X|$ has a single basepoint ([Dem80, Proposition 2, p. 40]), which is necessarily defined over the ground field. The case of degree 5 surfaces is a theorem formulated by Enriques in [Enr97] and proved independently by Swinnerton-Dyer, Shepherd-Barron and Skorobogatov; see [SD72, SB92, Sko93], respectively. If X is a del Pezzo surface of degree 7, then X_{k^s} is isomorphic to a blow-up of $\mathbb{P}^2_{k^s}$ at two distinct points, and the strict transform of the line on $\mathbb{P}^2_{k^s}$ passing through the two blow-up points is an exceptional curve that is $\mathrm{Gal}(k^s/k)$-stable. Contracting this curve yields a surface of degree 8 with a k-rational point, and we conclude by using the Lang-Nishimura lemma.

Del Pezzo surfaces of other degrees need not have k-rational points. Surfaces of degree at least 5, however, are known to satisfy the Hasse principle. For example, del Pezzo surfaces of degree 9 are forms of $\mathbb{P}^2_k$, i.e., Severi-Brauer surfaces, and thus satisfy the Hasse principle; see [Châ44]. If X is a del Pezzo surface of degree 8, then X_{k^s} is isomorphic either to a blow-up of $\mathbb{P}^2_{k^s}$ at a closed point or to $\mathbb{P}^1_{k^s} \times \mathbb{P}^1_{k^s}$. In the former case, the unique exceptional curve is fixed by the action of $\mathrm{Gal}(k^s/k)$; contracting this curve yields a Severi-Brauer surface with a k-rational point, and we conclude by using the Lang-Nishimura lemma. For the latter case, see [CT72b, p. 19]. The case of surfaces of degree 6 is a theorem of Manin, though we refer the reader to a beautiful and elementary proof by Colliot-Thélène in [CT72a].

Del Pezzo surfaces of degrees 2, 3 and 4 can fail to satisfy the Hasse principle, as the following examples show.

Example 1.4.1 ([KT04, Example 1])**.** The hypersurface given by

$$w^2 = -6x^4 - 3y^4 + 2z^4$$

in the weighted projective space $\mathbb{P}_{\mathbb{Q}}(1,1,1,2)$ is a del Pezzo surface of degree 2 which is locally soluble, but which lacks $\mathbb{Q}$-points.

Example 1.4.2 ([CG66])**.** The cubic surface in $\mathbb{P}^3_{\mathbb{Q}}$ given by

$$5x^3 + 9y^3 + 10z^3 + 12w^3 = 0$$

is a del Pezzo surface of degree 3 which is locally soluble, but which lacks $\mathbb{Q}$-points.

Example 1.4.3 ([BSD75, Theorem 3])**.** The variety in $\mathbb{P}^4_{\mathbb{Q}}$ defined by the equations

$$uv = x^2 - 5y^2$$
$$(u+v)(u+2v) = x^2 - 5z^2$$

is a del Pezzo surface of degree 4 which is locally soluble, but which lacks $\mathbb{Q}$-points.

All such known counterexamples can be explained by a Brauer-Manin obstruction; see §2.3.

The state of affairs for rational conic bundles is not a good one. The strongest known result is due to Salberger. In [Sal88], he shows that if $X \to \mathbb{P}^1_k$ is a rational conic bundle over a global field k, such that $\mathrm{H}^1\big(\mathrm{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big) = 0$, then X has a zero-cycle of degree 1 if and only if X_{k_v} has a zero-cycle of degree 1 for all $v \in \Omega_k$. A similar claim for k-rational points is unknown as of this writing.

There is a smattering of (hard-to-prove) results for surfaces X of the form (1.2). For example, if $\deg(P(x)) = 2$ then the conic bundles satisfy the Hasse principle by the Hasse-Minkowski theorem on quadratic forms. If $P(x)$ is a monic polynomial of degree 4, then we call X a Châtelet surface. Such surfaces need not satisfy the Hasse principle.

Example 1.4.4 ([Isk71])**.** The Châtelet surface over $\mathbb{Q}$ given by a smooth projective model of

$$y^2 + z^2 = (3 - x^2)(x^2 - 2)$$

does not satisfy the Hasse principle. A proof of this of fact, phrased in terms of Brauer-Manin obstructions, can be found in [Sko01, p. 145].

By generalizing Example 1.4.4, Poonen recently constructed Châtelet surfaces over any global field of characteristic not 2 which violate the Hasse principle; see [Poo07, Proposition 5.1 and §11]. Viray extended the construction to global fields of characteristic 2 in [Vir09].

In the landmark two-part paper [CTSSD87a, CTSSD87b], Colliot-Thélène, Sansuc and Swinnerton-Dyer show that the Brauer-Manin obstruction to the Hasse principle and to weak approximation on Châtelet surfaces is the only obstruction; see Chapter 2 for the necessary background material. In [SD99], Swinnerton-Dyer proves an analogous result for surfaces such that $P(x)$ is the product of polynomials of degrees 2 and 4. A streamlined, concise proof of these results is written up in [Sko01, Chapter 7].

Zariski density of k-rational points. A k-variety X is said to be unirational if there exists a dominant rational map $\mathbb{P}^m_k \dashrightarrow X$ for some positive integer m. The following theorem, a proof of which can be found in [Man74, Theorems 29.4 and 30.1], shows that k-points are Zariski dense for a large class of del Pezzo surfaces.

Theorem 1.4.5 (Segre-Manin)**.** *Let X be a del Pezzo surface of degree d over a field k of characteristic zero. Assume that $X(k) \neq \emptyset$, and if $d = 2$ then assume further that X has a k-rational point that does not lie on any exceptional curve of X. Then X is unirational; in particular, $X(k)$ is Zariski dense in X. Furthermore, if $d \geq 5$ then X is k-birational to $\mathbb{P}^2_k$.* □

Remark 1.4.6. If k has positive characteristic then X is still unirational provided that either

1. k contains more than 22 elements and X has degree at least 4, or that

2. k contains more than 34 elements and X has degree at least 3.

See [Man74, Theorem 30.1].

There is no proven analogous result to Theorem 1.4.5 for rational conic bundles over general fields. However, over a local field k, a rational conic bundle with a k-point is unirational, whence k-rational points are Zariski dense; see [Isk67] for a proof in the case when $k = \mathbb{R}$ and [Yan85] for nonarchimedean k.

The problem of unirationality for low degree nice geometrically rational surfaces remains wide open, and according to Manin and Tsfasman, it is "extremely difficult" ([MT86, p. 64])

Weak approximation. By Lemma 1.2.1 and Theorem 1.4.5, it follows that a locally soluble del Pezzo surface X of degree at least 5 satisfies weak approximation (note that $X(k) \neq \emptyset$ because $X(\mathbb{A}_k) \neq \emptyset$ and X satisfies the Hasse principle). On the other hand, there are examples of del Pezzo surfaces of degrees 2, 3 and 4, for which weak approximation fails, even when k-rational points are Zariski dense, as follows.

Example 1.4.7 ([KT08, Example 2])**.** The surface X given by

$$w^2 = -2x^4 - y^4 + 18z^4$$

in the weighted projective space $\mathbb{P}_{\mathbb{Q}}(1,1,1,2)$ is a del Pezzo surface of degree 2. The $\mathbb{Q}$-point $[x : y : z : w] = [1/2 : 0 : 1/2 : 1]$ on X is not on any exceptional curve; by Theorem 1.4.5, $X(\mathbb{Q})$ is Zariski dense. However, X does not satisfy weak approximation.

Example 1.4.8 ([SD62])**.** The cubic surface X in $\mathbb{P}^3_{\mathbb{Q}}$ given by

$$w(x^2 + y^2) = (4z - 7w)(z^2 - 2w^2)$$

is a del Pezzo surface of degree 3 that does not satisfy weak approximation. Theorem 1.4.5, however, shows $X(\mathbb{Q})$ is Zariski dense (note that $X(\mathbb{Q}) \neq \emptyset$; for example, $[x : y : z : w] = [1 : 1 : 0 : 0] \in X(\mathbb{Q})$).

Example 1.4.9 ([CTSSD87b, Example 15.5])**.** The variety in $\mathbb{P}^4_{\mathbb{Q}}$ defined by the equations

$$uv = x^2 + y^2$$
$$(4u - 3v)(4u - v) = x^2 + z^2$$

is a del Pezzo surface of degree 4 that does not satisfy weak approximation. Theorem 1.4.5, however, shows $X(\mathbb{Q})$ is Zariski dense (note that $X(\mathbb{Q}) \neq \emptyset$; for example, $[u : v : x : y : z] = [1 : 4 : 0 : 2 : 0] \in X(\mathbb{Q})$).

The state of the art results regarding weak approximation on rational conic bundles were already mentioned in our survey on "Existence of a smooth k-point" above. Example 1.4.8 can be used to construct a Châtelet surface with a rational point that does not satisfy weak approximation, namely, the Châtelet surface over $\mathbb{Q}$ given by

$$y^2 + z^2 = (4x - 7)(x^2 - 2).$$

Table 1.1 encapsulates the results we have hitherto presented for del Pezzo surfaces over number fields. A check mark (✓) in the first two rows indicates that the relevant arithmetic phenomenon holds for the indicated class of surfaces. A check mark in the Zariski density row means that if there is a rational point, then rational points are Zariski dense; the dagger (†) in the degree 2 case is there to remind the reader of the (presumably extraneous) hypothesis of Theorem 1.4.5 on these surfaces. An entry with a reference indicates the existence of a counterexample to the arithmetic phenomenon which can be found in the paper cited.

Phenomenon	$d \geq 5$	$d = 4$	$d = 3$	$d = 2$	$d = 1$
Hasse principle	✓	[BSD75]	[CG66]	[KT04]	✓
Weal approximation	✓	[CTSSD87b]	[SD62]	[KT08]	?
Zariski density	✓	✓	✓	✓†	?

Table 1.1: Arithmetic phenomena on del Pezzo surfaces over number fields.

1.5 Del Pezzo surfaces of degree 1: Main results

Let X be a del Pezzo surface of degree 1 over a number field k. Bearing in mind the results of §1.4, especially Table 1.1, we ask the following natural questions:

1. Are k-rational points dense in X for the Zariski topology?
2. Is there an X which is a (minimal) counterexample to weak approximation? If so, can we write down an explicit example?

There are, of course, "artificial" examples of del Pezzo surfaces of degree 1 that do not satisfy weak approximation. Take, for instance, the surface in Example 1.4.7, and blow up the point $[1/2 : 0 : 1/2 : 1]$ on it. By Lemma 1.2.1, the resulting surface is a del Pezzo surface of degree 1 that does not satisfy weak approximation. To avoid such examples, we will insist that our surfaces be k-minimal. Del Pezzo surfaces X with $\operatorname{Pic} X \cong \mathbb{Z}$ are minimal. The converse is true if $d \notin \{1, 2, 4\}$; see [Man74, Rem. 28.1.1].

To state the main results contained in this thesis we fix the following notation. Let $k[x, y, z, w]$ be the weighted graded ring where the variables x, y, z, w have weights $1, 1, 2, 3$, respectively. Set $\mathbb{P}_k(1, 1, 2, 3) := \operatorname{Proj} k[x, y, z, w]$. Let $I \subseteq k[x, y, z, w]$ be a homogeneous ideal. Then $V(I) := \operatorname{Proj} k[x, y, z, w]/I$. If $I = (f_1, \cdots f_n)$ we write $V(f_1, \ldots, f_n)$ instead of $V((f_1, \ldots, f_n))$.

Every del Pezzo surface of degree 1 over k is isomorphic to a smooth sextic hypersurface in $\mathbb{P}_k(1, 1, 2, 3)$. Conversely, any smooth sextic in $\mathbb{P}_k(1, 1, 2, 3)$ is a del Pezzo surface of degree 1 over k; see §2.2.3.

1.5.1 Zariski density of rational points

Definition 1.5.1. Let $F(x, y) \in \mathbb{Z}[x, y]$ be a homogeneous binary form, not divisible by a square of a nonunit in $\mathbb{Z}[x, y]$. We say that F has a fixed prime divisor if there is a prime number p such that $F(x, y) \in p\mathbb{Z}$ for all $x, y \in \mathbb{Z}$.

Remark 1.5.2. If $F(x, y) \in \mathbb{Z}[x, y]$ is a homogeneous binary form with content 1, then $F \bmod p$ has at most $\deg F$ zeroes in $\mathbb{P}^1(\mathbb{F}_p)$. Hence, if p is a fixed prime divisor of F, then $p + 1 \leq \deg(F)$.

Theorem 1.5.3. *Let $F(x, y) \in \mathbb{Z}[x, y]$ be a homogeneous binary form of degree 6. Let X be the del Pezzo surface of degree 1 over $\mathbb{Q}$ given by*

$$w^2 = z^3 + F(x, y) \tag{1.3}$$

in $\mathbb{P}_{\mathbb{Q}}(1, 1, 2, 3)$. Applying a linear transformation, we may assume that the coefficients of x^6 and y^6 are nonzero, without so changing the isomorphism class of X. Let c be the content of F and write $F(x, y) = cF_1(x, y)$ for some $F_1(x, y) \in \mathbb{Z}[x, y]$. Suppose that F_1 has no fixed prime divisors and that $F_1 = \prod_i f_i$, where the $f_i \in \mathbb{Z}[x, y]$ are irreducible homogeneous forms. Assume further that

$$\mu_3 \not\subseteq \mathbb{Q}[t]/f_i(t, 1) \qquad \textit{for some } i, \tag{1.4}$$

where μ_3 *is the group of third roots of unity. Finally, assume that Tate-Shafarevich groups of elliptic curves over* $\mathbb{Q}$ *with* j*-invariant* 0 *are finite. Then the rational points of* X *are dense for the Zariski topology.*

Theorem 1.5.4. *Let* $G[x, y] \in \mathbb{Z}[x, y]$ *be a homogeneous binary form of degree* 4*. Let* X *be the del Pezzo surface of degree* 1 *over* $\mathbb{Q}$ *given by*

$$w^2 = z^3 + G(x, y)z \tag{1.5}$$

in $\mathbb{P}_{\mathbb{Q}}(1, 1, 2, 3)$*. Applying a linear transformation, we may assume that the coefficients of* x^4 *and* y^4 *are nonzero, without so changing the isomorphism class of* X*. Let* c *be the content of* G *and write* $G(x, y) = cG_1(x, y)$ *for some* $G_1(x, y) \in \mathbb{Z}[x, y]$*. Suppose that* G_1 *has no fixed prime divisors and that* $G_1 = \prod_i g_i$*, where the* $g_i \in \mathbb{Z}[x, y]$ *are irreducible homogeneous forms. Assume further that*

$$\mu_4 \not\subseteq \mathbb{Q}[t]/g_i(t, 1) \qquad \textit{for some } i, \tag{1.6}$$

where μ_4 *is the group of fourth roots of unity. Finally, assume that Tate-Shafarevich groups of elliptic curves over* $\mathbb{Q}$ *with* j*-invariant* 1728 *are finite. Then the rational points of* X *are dense for the Zariski topology.*

The idea of the proof of Theorems 1.5.3 and 1.5.4 is as follows. Blowing-up the canonical point of a del Pezzo surface of degree 1 gives an elliptic surface $f\colon \mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$. Assuming finiteness of Tate-Shafarevich groups, Nekovář, Dokchitser and Dokchitser have shown that the root number of an elliptic curve $E/\mathbb{Q}$ is $(-1)^{\mathrm{rank}(E)}$ (the parity conjecture; see [Nek01, DD07]). It thus suffices to show that there are infinitely many fibers of f over $\mathbb{Q}$ with negative root number (i.e., odd Mordell-Weil rank). In [Roh93], Rohrlich pioneered the study of variations of root numbers on algebraic families of elliptic curves. We use his formulas for local root numbers, together with those of Halberstadt and Rizzo [Hal98, Riz03] to compute root numbers of elliptic curves associated to the del Pezzo surfaces of degree 1 of Theorems 1.5.3 and 1.5.4. We then modify a sieve of Gouvêa, Mazur, and Greaves [GM91, Gre92] to search for infinitely many pairs of fibers with *opposite* root numbers. This gives infinitely many fibers with odd rank, which proves the theorem.

The idea of studying density of rational points on an elliptic surface by looking at variations in the root numbers of fibers is not new; the novelty in our approach lies in the combination of sieving techniques from analytic number theory with explicit formulas for root numbers. The reader is especially invited to look at [GM97] where the question of *potential density* of rational points, i.e., Zariski density after a finite extension of the ground field, is studied for elliptic surfaces with non-constant j-invariant. In contrast, the elliptic surfaces we study in this thesis are all isotrivial.

We obtain the following corollary to Theorem 1.5.3, which addresses the question of Zariski density of rational points for "diagonal" del Pezzo surface of degree 1 over $\mathbb{Q}$.

Corollary 1.5.5. *Let X be the del Pezzo surface of degree 1 over $\mathbb{Q}$ given as a sextic in the weighted projective space $\mathbb{P}_{\mathbb{Q}}(1,1,2,3)$ by*

$$w^2 = z^3 + Ax^6 + By^6, \tag{1.7}$$

where A and B are nonzero integers. Assume that Tate-Shafarevich groups of elliptic curves over $\mathbb{Q}$ with j-invariant 0 are finite. If $3A/B$ is not a rational square, or if A and B are relatively prime and $9 \nmid AB$, then the rational points of X are Zariski dense.

Remark 1.5.6. The restriction in (1.7) that A and B are integers is not severe. If A and B are rational numbers, then one can clear denominators and rescale the variables to obtain an equation of the the form (1.7). A similar comment applies for the restriction that $F(x,y) \in \mathbb{Z}[x,y]$ in Theorem 1.5.3 and that $G(x,y) \in \mathbb{Z}[x,y]$ in Theorem 1.5.4.

Using our sieving technique, we will also show that the surfaces of Theorems 1.5.3 and 1.5.4 satisfy a variant of weak-weak approximation. We refer the reader to §3.5 for details.

1.5.2 Weak approximation

We construct the following counterexamples to weak approximation.

Theorem 1.5.7. *Let $p \geq 5$ be a rational prime number such that $p \not\equiv 1 \bmod 12$. Let X be the del Pezzo surface of degree 1 over $\mathbb{Q}$ given by*

$$w^2 = z^3 + p^3x^6 + p^3y^6$$

in $\mathbb{P}_{\mathbb{Q}}(1,1,2,3)$. Then X is $\mathbb{Q}$-minimal and there is a Brauer-Manin obstruction to weak approximation on X. Moreover, the obstruction arises from a cyclic algebra class in $\operatorname{Br} X/\operatorname{Br}\mathbb{Q}$.

Remark 1.5.8. By Corollary 1.5.5, the above counterexamples to weak approximation have a Zariski dense set of points, at least under the assumption that Tate-Shafarevich groups of elliptic curves are finite.

To prove Theorem 1.5.7, we begin with an explicit study of the geometry of "diagonal" del Pezzo surfaces of degree 1 over an arbitrary field k with $\operatorname{char} k \neq 2, 3$. These are sextic surfaces of the form

$$w^2 = z^3 + Ax^6 + By^6 \tag{1.8}$$

in the weighted projective space $\mathbb{P}_k(1,1,2,3)$, where $A, B \in k^*$. The conditions A, $B \in k^*$ and $\operatorname{char} k \neq 2, 3$, taken together, are equivalent to the smoothness of these surfaces. We start by finding an explicit description of generators for the geometric Picard group for the surfaces (1.8). More generally, we find explicit equations for all 240 exceptional curves on *any* del Pezzo surface of degree 1 over any field.

Theorem 1.5.9. *Let X be a del Pezzo surface of degree 1 over a field k, given as a smooth sextic hypersurface $V(f(x,y,z,w))$ in $\mathbb{P}_k(1,1,2,3)$. Let*

$$\Gamma = V(z - Q(x,y), w - C(x,y)) \subseteq \mathbb{P}_{k^s}(1,1,2,3),$$

where $Q(x,y)$ and $C(x,y)$ are homogenous forms of degrees 2 and 3, respectively, in $k^s[x,y]$. If Γ is a divisor on X_{k^s}, then it is an exceptional curve of X. Conversely, every exceptional curve on X is a divisor of this form.

With explicit generators for $\operatorname{Pic} X_{k^s}$, we may compute the cohomology group $\mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big)$, which is a k-birational invariant of X; see [Man74, Theorem 23.3]. We derive the following theorem, analogous to [KT04, Thm. 1].

Theorem 1.5.10. *Let k be a field with $\operatorname{char} k \neq 2, 3$. Let X be a minimal del Pezzo surface of degree 1 over k of the form (1.8). Then $\mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic}(X_{k^s})\big)$ is isomorphic to one of the following fourteen groups:*

$$\{1\}; \quad (\mathbb{Z}/2\mathbb{Z})^i, \quad i \in \{1,2,3,4,6,8\}; \quad (\mathbb{Z}/3\mathbb{Z})^j, \quad j \in \{1,2,3,4\};$$
$$(\mathbb{Z}/6\mathbb{Z})^k \quad k \in \{1,2\}; \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$

Each group occurs for some field k. When $k = \mathbb{Q}$ only the following seven groups occur:

$$\{1\}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$
$$\mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/6\mathbb{Z}.$$

Remark 1.5.11. In [Cor07, Theorem 4.1], Patrick Corn determines all the possible groups that $\mathrm{H}^1\big(\operatorname{Gal} k^s/k, \operatorname{Pic} X_{k^s}\big)$ can be isomorphic to, for del Pezzo surfaces of degree 1. The advantage of our work is that we can compute this cohomological invariant as a function of A and B for surfaces of the form (1.8).

If, furthermore, k is a global field, then we may compute the group $\operatorname{Br} X/\operatorname{Br} k$, of arithmetic interest, via the isomorphism

$$\operatorname{Br} X/\operatorname{Br} k \xrightarrow{\sim} \mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big), \tag{1.9}$$

obtained from the Hochschild-Serre spectral sequence; see §2.3.1 for the definition of $\operatorname{Br} X$ and §2.3.4 for details on the Hochschild-Serre spectral sequence and the isomorphism (1.9).

To prove a statement like Theorem 1.5.7, we have to identify elements of $\operatorname{Br} X/\operatorname{Br} k$ explicitly. Given a cohomology class in $\mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big)$, it can be difficult to identify the corresponding element in $\operatorname{Br} X/\operatorname{Br} k$ guaranteed by the isomorphism (1.9). We present a simple strategy to search for cohomology classes in $\mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big)$ which correspond to *cyclic algebras* in the image of the natural map

$$\operatorname{Br} X/\operatorname{Br} k \to \operatorname{Br} k(X)/\operatorname{Br} k,$$

where X is a locally soluble smooth geometrically integral variety over a global field k. We hope that Theorem 4.4.3 will be of use to others wishing to calculate Brauer-Manin obstructions to the Hasse principle and weak approximation via cyclic algebras on this wide class of varieties.

Chapter 2

Background material

In this chapter we review some standard material from the theory of del Pezzo surfaces and Brauer-Manin obstructions. We begin by outlining results of Coombes which allow us to work with imperfect base fields.

The reader who is mainly interested in Chapter 3 is encouraged to skip §2.3; the material in that section is relevant only for the results of Chapter 4.

2.1 Del Pezzo surfaces are separably split

Throughout this section, k denotes a separably closed field and $\bar{k}$ a fixed algebraic closure of k. Recall that a collection of closed points in $\mathbb{P}^2(k)$ is said to be in general position if no 3 points lie on a line, no 6 points lie on a conic, and no 8 points lie on a singular cubic, with one of the points at the singularity. Our goal is to prove the following strengthening of [Man74, Theorem 24.4].

Theorem 2.1.1. *Let X be a del Pezzo surface of degree d over k. Then either X is isomorphic to the blow-up of $\mathbb{P}^2_k$ at $9-d$ points in general position in $\mathbb{P}^2(k)$, or $d=8$ and X is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$.*

We need two results of Coombes, as follows.

Proposition 2.1.2 ([Coo88, Proposition 5])**.** *Let $f\colon X \to Y$ be a birational morphism of smooth projective surfaces over k. Then f factors as*

$$X = X_0 \to X_1 \to \cdots \to X_r = Y,$$

where each map $X_i \to X_{i+1}$ is a blow-up at a closed k-point of X_{i+1}. □

The above proposition is well-known if we replace k with $\bar{k}$. The main step in the proof of Proposition 2.1.2 is to show that the blow-up at a closed point whose residue field is a nontrivial purely inseparable extension of k cannot give rise to a *smooth* surface. Using Iskovskikh's classification theorem (Theorem 1.2.3), Coombes deduces the following proposition.

Proposition 2.1.3 ([Coo88, Proposition 7]). *The minimal smooth projective rational surfaces over k are $\mathbb{P}^2_k$ and the Hirzebruch surfaces $\mathbb{F}_n := \mathbb{P}\big(\mathscr{O}_{\mathbb{P}^1_k} \oplus \mathscr{O}_{\mathbb{P}^1_k}(n)\big)$, where either $n = 0$ or $n \geq 2$.* □

Finally, we need the following lemma.

Lemma 2.1.4 ([Man74, Theorem 24.3(ii)]). *Let X be a del Pezzo surface over an algebraically closed field. Then every irreducible curve with negative self-intersection in exceptional.* □

Proof of Theorem 2.1.1. Let $f\colon X \to Y$ be a birational k-morphism with Y minimal, and write

$$X = X_0 \to X_1 \to \cdots \to X_r = Y \tag{2.1}$$

for a factorization of f as in Proposition 2.1.2. By Proposition 2.1.3 we need only consider the following cases:

1. $Y = \mathbb{P}^2_k$. We claim that no point that is blown-up in one step of the factorization (2.1) may lie on the exceptional divisor of a previous blow-up: otherwise $X_{\bar{k}}$ would contain a curve with self-intersection less than -1, contradicting Lemma 2.1.4. Hence X is the blow-up of $\mathbb{P}^2_k$ at r distinct closed k-points. We conclude that $d = K_X^2 = 9 - r$, as claimed. Suppose that 3 of these points lie on a line L. Let $f_{\bar{k}}^{-1}L_{\bar{k}}$ denote the strict transform of $L_{\bar{k}}$ for the base-extension $f_{\bar{k}}\colon X_{\bar{k}} \to Y_{\bar{k}}$. Then $(f_{\bar{k}}^{-1}L_{\bar{k}}, f_{\bar{k}}^{-1}L_{\bar{k}}) < -1$, but this is impossible by Lemma 2.1.4. Similarly, if 6 of the blown-up points lie on a conic Q, or if 8 points lie on a singular cubic C with one of the points at the singularity, then $(f_{\bar{k}}^{-1}Q_{\bar{k}}, f_{\bar{k}}^{-1}Q_{\bar{k}}) < -1$, or $(f_{\bar{k}}^{-1}C_{\bar{k}}, f_{\bar{k}}^{-1}C_{\bar{k}}) < -1$, respectively, which is not possible. Hence the blown-up points are in general position.

2. $Y = \mathbb{P}^1_k \times \mathbb{P}^1_k$. If $X = Y$ then X is a del Pezzo surface of degree 8. Otherwise, we may contract the two nonintersecting (-1)-curves of X_{r-1} and obtain a birational

morphism $\phi\colon X_{r-1} \to \mathbb{P}^2_k$. We may use the map ϕ to construct a new birational morphism $X \to \mathbb{P}^2_k$, given by

$$X = X_0 \to X_1 \to \cdots \to X_{r-1} \xrightarrow{\phi} \mathbb{P}^2_k,$$

and thus we may reduce this case to the previous case.

3. $Y = \mathbb{F}_n, n \geq 2$. There is a curve $C \subseteq (\mathbb{F}_n)_{\bar{k}}$ whose divisor class satisfies $(C, C) < -1$. Let $f_{\bar{k}}^{-1}(C)$ denote the strict transform of C in $X_{\bar{k}}$ for the base-extension $f_{\bar{k}}\colon X_{\bar{k}} \to (\mathbb{F}_n)_{\bar{k}}$. Then $(f_{\bar{k}}^{-1}C, f_{\bar{k}}^{-1}C) < -1$, but this is impossible by Lemma 2.1.4. $\square$

2.2 Further properties of del Pezzo surfaces

We review some well known facts about del Pezzo surfaces over a field k, beyond those stated in §1.3 and in the previous section. The basic references on the subject are [Man74], [Dem80] and [Kol96, III.3].

2.2.1 The Picard group

Let X be a del Pezzo surface over a field k of degree d. Recall that an exceptional curve on X is an irreducible curve C on $X_{\bar{k}}$ such that $(C, C) = (C, K_X) = -1$. Theorem 2.1.1 shows that exceptional curves on X are already defined over k^s. The number of exceptional curves on X varies with d as shown in Table 2.1.

We have seen that if $X_{k^s} \not\cong \mathbb{P}^1_{k^s} \times \mathbb{P}^1_{k^s}$ then X_{k^s} is isomorphic to a blow-up of $\mathbb{P}^2_{k^s}$ at $r := 9 - d$ closed points $\{P_1 \dots, P_r\}$ in general position. It follows that the group $\operatorname{Pic} X_{k^s}$ is isomorphic to $\mathbb{Z}^{10-d}$ (see [Har77, Proposition V.3.2]); if $d \leq 7$ then it is generated by the classes of exceptional curves. Let e_i be the class of an exceptional curve corresponding to P_i under the blow-up map, and let ℓ be the class of the pullback of a line in $\mathbb{P}^2_{k^s}$ not passing through any of the P_i. Then $\{e_1, \dots, e_r, \ell\}$ is a basis for $\operatorname{Pic} X_{k^s}$. Note that

$$(e_i, e_j) = -\delta_{ij}, \quad (e_i, \ell) = 0, \quad (\ell, \ell) = 1,$$

where δ_{ij} is the usual Kronecker delta function. With respect to this basis, the anticanonical class is given by $-K_X = 3\ell - \sum e_i$.

$d(X)$	7	6	5	4	3	2	1
# of exceptional curves	3	6	10	16	27	56	240

Table 2.1: Number of exceptional curves on a del Pezzo surface X

2.2.2 Galois action on the Picard group

Let X be a smooth projective geometrically rational surface over a global field k. The Galois group $\mathrm{Gal}(k^s/k)$ acts on $\mathrm{Pic}\, X_{k^s}$ as follows. For $\sigma \in \mathrm{Gal}(k^s/k)$, let $\tilde{\sigma}\colon \mathrm{Spec}\, k^s \to \mathrm{Spec}\, k^s$ be the corresponding morphism. Then $\mathrm{id}_X \times \tilde{\sigma} \in \mathrm{Aut}\, X_{k^s}$ induces an automorphism $(\mathrm{id}_X \times \tilde{\sigma})^*$ of $\mathrm{Pic}\, X_{k^s}$. This gives a group homomorphism

$$\mathrm{Gal}(k^s/k) \to \mathrm{Aut}(\mathrm{Pic}\, X_{k^s}) \qquad \sigma \mapsto (\mathrm{id}_X \times \tilde{\sigma})^*.$$

The action of $\mathrm{Gal}(k^s/k)$ on $\mathrm{Pic}(X_{k^s})$ fixes the canonical class K_X and preserves the intersection pairing; see [Man74, Theorem 23.8].

Let K be the smallest extension of k in k^s over which all exceptional curves of X are defined. We say that K is the splitting field of X. The natural action of $\mathrm{Gal}(k^s/k)$ on $\mathrm{Pic}\, X_{k^s} \cong \mathrm{Pic}\, X_K$ factors through the quotient $\mathrm{Gal}(K/k)$, giving a homomorphism

$$\phi_X\colon \mathrm{Gal}(K/k) \to \mathrm{Aut}(\mathrm{Pic}\, X_K). \tag{2.2}$$

If we have equations for an exceptional curve C of X, then an element $\sigma \in \mathrm{Gal}(K/k)$ acts on C by applying σ to each coefficient. The curve ${}^{\sigma}C$ is itself an exceptional curve of X.

If, furthermore, X is a del Pezzo surface of degree 1, then the image of ϕ_X is isomorphic to a subgroup of the Weyl group $W(E_8)$ (which is a finite group of order $696,729,600$); see [Man74, Theorem 23.9]. To keep computations reasonable in Chapter 4 when searching for counterexamples to weak approximation, we work with surfaces X for which $\mathrm{im}\, \phi_X$ is small. On the other hand, the image cannot be too small: for example, if $\mathrm{im}\, \phi_X = \{1\}$, then X is k-birational to $\mathbb{P}^2_k$, so it satisfies weak approximation, by Lemma 1.2.1.

2.2.3 Anticanonical models

For any scheme X and line sheaf $\mathscr{L}$ on X, we may construct the graded ring

$$R(X, \mathscr{L}) := \bigoplus_{m \geq 0} H^0(X, \mathscr{L}^{\otimes m}).$$

When $\mathscr{L} = \omega_X^{\otimes -1}$, we call $R(X, \omega_X^{\otimes -1})$ the **anticanonical ring** of X. If X is a del Pezzo surface then X is isomorphic to the scheme $\operatorname{Proj} R(X, \omega_X^{\otimes -1})$ [Kol96, Theorem III.3.5]. This scheme is known as the **anticanonical model** of the del Pezzo surface.

The construction of anticanonical models is reminiscent of the procedure that yields a Weierstrass model of an elliptic curve. In fact, we can use the Riemann-Roch theorem for surfaces to prove the following dimension formula for a del Pezzo surface X over k of degree d:

$$h^0\big(X, -mK_X\big) = \frac{m(m+1)}{2}d + 1; \tag{2.3}$$

see [Kol96, Corollary III.3.2.5] or [CO99]. If X has degree 1, then the anticanonical model for X is a smooth sextic hypersurface in $\mathbb{P}_k(1,1,2,3)$, and we may compute such a model, up to isomorphism, as follows:

1. Choose a basis $\{x, y\}$ for the 2-dimensional k-vector space $\mathrm{H}^0\big(X, -K_X\big)$.

2. The elements x^2, xy, y^2 of $\mathrm{H}^0\big(X, -2K_X\big)$ are linearly independent. However, $h^0\big(X, -2K_X\big) = 4$; choose an element z to get a basis $\{x^2, xy, y^2, z\}$ for this k-vector space.

3. The elements $x^3, x^2y, xy^2, y^3, xz, yz$ of $\mathrm{H}^0\big(X, -3K_X\big)$ are linearly independent, but $h^0\big(X, -3K_X\big) = 7$. Choose an element w to get a basis $\{x^3, x^2y, xy^2, y^3, xz, yz, w\}$ for this k-vector space.

4. The vector space $\mathrm{H}^0\big(X, -6K_X\big)$ is 22-dimensional, so the 23 elements

$$\{x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, x^4z, x^3yz, x^2y^2z, xy^3z,$$
$$y^4z, x^2z^2, xyz^2, y^2z^2, z^3, x^3w, x^2yw, xy^2w, y^3w, xzw, yzw, w^2\}$$

must be k-linearly dependent. Let $f(x, y, z, w) = 0$ be a linear dependence relation among these elements. Then an anticanonical model of X is $\operatorname{Proj} k[x, y, z, w]/(f)$, where x, y, z, w are variables with weights $1, 1, 2$ and 3 respectively. This way X may be described as the (smooth) sextic hypersurface $V(f)$ in $\mathbb{P}_k(1,1,2,3)$.

For more details on this construction, see [CO99, pp.1199–1201].

Remark 2.2.1. If k is a field of characteristic not equal to 2 or 3, then in step (4) above we may complete the square with respect to the variable w and the cube with respect to the variable z to obtain an equation $f(x, y, z, w) = 0$ involving only the monomials

$$\{x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, x^4z, x^3yz, x^2y^2z, xy^3z, y^4z, z^3, w^2\}.$$

Moreover, we may also rescale the variables so that the coefficients of w^2 and z^3 are ± 1.

Remark 2.2.2. If X has degree $d \geq 3$, then the anticanonical model recovers the usual description of X as a smooth degree d surface in $\mathbb{P}^d_k$. In particular, when $d = 3$ we get a smooth cubic surface in $\mathbb{P}^3_k$. If X has degree 2 then the anticanonical model is a quartic hypersurface in the weighted projective space $\mathbb{P}_k(1, 1, 1, 2)$; such a surface can then be thought of as a double cover of a $\mathbb{P}^2_k$ ramified along a quartic curve.

Remark 2.2.3. If we write a del Pezzo surface X of degree 1 over a field k as the smooth sextic hypersurface $V(f(x, y, z, w))$ in $\mathbb{P}_k(1, 1, 2, 3)$, then $\{x, y\}$ is a basis for $\mathrm{H}^0\big(X, -K_X\big)$ and $\{x^2, xy, y^2, z\}$ is a basis for $\mathrm{H}^0\big(X, -2K_X\big)$. In particular, the base point of $|-K_X|$ is $[0 : 0 : 1 : 1]$.

2.2.4 Del Pezzo surfaces of degree 1 and elliptic surfaces

Let X be a del Pezzo surface of degree 1 over a field k. Recall that the anticanonical linear system $|-K_X|$ contains a single (k-rational) base-point (see §1.4); we call this point the anticanonical point of X. By (2.3) we have $h^0(X, -K_X) = 2$, and thus the linear system $|-K_X|$ gives rise to a rational map $f \colon X \dashrightarrow \mathbb{P}^1_k$; this map is regular everywhere except at the anticanonical point O. Blowing-up O to resolve the indeterminacy of f we obtain a commutative diagram

$$\begin{tikzcd} & \mathcal{E} \arrow[dl] \arrow[d, "\rho"] \\ X \arrow[r, dashed, "f"] & \mathbb{P}^1_k \end{tikzcd}$$

Almost all of the fibers of ρ are nonsingular genus 1 curves. The morphism ρ restricts to an isomorphism between the exceptional divisor of $\mathcal{E}$ and $\mathbb{P}^1_k$. This gives a distinguished section $\mathscr{O} \colon \mathbb{P}^1_k \to \mathcal{E}$ of ρ, making $(\rho, \mathscr{O})$ into an elliptic surface.

Concretely, if X is given by a smooth sextic

$$w^2 = z^3 + F(x, y)z^2 + G(x, y)z + H(x, y)$$

in $\mathbb{P}_k(1, 1, 2, 3)$, then $O = [0 : 0 : 1 : 1]$; see Remark 2.2.3. In this case, $\mathcal{E}$ is the subscheme of $\mathbb{P}_k(1, 1, 2, 3) \times \mathbb{P}^1_k = \mathrm{Proj}(k[x, y, z, w]) \times \mathrm{Proj}(k[m, n])$ cut out by the equations

$$w^2 = z^3 + F(x, y)z^2 + G(x, y)z + H(x, y) \quad \text{and} \quad nx - my = 0. \tag{2.4}$$

The map $\rho\colon \mathcal{E} \to \mathbb{P}^1_k$ is then given by $([x:y:z:w],[m:n]) \mapsto [m:n]$. Note that for points away from the exceptional divisor we have $[m:n]=[x:y]$.

Let t be the rational function m/n, so that $x = ty$ on $\mathcal{E}$. The generic fiber $E/k(t)$ of ρ is the curve

$$E\colon w^2 = z^3 + y^2F(t,1)z^2 + y^4G(t,1)z + y^6H(t,1) \tag{2.5}$$

in $\mathrm{Proj}(k(t)[y,z,w])$. On the *affine* chart $\mathrm{Spec}(k(t)[z/y^2, w/y^3])$ of this weighted ambient space, the curve (2.5) is isomorphic to the affine curve

$$(w/y^3)^2 = (z/y^2)^3 + F(t,1)(z/y^2)^2 + G(t,1)(z/y^2) + H(t,1).$$

Relabelling the variables, we find that the elliptic curve $E/k(t)$ is given by the Weierstrass model

$$y^2 = x^3 + F(t,1)x^2 + G(t,1)x + H(t,1).$$

Similarly, we can also check that the fiber of ρ above $[m:n] \in \mathbb{P}^2_k(k)$ is isomorphic to the curve in $\mathbb{P}^2_k$ with affine equation given by

$$y^2 = x^3 + F(m,n)x^2 + G(m,n)x + H(m,n).$$

2.3 Brauer-Manin obstructions

Let X be a nice variety over a global field k. We have seen that the inclusion $X(k) \subseteq X(\mathbb{A}_k)$ gives a necessary condition for the existence of a k-rational point on X, namely, $X(\mathbb{A}_k) \neq \emptyset$. This condition is relatively easy to check in practice. For example, the Lang-Weil bounds, together with Hensel's lemma ensure the existence of k_v-points for all but finitely many $v \in \Omega_k$. We have also seen that the condition $X(\mathbb{A}_k) \neq \emptyset$ need not guarantee the existence of a k-rational point on X. To explain some counterexamples to the Hasse principle, in [Man71] Manin introduced an obstruction based on a set $X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k)$ containing the closure for the adèlic topology of $X(k)$ in $X(\mathbb{A}_k)$:

$$\overline{X(k)} \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k). \tag{2.6}$$

Definition 2.3.1. Let X be a nice variety over a global field k. We say there is a

- Brauer-Manin obstruction to the Hasse principle if $X(\mathbb{A}_k) \neq \emptyset$ but $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$;
- Brauer-Manin obstruction to weak approximation if $X(\mathbb{A}_k) \setminus X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

2.3.1 The Brauer group of a scheme

The set $X(\mathbb{A}_k)^{\mathrm{Br}}$ is defined using the Brauer group of X, which is in turn defined using either Azumaya algebras or étale cohomology, as follows.

Definition 2.3.2. An Azumaya algebra on a scheme X is an $\mathscr{O}_X$-algebra $\mathcal{A}$ that is coherent and locally free as an $\mathscr{O}_X$-module, such that the fiber $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathscr{O}_{X,x}} k(x)$ is a central simple algebra over the residue field $k(x)$ for each $x \in X$.

Two Azumaya algebras $\mathcal{A}$ and $\mathcal{B}$ on X are similar if there exist locally free coherent $\mathscr{O}_X$-modules $\mathcal{E}$ and $\mathcal{F}$ such that

$$\mathcal{A} \otimes_{\mathscr{O}_X} \mathrm{End}_{\mathscr{O}_X}(\mathcal{E}) \cong \mathcal{B} \otimes_{\mathscr{O}_X} \mathrm{End}_{\mathscr{O}_X}(\mathcal{F}).$$

Definition 2.3.3. The Azumaya Brauer group of a scheme X is the set of similarity classes of Azumaya algebras on X, with multiplication induced by tensor product of sheaves. We denote this group by $\mathrm{Br}_{\mathrm{Az}} X$.

The inverse of $[\mathcal{A}] \in \mathrm{Br}_{\mathrm{Az}} X$ is the class $[\mathcal{A}^{\mathrm{op}}]$ of the opposite algebra of $\mathcal{A}$; the identity element is $[\mathscr{O}_X]$ (see [Gro68a, p. 47]).

Definition 2.3.4. The Brauer group of a scheme X is $\mathrm{Br}\, X := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$.

Remark 2.3.5. If F is a field, then $\mathrm{Br}\,\mathrm{Spec}\, F = \mathrm{Br}\, F$, the usual Brauer group of a field. The Brauer group is a contravariant functor on schemes, with values in the category of abelian groups.

For any scheme X there is a natural inclusion

$$\mathrm{Br}_{\mathrm{Az}} X \hookrightarrow \mathrm{Br}\, X;$$

see [Mil80, Theorem IV.2.5]. The following result of Gabber, a proof of which can be found in [dJ], determines the image of this injection for a scheme with some kind of polarization.

Theorem 2.3.6 (Gabber, de Jong)**.** *If X is a scheme endowed with an ample invertible sheaf then the natural map $\mathrm{Br}_{\mathrm{Az}} X \hookrightarrow \mathrm{Br}\, X$ induces an isomorphism*

$$\mathrm{Br}_{\mathrm{Az}} X \xrightarrow{\sim} (\mathrm{Br}\, X)_{\mathrm{tors}}. \qquad \square$$

If X is an integral scheme with function field $k(X)$, then the inclusion $\operatorname{Spec} k(X) \to X$ gives rise to a map $\operatorname{Br} X \to \operatorname{Br} k(X)$ via functoriality of étale cohomology. If further X is regular and quasi-compact then this induced map is injective; see [Mil80, Example III.2.22]. On the other hand, the group $\operatorname{Br} k(X)$ is torsion, because it is a Galois cohomology group. These two facts imply the following corollary of Theorem 2.3.6.

Corollary 2.3.7. *Let X be a regular quasiprojective variety over a field. Then*

$$\operatorname{Br}_{Az} X \cong \operatorname{Br} X. \qquad \square$$

Finally, we note that the Brauer group of well-behaved low dimensional schemes over a field is a birational invariant.

Theorem 2.3.8 ([Gro68b, Corollaire III.7.5])**.** *Let X be a nice k-variety of dimension at most 2. Then $\operatorname{Br} X$ depends only on the birational class of X.* $\square$

Corollary 2.3.9. *Let X be a nice geometrically rational surface over a field k. Then $\operatorname{Br} X_{k^s} = 0$.*

Proof. This follows directly from Theorem 2.3.8 and the fact that $\operatorname{Br} \mathbb{P}^2_{k^s} = 0$; see [Mil70, p. 305]. $\square$

2.3.2 The Brauer-Manin set

Let X be a nice variety over a global field k. For each $\mathcal{A} \in \operatorname{Br} X$ and each field extension K/k there is a specialization map

$$\operatorname{ev}_{\mathcal{A}} \colon X(K) \to \operatorname{Br} K, \qquad x \mapsto \mathcal{A}_x \otimes_{\mathscr{O}_{X,x}} K.$$

These specialization maps may be put together to construct a pairing

$$\phi : \operatorname{Br} X \times X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}, \qquad (\mathcal{A}, (x_v)) \mapsto \sum_{v \in \Omega_k} \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(x_v)), \tag{2.7}$$

where $\operatorname{inv}_v \colon \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z}$ is the usual invariant map from local class field theory. The sum in (2.7) is in fact finite because for $(x_v) \in X(\mathbb{A}_k)$ we have $\operatorname{ev}_{\mathcal{A}}(x_v) = 0 \in \operatorname{Br} k_v$ for all but finitely many v; see [Sko01, p. 101]. For $\mathcal{A} \in \operatorname{Br} X$ we obtain a commutative diagram

$$\begin{tikzcd}
 & X(k) \arrow[r] \arrow[d, "\operatorname{ev}_{\mathcal{A}}"'] & X(\mathbb{A}_k) \arrow[d, "\operatorname{ev}_{\mathcal{A}}"'] \arrow[dr, "\phi(\mathcal{A},-)"] & & \\
0 \arrow[r] & \operatorname{Br} k \arrow[r] & \bigoplus_v \operatorname{Br} k_v \arrow[r, "\sum_v \operatorname{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd} \tag{2.8}$$

where the bottom row is the usual exact sequence from class field theory.

We are now ready to define the intermediate set in (2.6).

Definition 2.3.10. Let X be a nice variety over a global field k, and let $\mathcal{A} \in \operatorname{Br} X$. Let

$$X(\mathbb{A}_k)^{\mathcal{A}} := \big\{(x_v) \in X(\mathbb{A}_k) : \phi(\mathcal{A}, (x_v)) = 0\big\}.$$

We call

$$X(\mathbb{A}_k)^{\mathrm{Br}} := \bigcap_{\mathcal{A} \in \operatorname{Br} X} X(\mathbb{A})^{\mathcal{A}}$$

the Brauer-Manin set of X.

Remark 2.3.11. The commutativity of the diagram (2.8), together with the fact that the bottom row is a complex, implies that $X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}}$. Moreover, if $\mathbb{Q}/\mathbb{Z}$ is given the discrete topology, then the map $\phi(\mathcal{A}, -)\colon X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ is continuous, so $X(\mathbb{A}_k)^{\mathcal{A}}$ is a closed subset of $X(\mathbb{A}_k)$; see [Har04, §3.1]. This shows that $\overline{X(k)} \subseteq X(\mathbb{A}_k)^{\mathrm{Br}}$.

Remark 2.3.12. The structure map $X \to \operatorname{Spec} k$ gives rise to a map $\operatorname{Br} k \to \operatorname{Br} X$ which is injective if $X(\mathbb{A}_k) \neq \emptyset$; see §2.3.4 below. The exactness of the bottom row of (2.8) then implies that to compute $\bigcap_{\mathcal{A} \in \operatorname{Br} X} X(\mathbb{A}_k)^{\mathcal{A}}$ it is enough to calculate the intersection over a set of representatives for the group $\operatorname{Br} X/\operatorname{Br} k$.

2.3.3 Conjectures of Colliot-Thélène and Sansuc

Hasse principle and weak approximation. All the counterexamples to the Hasse principle and weak approximation in Chapter 1 can be explained by a Brauer-Manin obstruction. In [CTS80, question $\mathbf{k}_1$], Colliot-Thélène and Sansuc ask whether for a smooth projective geometrically rational surface X over a global field k, the Brauer-Manin obstruction to the Hasse principle is the "only one," i.e.,

$$\text{does the implication } X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset \implies X(k) \neq \emptyset \text{ hold?} \tag{2.9}$$

One may ask a similar question for weak approximation:

$$\text{does the equality } \overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}} \text{ hold?} \tag{2.10}$$

Colliot-Thélène and Sansuc had affirmative answers to these questions in mind, based on evidence eventually published in the papers [CTCS80, CTS82]. In the case of weak approximation, equality (2.10) is known to be true for del Pezzo surfaces of degree at least 4 (the case of degree 4 is a hard theorem of Salberger and Skorobogatov [SS91]). Numerical evidence for the case of the Hasse principle on surfaces of degrees 3 and 2 has been gathered in [CTKS87, Cor05].

In [CT03], these questions about the uniqueness of the Brauer-Manin obstruction are generalized and the following far-reaching conjecture is proposed.

Conjecture 2.3.13 (Colliot-Thélène)**.** Let X be a smooth proper geometrically integral variety over a global field k. Suppose that X is geometrically rationally connected. Then the Brauer-Manin obstruction to the Hasse principle and weak approximation for X is the only one.

Not all counterexamples to the Hasse-principle and weak approximation can be explained by a Brauer-Manin obstruction. Skorobogatov gave the first unconditional examples of the insufficiency of this obstruction: he produced a bi-elliptic surface that has no rational points and which nonetheless has a nonempty Brauer-Manin set; see [Sko99]. Harari and Skorobogatov have constructed examples of Enriques surfaces that fail to satisfy weak approximation, but for which the containment $\overline{X(k)} \subseteq X(\mathbb{A}_k)^{\mathrm{Br}}$ is strict; see [Har00, HS05]. Recently, Poonen constructed certain Châtelet surface bundles whose lack of rational points cannot be explained directly by any known cohomologically constructed obstruction; see [Poo08].

Weak-weak approximation. Unirational varieties are expected to satisfy weak-weak approximation.

Conjecture 2.3.14 (Colliot-Thélène [Ser08, p. 30])**.** Let X be a smooth proper geometrically integral variety over a number field k. If X is unirational then it satisfies weak-weak approximation.

Colliot-Thélène and Ekedahl have shown that if Conjecture 2.3.14 holds then the inverse Galois problem could be solved over $\mathbb{Q}$, i.e., every finite group is a Galois group over $\mathbb{Q}$; see [Ser08, Theorem 3.5.9].

2.3.4 The Hochschild-Serre spectral sequence in étale cohomology

Let X be a nice locally soluble variety over a global field k. By Remark 2.3.12, to compute $X(\mathbb{A}_k)^{\mathrm{Br}}$ it suffices to compute the intersection of $X(\mathbb{A})^{\mathcal{A}}$ over a set of representatives for the group $\operatorname{Br} X/\operatorname{Br} k$. If $\operatorname{Br} X_{k^s} = 0$, then the Hochschild-Serre spectral sequence in étale cohomology provides a tool for computing the group $\operatorname{Br} X/\operatorname{Br} k$.

Let K be a finite Galois extension of k, with Galois group G. The Hochschild-Serre spectral sequence

$$E_2^{p,q} := \mathrm{H}^p\big(G, \mathrm{H}^q_{\text{ét}}(X_K, \mathbb{G}_m)\big) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}\big(X, \mathbb{G}_m\big) =: L^{p+q}$$

gives rise to the usual "low-degree" long exact sequence

$$0 \to E_2^{1,0} \to L^1 \to E_2^{0,1} \to E_2^{2,0} \to \ker\left(L^2 \to E_2^{0,2}\right) \to E_2^{1,1} \to E_2^{3,0}$$

which in our case is

$$\begin{aligned} 0 \to \operatorname{Pic} X \to (\operatorname{Pic} X_K)^G \to \mathrm{H}^2\big(G, K^*\big) \to \ker(\operatorname{Br} X \to \operatorname{Br} X_K) \\ \to \mathrm{H}^1\big(G, \operatorname{Pic} X_K\big) \to \mathrm{H}^3\big(G, K^*\big). \end{aligned} \tag{2.11}$$

Taking the direct limit over all finite Galois extensions of k gives the exact sequence

$$\begin{aligned} 0 \to \operatorname{Pic} X \to (\operatorname{Pic} X_{k^s})^{\operatorname{Gal}(k^s/k)} \to \operatorname{Br} k \to \ker(\operatorname{Br} X \to \operatorname{Br} X_{k^s}) \\ \to \mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big) \to \mathrm{H}^3\big(\operatorname{Gal}(k^s/k), k^{s*}\big). \end{aligned} \tag{2.12}$$

Furthermore, if k is a global field, then $\mathrm{H}^3\big(\operatorname{Gal}(k^s/k), k^{s*}\big) = 0$; this fact is due to Tate—see [NSW08, 8.3.11(iv), 8.3.17].

For each $v \in \Omega_k$, local solubility of X gives a morphism $\operatorname{Spec} k_v \to X$ that splits the base extension $\pi_v \colon X_{k_v} \to \operatorname{Spec} k_v$ of the structure map of X. Thus, by functoriality of the Brauer group, the natural maps $\pi_v^* \colon \operatorname{Br} k_v \to \operatorname{Br} X_{k_v}$ split for every $v \in \Omega_k$. The exactness of the bottom row of (2.8) then shows that the natural map $\operatorname{Br} k \to \operatorname{Br} X$ coming from the structure morphism of X is injective. Moreover, if X is a del Pezzo surface, then $\operatorname{Br} X_{k^s} = 0$ by Corollary 2.3.9 and thus (2.12) gives rise the to short exact sequence

$$0 \to \operatorname{Br} k \to \operatorname{Br} X \to \mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big) \to 0.$$

If K is a splitting field for X then the inflation map

$$\mathrm{H}^1\big(\operatorname{Gal}(K/k), \operatorname{Pic} X_K\big) \to \mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big)$$

is an isomorphism, because the cokernel maps into the first cohomology group of a free $\mathbb{Z}$-module with trivial action by a profinite group, which is trivial. Hence

$$\operatorname{Br} X/\operatorname{Br} k \cong \mathrm{H}^1\big(\operatorname{Gal}(K/k), \operatorname{Pic} X_K\big). \tag{2.13}$$

Finally, we note that since $X(\mathbb{A}_k) \neq \emptyset$, if H is a subgroup of G, then by (2.11) and the injectivity of the map $\operatorname{Br} k \to \operatorname{Br} X$, we know that

$$\operatorname{Pic} X_{K^H} \xrightarrow{\sim} (\operatorname{Pic} X_K)^H, \tag{2.14}$$

where K^H is the fixed field of K by H. It will be important for us in Chapter 4 to make this isomorphism explicit. This is the subject of the next section.

2.3.5 Galois descent of line bundles

To make the isomorphism (2.14) explicit we need the theory of Galois descent of line sheaves, which is a special case of the theory of descent of quasi-coherent sheaves over faithfully flat and quasi-compact morphisms. Good references for Galois descent are [BLR90] and [KT06]. For the general theory of descent see [Gro03].

Let K/k be a finite Galois extension of global fields. For every element $\sigma \in \operatorname{Gal}(K/k)$ let $\tilde{\sigma} : \operatorname{Spec} K \to \operatorname{Spec} K$ denote the corresponding morphism. Let X be a k-scheme, and suppose we are given a line bundle $\widetilde{\mathscr{F}}$ on the K-scheme X_K, together with a collection of isomorphisms[1] $f_\sigma : \widetilde{\mathscr{F}} \to \tilde{\sigma}^* \widetilde{\mathscr{F}}$ such that

$$f_{\tau\sigma} = {}^{\sigma}f_\tau \circ f_\sigma \qquad \text{for all } \sigma, \tau \in \operatorname{Gal}(K/k), \tag{2.15}$$

where ${}^{\sigma}f_\tau := \tilde{\sigma}^* f_\tau$. Then there exists a sheaf $\mathscr{F}$ on X, and an isomorphism $\lambda \colon \mathscr{F}_K \to \widetilde{\mathscr{F}}$ such that $f_\sigma = {}^{\sigma}\lambda \circ \lambda^{-1}$ for all σ. Together, the equalities (2.15) are referred to as the cocycle condition.

If X is a geometrically integral k-scheme, then $\widetilde{\mathscr{F}} = \mathscr{O}_{X_K}(D)$ for some divisor $D \in \operatorname{Div} X_K$, and f_σ can be regarded as a function (up to multiplication by a scalar) whose associated divisor is $D - {}^{\sigma}D$. If $X(k) \neq \emptyset$ then one may use a point in $P \in X(k)$ to normalize the functions so that f_σ acts as the identity in the fiber of $\widetilde{\mathscr{F}}$ at P. We usually do not know if $X(k)$ is empty or not, but in the case of del Pezzo surfaces of degree 1 over k we have the anticanonical point.

To obtain a divisor for the descended line bundle, we take a rational section ξ of $\widetilde{\mathscr{F}}$ and we "average it" over the Galois group G to obtain a rational section of $\mathscr{F}$

$$\mathfrak{s} := \sum_{\sigma \in G} {}^{\sigma^{-1}}(f_\sigma(\xi)).$$

[1] Here we use a slight abuse of notation: we write $\tilde{\sigma}^*$ for the automorphism of X_K induced by the automorphism $\tilde{\sigma}^*$ of $\operatorname{Spec} K$.

Note that it may be necessary to change the choice of ξ to make $\mathfrak{s}$ nonzero. The divisor of zeroes of $\mathfrak{s}$, with respect to local trivializations for $\mathscr{F}$, gives a line bundle isomorphic to the descended line bundle. We often use the rational section $\xi = 1$, and since f_σ acts by multiplication, we obtain $\mathfrak{s} = \sum_{\sigma \in G} {}^{\sigma^{-1}}(f_\sigma)$ in this case.

Chapter 3

Zariski density of rational points on del Pezzo surfaces of degree 1

In this chapter we study the question of Zariski density of $\mathbb{Q}$-rational points for del Pezzo surfaces of degree 1 of the form

$$w^2 = z^3 + F(x, y) \quad \text{or} \quad w^2 = z^3 + G(x, y)z$$

in $\mathbb{P}_{\mathbb{Q}}(1, 1, 2, 3)$, where F and G are homogeneous integral forms of degree 6 and 4, respectively. Our goal is to prove Theorems 1.5.3 and 1.5.4, as well as Corollary 1.5.5. Blowing up the anticanonical point $[0 : 0 : 1 : 1]$ on these surfaces gives elliptic surfaces $\mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$, where the fiber above the point $[m : n] \in \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$ is isomorphic to the plane cubic curve

$$y^2 = x^3 + F(m, n) \quad \text{or} \quad y^2 = x^3 + G(m, n)x, \tag{3.1}$$

respectively; see §2.2.4. This cubic curve is an elliptic curve for all but finitely many $[m : n]$. We investigate the parity of the Mordell-Weil rank of these elliptic curves in an effort to produce many rational points on the original del Pezzo surfaces. Our main tools are a detailed study of the root numbers of the elliptic curves

$$y^2 = x^3 + \alpha \quad \text{and} \quad y^2 = x^3 + \alpha x \qquad (\alpha \neq 0), \tag{3.2}$$

and a "pseudo squarefree" sieve that allows us to produce infinite families of elliptic curves of the form (3.1) with *opposite Mordell-Weil parity*; see Remarks 3.1.7 and 3.1.11.

Following a suggestion of Colliot-Thélène, we prove in §3.5 that the surfaces of Theorems 1.5.3 and 1.5.4 satisfy a variant of weak-weak approximation.

Throughout, for a prime $p \in \mathbb{Z}$ we denote the corresponding p-adic valuation by v_p. If a is a nonzero integer then $\left(\dfrac{a}{p}\right)$ will denote the usual Legendre symbol; if m is an odd positive integer then $\left(\dfrac{a}{m}\right)$ will denote the usual Jacobi symbol.

3.1 Root numbers and flipping

Let E be an elliptic curve over $\mathbb{Q}$. The root number $W(E)$ of E is defined as a product of local factors

$$W(E) = \prod_{p \leq \infty} W_p(E),$$

where p runs over the rational prime numbers and infinity, $W_p(E) \in \{\pm 1\}$ and $W_p(E) = +1$ for all but finitely many p. The local root number $W_p(E)$ of E at p is defined in terms of epsilon factors of Weil-Deligne representations of $\mathbb{Q}_p$; it is an invariant of the isomorphism class of the base extension $E_{\mathbb{Q}_p}$ of E. For a definition of these local factors see [Del73, Tat79]. If p is a prime of good reduction for E then $W_p(E) = +1$; furthermore, $W_\infty(E) = -1$ (see [Roh93]). The computation of $W_p(E)$ for primes of bad reduction in terms of data associated to a Weierstrass model of E has been studied by various authors, particularly by Rohrlich, Halberstadt and Rizzo [Roh93, Hal98, Riz03]. We build on their work to give formulas for the root numbers of elliptic curves as in (3.2).

Conjecturally, the root number $W(E)$ of an elliptic curve is the sign in the conjectural functional equation for the L-series $L(E, s)$ of E:

$$(2\pi)^{-s}\Gamma(s)N^{s/2}L(E,s) = W(E)(2\pi)^{2-s}\Gamma(2-s)N^{(2-s)/2}L(E, 2-s),$$

where N is the conductor of E. According to the Birch–Swinnerton-Dyer conjecture,

$$W(E) = (-1)^{\mathrm{rank}(E)}. \tag{3.3}$$

Equality (3.3) is itself known as the parity conjecture. By work of Nekovář, Dokchitser and Dokchitser, the finiteness of Tate-Shafarevich groups is enough to prove the parity conjecture [Nek01, DD07]. Our results on Mordell-Weil ranks of the fibers of elliptic surfaces are thus all conditional on the finiteness of Tate-Shafarevich groups of elliptic curves over $\mathbb{Q}$.

3.1.1 The root number of $E_\alpha/\mathbb{Q} : y^2 = x^3 + \alpha$

Let α be a nonzero integer. We give a closed formula for the root number of the elliptic curve $E_\alpha/\mathbb{Q} : y^2 = x^3 + \alpha$, in terms of α. Throughout, we write $W(\alpha)$ for this root number and $W_p(\alpha)$ for the local root number of E_α at p. We begin by determining $W_2(\alpha)$ and $W_3(\alpha)$.

Lemma 3.1.1. *Let α be a nonzero integer. Define α_2 and α_3 by $\alpha = 2^{v_2(\alpha)}\alpha_2 = 3^{v_3(\alpha)}\alpha_3$. Then*

$$W_2(\alpha) = \begin{cases} -1 & \text{if } v_2(\alpha) \equiv 0 \text{ or } 2 \bmod 6; \\ & \text{or if } v_2(\alpha) \equiv 1, 3, 4 \text{ or } 5 \bmod 6 \text{ and } \alpha_2 \equiv 3 \bmod 4; \\ +1 & \text{otherwise,} \end{cases}$$

$$W_3(\alpha) = \begin{cases} -1 & \text{if } v_3(\alpha) \equiv 1 \text{ or } 2 \bmod 6 \text{ and } \alpha_3 \equiv 1 \bmod 3; \\ & \text{or if } v_3(\alpha) \equiv 4 \text{ or } 5 \bmod 6 \text{ and } \alpha_3 \equiv 2 \bmod 3; \\ & \text{or if } v_3(\alpha) \equiv 0 \bmod 6 \text{ and } \alpha_3 \equiv 5 \text{ or } 7 \bmod 9; \\ & \text{or if } v_3(\alpha) \equiv 3 \bmod 6 \text{ and } \alpha_3 \equiv 2 \text{ or } 4 \bmod 9, \\ +1 & \text{otherwise.} \end{cases}$$

Proof. According to [Riz03, §1.1], to determine the local root number at p of an elliptic curve given in Weierstrass form, we must find the smallest vector with nonnegative entries

$$(a, b, c) := (v_p(c_4), v_p(c_6), v_p(\Delta)) + k(4, 6, 12) \tag{3.4}$$

for $k \in \mathbb{Z}$, where c_4, c_6 and Δ are the usual quantities associated to a Weierstrass equation (see [Sil92, Ch. III]). For the curves in question we have

$$c_4 = 0, \quad c_6 = -2^5 \cdot 3^3 \cdot \alpha, \quad \text{and} \quad \Delta = -2^4 \cdot 3^3 \cdot \alpha^2,$$

whence

$$(v_p(c_4), v_p(c_6), v_p(\Delta)) = (\infty, v_p(\alpha), 2v_p(\alpha)) + \begin{cases} (0, 5, 4) & \text{if } p = 2, \\ (0, 3, 3) & \text{if } p = 3, \end{cases}$$

Now it is a simple matter of using the tables in [Riz03, §1.1] to compute local root numbers. We illustrate the computation of $W_2(\alpha)$ in one example. Suppose that $v_2(\alpha) \equiv 4 \bmod 6$. Then $(a, b, c) = (\infty, 3, 0)$, and according to the entries under $(\geq 4, 3, 0)$ in Table III of [Riz03], we have $W_2(\alpha) = -1$ if and only if $c_6' := c_6/2^{v_2(c_6)} \equiv 3 \bmod 4$, i.e., if and only if $\alpha_2 \equiv 3 \bmod 4$. All other local root number computations are similar and we omit the details. □

Remark 3.1.2. We take the opportunity to note that the entry $(\geq 5, 6, 9)$ in Table II of [Riz03] has a typo. The "special condition" should read $c_6' \not\equiv \pm 4 \bmod 9$.

Remark 3.1.3. In [Liv95], Liverance gives a closed formula for the global root number of curves of the form $y^2 = x^3 + \alpha$, where α is a sixth-power free integer. However, what he calls w_2 and w_3 in his formula are *not* the local root numbers at 2 and 3, respectively, for these curves.

The elliptic curve E_α has potential good reduction at every non-archimedean place. We will use the following proposition, due to Rohrlich, which gives a formula for the local root numbers of an elliptic curve at primes $p \geq 5$ of potential good reduction.

Proposition 3.1.4 ([Roh93, Proposition 2])**.** *Let $p \geq 5$ be a rational prime, and let $E/\mathbb{Q}_p$ be an elliptic curve with potential good reduction. Write $\Delta \in \mathbb{Q}_p^*$ for the discriminant of any generalized Weierstrass equation for E over $\mathbb{Q}_p$. Let*

$$e := \frac{12}{\gcd(v_p(\Delta), 12)}.$$

Then

$$W_p(E) = \begin{cases} 1 & \text{if } e = 1, \\ \left(\dfrac{-1}{p}\right) & \text{if } e = 2 \text{ or } 6, \\ \left(\dfrac{-3}{p}\right) & \text{if } e = 3, \\ \left(\dfrac{-2}{p}\right) & \text{if } e = 4. \end{cases}$$ □

Proposition 3.1.5 (Root numbers for $y^2 = x^3 + \alpha$)**.** *Let α be a nonzero integer, and let*

$$R(\alpha) = W_2(\alpha)\left(\frac{-1}{\alpha_2}\right)W_3(\alpha)(-1)^{v_3(\alpha)}. \tag{3.5}$$

Then

$$W(\alpha) = -R(\alpha)\prod_{\substack{p^2 \mid \alpha \\ p \geq 5}} \begin{cases} 1 & \text{if } v_p(\alpha) \equiv 0, 1, 3, 5 \bmod 6, \\ \left(\dfrac{-3}{p}\right) & \text{if } v_p(\alpha) \equiv 2, 4 \bmod 6. \end{cases} \tag{3.6}$$

Let β be another nonzero integer, and suppose that $\alpha \equiv \beta \bmod 2^{v_2(\alpha)+2} \cdot 3^{v_3(\alpha)+2}$. Then $R(\alpha) = R(\beta)$.

Proof. Since $\Delta(E_\alpha) = -2^4 3^3 \alpha^2$, applying Proposition 3.1.4 we obtain

$$W(\alpha) = -W_2(\alpha)W_3(\alpha) \prod_{\substack{p \mid \alpha \\ p \geq 5}} \begin{cases} 1 & \text{if } v_p(\alpha) \equiv 0 \bmod 6, \\ \left(\dfrac{-1}{p}\right) & \text{if } v_p(\alpha) \equiv 1, 3, 5 \bmod 6, \\ \left(\dfrac{-3}{p}\right) & \text{if } v_p(\alpha) \equiv 2, 4 \bmod 6. \end{cases} \tag{3.7}$$

Let r be the product of the primes $p \geq 5$ such that $v_p(\alpha) = 1$, let $b = \alpha/r$ and set

$$\alpha_2 := \frac{\alpha}{2^{v_2(\alpha)}}, \qquad b_2 := \frac{b}{2^{v_2(b)}}.$$

Note that $r = \alpha_2/b_2 = \alpha/b$. We may rewrite (3.7) as

$$W(\alpha) = -W_2(\alpha)W_3(\alpha)\left(\frac{-1}{r}\right) \prod_{\substack{p \mid b \\ p \geq 5}} \begin{cases} 1 & \text{if } v_p(\alpha) \equiv 0 \bmod 6, \\ \left(\dfrac{-1}{p}\right) & \text{if } v_p(\alpha) \equiv 1, 3, 5 \bmod 6, \\ \left(\dfrac{-3}{p}\right) & \text{if } v_p(\alpha) \equiv 2, 4 \bmod 6. \end{cases} \tag{3.8}$$

On the other hand, we have

$$\left(\frac{-1}{r}\right) = \left(\frac{-1}{\alpha_2/b_2}\right) = \left(\frac{-1}{\alpha_2}\right) \cdot \left(\frac{-1}{b_2}\right) = \left(\frac{-1}{\alpha_2}\right) \cdot \left(\frac{-1}{3}\right)^{v_3(\alpha)} \cdot \prod_{\substack{p \mid b \\ p \geq 5}} \left(\frac{-1}{p}\right)^{v_p(\alpha)},$$

so we can write (3.8) as

$$\begin{aligned} W(\alpha) &= -\left[W_2(\alpha)\left(\frac{-1}{\alpha_2}\right)W_3(\alpha)(-1)^{v_3(\alpha)}\right] \prod_{\substack{p \mid b \\ p \geq 5}} \begin{cases} \left(\dfrac{-1}{p}\right)^{v_p(\alpha)} & \text{if } v_p(\alpha) \equiv 0 \bmod 6, \\ \left(\dfrac{-1}{p}\right)^{1+v_p(\alpha)} & \text{if } v_p(\alpha) \equiv 1, 3, 5 \bmod 6, \\ \left(\dfrac{-3}{p}\right) \cdot \left(\dfrac{-1}{p}\right)^{v_p(\alpha)} & \text{if } v_p(\alpha) \equiv 2, 4 \bmod 6, \end{cases} \\ &= -R(\alpha) \prod_{\substack{p^2 \mid \alpha \\ p \geq 5}} \begin{cases} 1 & \text{if } v_p(\alpha) \equiv 0, 1, 3, 5 \bmod 6, \\ \left(\dfrac{-3}{p}\right) & \text{if } v_p(\alpha) \equiv 2, 4 \bmod 6. \end{cases} \end{aligned}$$

as desired, because $p \mid b,\ p \geq 5 \iff p^2 \mid \alpha,\ p \geq 5$.

To prove the last claim of the theorem, note that if $\alpha \equiv \beta \bmod 2^{v_2(\alpha)+2} \cdot 3^{v_3(\alpha)+2}$ then $v_2(\alpha) = v_2(\beta)$, $v_3(\alpha) = v_3(\beta)$ and we have

$$\frac{\alpha}{2^{v_2(\alpha)}} \equiv \frac{\beta}{2^{v_2(\beta)}} \bmod 4 \quad \text{and} \quad \frac{\alpha}{3^{v_3(\alpha)}} \equiv \frac{\beta}{3^{v_3(\beta)}} \bmod 9.$$

The claim now follows from Lemma 3.1.1 $\square$

The following corollary describes conditions on two nonzero integers α and β which guarantee that the elliptic curves $y^2 = x^3 + \alpha$ and $y^2 = x^3 + \beta$ have *opposite* root numbers. This is one of the key inputs to the proof of Theorem 1.5.3.

Corollary 3.1.6 (Flipping I). *Let α, β be nonzero integers such that*

1. $\alpha \equiv \beta \bmod 2^{v_2(\alpha)+2} \cdot 3^{v_3(\alpha)+2}$,

2. $\alpha = c\ell$, *where ℓ is squarefree and* $\gcd(c, \ell) = 1$,

3. $\beta = cq^{2+6k}\eta$, *where η is square free,* $\gcd(c, \eta) = \gcd(q, c\eta) = 1$, $k \geq 0$, $q \geq 5$ *is prime and* $q \equiv 2 \bmod 3$.

Then $W(\alpha) = -W(\beta)$.

Proof. The first condition ensures that $R(\alpha) = R(\beta)$. Since ℓ is squarefree and $\gcd(c, \ell) = 1$, the only primes greater than 3 contributing to $W(\alpha)$ are those whose square divides c. Similarly, since η is squarefree and $\gcd(c, \eta) = \gcd(q, \eta) = 1$, the only primes greater than 3 contributing to $W(\beta)$ are those whose square divides c, and q. Since $\gcd(q, c) = 1$, $q \geq 5$ and $q \equiv 2 \bmod 3$, we have

$$W(\beta) = \left(\frac{-3}{q}\right) W(\alpha) = -W(\alpha)$$

□

Remark 3.1.7. To prove Zariski density of rational points on the elliptic surface $\mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ associated to a del Pezzo of degree 1 as in Theorem 1.5.3, it is enough to do the following. First, prove that there exist infinite sets $\mathscr{F}_1$ and $\mathscr{F}_2$ of coprime pairs of integers such that whenever $(m_1, n_1) \in \mathscr{F}_1$ and $(m_2, n_2) \in \mathscr{F}_2$ then

1. $\alpha := F(m_1, n_1)$ and $\beta := F(m_2, n_2)$ are nonzero integers.

2. $\alpha \equiv \beta \bmod 2^{v_2(\alpha)+2} \cdot 3^{v_3(\alpha)+2}$,

3. $\alpha = c\ell$, where ℓ is squarefree and $\gcd(c, \ell) = 1$,

4. $\beta = cq^{2+6k}\eta$, where η is square free, $\gcd(c, \eta) = \gcd(q, c\eta) = 1$, $k \geq 0$, $q \geq 5$ is prime and $q \equiv 2 \bmod 3$.

Then, by Corollary 3.1.6, we know that either

$$W(F(m, n)) = -1 \text{ for all } (m, n) \in \mathscr{F}_1,$$

or

$$W(F(m,n)) = -1 \text{ for all } (m,n) \in \mathscr{F}_2.$$

Hence, there are infinitely many closed fibers of $\mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ with negative root number. Assuming the parity conjecture, this gives an infinite number of closed fibers with infinitely many points, and hence a Zariski dense set of rational points on $\mathcal{E}$.

3.1.2 The root number of $E_\alpha/\mathbb{Q} : y^2 = x^3 + \alpha x$

Next, we give a closed formula for the root number of the elliptic curve $E_\alpha/\mathbb{Q} : y^2 = x^3 + \alpha x$, in terms of the nonzero integer α. The proofs mirror those of §3.1.1, and thus we provide only a few details for them. Throughout this section, we write $W(\alpha)$ for the root number of E_α and $W_p(\alpha)$ for the local root number at p of E_α.

Lemma 3.1.8. *Let α be a nonzero integer. Define α_2 and α_3 by $\alpha = 2^{v_2(\alpha)}\alpha_2 = 3^{v_3(\alpha)}\alpha_3$. Then*

$$W_2(\alpha) = \begin{cases} -1 & \text{if } v_2(\alpha) \equiv 1 \text{ or } 3 \bmod 4 \text{ and } \alpha_2 \equiv 1 \text{ or } 3 \bmod 8; \\ & \text{or if } v_2(\alpha) \equiv 0 \bmod 4 \text{ and } \alpha_2 \equiv 1, 5, 9, 11, 13 \text{ or } 15 \bmod 16; \\ & \text{or if } v_2(\alpha) \equiv 2 \bmod 4 \text{ and } \alpha_2 \equiv 1, 3, 5, 7, 11 \text{ or } 15 \bmod 16; \\ +1 & \text{otherwise,} \end{cases}$$

$$W_3(\alpha) = \begin{cases} -1 & \text{if } v_3(\alpha) \equiv 2 \bmod 4, \\ +1 & \text{otherwise.} \end{cases}$$

Proof. We proceed as in the proof of Lemma 3.1.1, this time using the quantities

$$c_4 = -2^4 \cdot 3 \cdot \alpha, \quad c_6 = 0, \quad \text{and} \quad \Delta = -2^6 \cdot \alpha^3,$$

together with the tables in [Riz03, §1.1]. ☐

Proposition 3.1.9 (Root numbers for $y^2 = x^3 + \alpha x$)**.** *Let α be a nonzero integer, and let*

$$R(\alpha) = W_2(\alpha)\left(\frac{-1}{\alpha_2}\right)W_3(\alpha)(-1)^{v_3(\alpha)}. \tag{3.9}$$

Then

$$W(\alpha) = -R(\alpha)\prod_{\substack{p^2 \mid \alpha \\ p \geq 5}} \begin{cases} \left(\frac{-1}{p}\right) & \text{if } v_p(\alpha) \equiv 2 \bmod 4, \\ \left(\frac{2}{p}\right) & \text{if } v_p(\alpha) \equiv 3 \bmod 4. \end{cases}$$

Let β be another nonzero free integer, and suppose that $\alpha \equiv \beta \bmod 2^{v_2(\alpha)+4} \cdot 3^{v_3(\alpha)}$. Then $R(\alpha) = R(\beta)$.

Proof. Since $\Delta(E_\alpha) = -2^6 \cdot \alpha^3$, applying Proposition 3.1.4 we obtain

$$W(\alpha) = -W_2(\alpha)W_3(\alpha) \prod_{\substack{p \mid \alpha \\ p \geq 5}} \begin{cases} 1 & \text{if } v_p(\alpha) \equiv 0 \bmod 4, \\ \left(\dfrac{-2}{p}\right) & \text{if } v_p(\alpha) \equiv 1, 3 \bmod 4, \\ \left(\dfrac{-1}{p}\right) & \text{if } v_p(\alpha) \equiv 2 \bmod 4. \end{cases} \tag{3.10}$$

Now proceed as in the proof of Proposition 3.1.5. □

The following corollary, which parallels Corollary 3.1.6, describes conditions on two nonzero integers α and β that guarantee that the elliptic curves $y^2 = x^3 + \alpha x$ and $y^2 = x^3 + \beta x$ have *opposite* root numbers. This is one of the key inputs to the proof of Theorem 1.5.4.

Corollary 3.1.10 (Flipping II)**.** *Let α, β be nonzero integers such that*

1. $\alpha \equiv \beta \bmod 2^{v_2(\alpha)+4} \cdot 3^{v_3(\alpha)}$,
2. $\alpha = c\ell$, *where ℓ is squarefree and* $\gcd(c, \ell) = 1$,
3. $\beta = cq^{2+4k}\eta$, *where η is square free,* $\gcd(c, \eta) = \gcd(q, c\eta) = 1$, $k \geq 0$, $q \geq 5$ *is prime and* $q \equiv 3 \bmod 4$*; or* $\beta = cp^{3+4k}\eta$, *where η is square free,* $\gcd(c, \eta) = \gcd(q, c\eta) = 1$, $k \geq 0$, $q \geq 5$ *is prime and* $q \equiv 3$ *or* $5 \bmod 8$.

Then $W(\alpha) = -W(\beta)$. □

Remark 3.1.11. To prove Zariski density of rational points on the elliptic surface $\mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ associated to a del Pezzo of degree 1 as in Theorem 1.5.4, it is enough to do the following. First, prove that there exist infinite sets $\mathscr{F}_1$ and $\mathscr{F}_2$ of coprime pairs of integers such that whenever $(m_1, n_1) \in \mathscr{F}_1$ and $(m_2, n_2) \in \mathscr{F}_2$ then

1. $\alpha := G(m_1, n_1)$ and $\beta := G(m_2, n_2)$ are nonzero integers.
2. $\alpha \equiv \beta \bmod 2^{v_2(\alpha)+4} \cdot 3^{v_3(\alpha)}$,
3. $\alpha = c\ell$, where ℓ is squarefree and $\gcd(c, \ell) = 1$,

4. $\beta = cq^{2+4k} \cdot \eta$, where η is square free, $\gcd(c,\eta) = \gcd(q, c\eta) = 1$, $k \geq 0$, $q \geq 5$ is prime and $q \equiv 3 \bmod 4$; or $\beta = cq^{3+4k}\eta$, where η is square free, $\gcd(c,\eta) = \gcd(q, c\eta) = 1$, $k \geq 0$, $q \geq 5$ is prime and $q \equiv 3$ or $5 \bmod 8$.

Then, arguing as in Remark 3.1.7 (using Corollary 3.1.10) we find infinitely many closed fibers of $\mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ with negative root number. This gives a Zariski dense set of rational point for $\mathcal{E}$, assuming the parity conjecture.

3.2 The Modified Square-free Sieve

In this section we present a variation of a sieve by Gouvêa, Mazur and Greaves [GM91, Gre92]. It is the tool that allows us to identify families of fibers with negative root numbers on certain elliptic surfaces.

Let $F(m,n) \in \mathbb{Z}[m,n]$ be a binary homogeneous form of degree d, not divisible by the square of a nonunit in $\mathbb{Z}[m,n]$. Write $F = \prod_{i=1}^{t} f_i$, where the $f_i(m,n) \in \mathbb{Z}[m,n]$ are irreducible, and assume that $\deg f_i \leq 6$ for all i. Applying a unimodular transformation we may (and do) assume that the coefficients of m^d and n^d in $F(m,n)$ are nonzero. Call their respective coefficients a_d and a_0. Write $F(m,n) = a_d \prod(m - \theta_i n)$, where the θ_i are algebraic numbers and $1 \leq i \leq d$. Let

$$\Delta(F) = \left| a_0 a_d^{2d-1} \prod_{i \neq j} (\theta_i - \theta_j) \right|;$$

this is essentially the discriminant of the form F. It is nonzero if and only if F contains no square factors.

Fix a positive integer M, as well as a subset $\mathcal{S}$ of $(\mathbb{Z}/m\mathbb{Z})^2$. Our goal is to count pairs of integers (m,n) such that $(m \bmod M, n \bmod M) \in \mathcal{S}$ and $F(m,n)$ is not divisible by p^2 for any prime number p such that $p \nmid M$. This will allow us to give an asymptotic formula for the number of pairs of integers (m,n) with $0 \leq m,n \leq x$ such that

$$F(m,n) = \nu \cdot \ell,$$

where ν is a *fixed* integer and ℓ is a squarefree integer such that $\gcd(\nu,\ell) = 1$. The case $\nu = 1$ is handled by Gouvêa and Mazur in [GM91] under the additional assumption that $\deg f_i \leq 3$, and extended by Greaves in [Gre92] to the case $\deg f_i \leq 6$. We build upon their work to prove an asymptotic formula when $\nu > 1$.

We make use of the following (mild variation of an) arithmetic function studied by Gouvêa and Mazur: put $\rho(1) = 1$, and for $k \geq 2$ let

$$\rho(k) = \#\{(m,n) \in \mathbb{Z}^2 : 0 \leq m,n \leq k-1, F(m,n) \equiv 0 \bmod k\}.$$

By the Chinese remainder theorem, the function ρ is multiplicative; i.e., if k_1 and k_2 are relatively prime positive integers then $\rho(k_1 k_2) = \rho(k_1)\rho(k_2)$.

Lemma 3.2.1 ([GM91, Lemma 3(2)]). *For fixed F as above and squarefree ℓ, we have $\rho(\ell^2) = O(\ell^2 \cdot d_k(\ell))$ as $\ell \to \infty$, where $k = \deg(F) + 1$ and $d_k(\ell)$ denotes the number of ways in which ℓ can be expressed as a product of k factors. In particular, $\rho(p^2) = O(p^2)$ as $p \to \infty$.* $\square$

We can now state the main result of this section.

Theorem 3.2.2 (Modified squarefree sieve). *Let $F(m,n) \in \mathbb{Z}[m,n]$ be a homogeneous binary form of degree d. Assume that no square of a nonunit in $\mathbb{Z}[m,n]$ divides $F(m,n)$, and that no irreducible factor of F has degree greater than 6. Fix a positive integer M, as well as a subset $\mathcal{S}$ of $(\mathbb{Z}/m\mathbb{Z})^2$. Let $N(x)$ be the number of pairs of integers (m,n) with $0 \leq m,n \leq x$ such that $(m \bmod M, n \bmod M) \in \mathcal{S}$ and $F(m,n)$ is not divisible by p^2 for any prime p such that $p \nmid M$. Then, as $x \to \infty$, we have*

$$N(x) = Cx^2 + O\left(\frac{x^2}{(\log x)^{1/3}}\right),$$

where

$$C = \frac{|\mathcal{S}|}{M^2} \prod_{p \nmid M} \left(1 - \frac{\rho(p^2)}{p^4}\right).$$

Remark 3.2.3. By Lemma 3.2.1, $\rho(p^2) = O(p^2)$ as $p \to \infty$ for a fixed F, so the infinite product defining C converges.

Heuristically, the condition that $F(m,n)$ be squarefree outside a prescribed integer is well approximated by the condition that $F(m,n)$ not be divisible by the square of a prime that is "small relative to x." More precisely, let $\xi = (1/3)\log x$ and define the principal term

$$N'(x) = \{(m,n) \in \mathbb{Z}^2 :\ 0 \leq m,n \leq x,\ \ F(m,n) \not\equiv 0 \bmod p^2 \text{ for all } p \leq \xi, p \nmid M \\ \text{and } (m \bmod M, n \bmod M) \in \mathcal{S}\}.$$

Let $F = \prod_{i=1}^{t} f_i$ be a factorization of F into irreducible binary forms. Define the partial i-th error term $E_i(x)$ as follows:

$$E_0(x) = \#\{(m,n) \in \mathbb{Z}^2 : 0 \leq m,n \leq x,\ p \,|\, m \text{ and } p \,|\, n \text{ for some } p > \xi\},$$

and

$$E_i(x) = \#\{(m,n) \in \mathbb{Z}^2 : 0 \leq m,n \leq x,\ p^2 \,|\, f_i(m,n) \text{ for some } p > \xi\}.$$

The total error term is $E(x) := \sum_{i=0}^{t} E_i(x)$. The proof of [GM91, Prop. 2], essentially unchanged, shows the following.

Proposition 3.2.4. *If* $\xi > \max\{\Delta(F), M\}$ *then*

$$N'(x) - E(x) \leq N(x) \leq N'(x). \qquad \square$$

The proposition implies that

$$N(x) = N'(x) + O(E(x)).$$

For this reason that we think of ξ as giving us the notion of "small prime relative to x." The choice of $(1/3)\log x$ is somewhat flexible (see [GM91, §4]); what is important is that when ℓ is a squarefree integer divisible only by primes *smaller* than ξ then

$$\ell \leq \prod_{p<\xi} p = \exp\left(\sum_{p<\xi} \log p\right) \leq e^{2\xi} = x^{2/3}, \tag{3.11}$$

where the last inequality follows from the estimate

$$\sum_{p<\xi} \log p \leq \sum_{p<\xi} \log \xi = \pi(\xi)\log\xi < 2\xi,$$

and $\pi(x) = \#\{p \text{ prime} : p < x\}$; see [Sto03, p. 105].

In [Gre92], Greaves shows that as $x \to \infty$

$$E(x) = O\left(\frac{x^2}{(\log x)^{1/3}}\right)$$

Greaves' proof requires the hypothesis that no irreducible factor of F have degree greater than 6, which explains the presence of this hypothesis in Theorem 3.2.2. Theorem 3.2.2 thus follows from the next lemma.

Lemma 3.2.5. *With notation as above, as $x \to \infty$ we have*

$$N'(x) = Cx^2 + O\left(\frac{x^2}{\log x}\right)$$

Proof. Let ℓ be a squarefree integer divisible only by primes smaller than ξ, and such that $\gcd(\ell, M) = 1$. Let $N_\ell(M, \mathcal{S}; x)$ be the number of pairs of integers (m, n) such that

$$0 \le m, n \le x, \quad (m \bmod M, n \bmod M) \in \mathcal{S}, \quad \text{and } F(m,n) \equiv 0 \bmod \ell^2.$$

For a fixed congruence class modulo ℓ^2 of solutions $F(m_0, n_0) \equiv 0 \bmod \ell^2$, satisfying $(m_0 \bmod M, n_0 \bmod M) \in \mathcal{S}$, we count the number of representatives in the box $0 \le m, n, \le x$, and obtain

$$N_\ell(M, \mathcal{S}; x) = \frac{x^2 \cdot |\mathcal{S}|}{M^2} \cdot \frac{\rho(\ell^2)}{\ell^4} + O\left(x \cdot \frac{\rho(\ell^2)}{\ell^2}\right),$$

where the implied constant depends on F, M and $\mathcal{S}$, but not on ℓ or x. By the inclusion-exclusion principle we have

$$N'(x) = \sum_\ell \mu(\ell) N_\ell(M, \mathcal{S}; x),$$

where μ denotes the usual Möbius function and the sum runs over squarefree integers that are divisible only by primes smaller than ξ and that are relatively prime to M. Thus, by (3.11),

$$\begin{aligned} N'(x) &= \frac{x^2 \cdot |\mathcal{S}|}{M^2} \sum_\ell \mu(\ell) \frac{\rho(\ell^2)}{\ell^4} + O\left(x \cdot \sum_{\ell \le x^{2/3}} \frac{\rho(\ell^2)}{\ell^2}\right) \\ &= \frac{x^2 \cdot |\mathcal{S}|}{M^2} \prod_{p<\xi,\ p \nmid M} \left(1 - \frac{\rho(p^2)}{p^4}\right) + O\left(x \cdot \sum_{\ell \le x^{2/3}} \frac{\rho(\ell^2)}{\ell^2}\right) \end{aligned}$$

Assume that x is large enough so that $\xi > M$. Then, by Lemma 3.2.1, we have

$$\begin{aligned} \prod_{p \ge \xi} \left(1 - \frac{\rho(p^2)}{p^4}\right) &= \prod_{p \ge \xi} \left(1 - O\left(\frac{1}{p^2}\right)\right) = 1 - \sum_{p \ge \xi} O\left(\frac{1}{p^2}\right) \\ &= 1 - O\left(\int_{t \ge \xi} \frac{1}{t^2}\, dt\right) = 1 - O\left(\frac{1}{\xi}\right) \end{aligned}$$

Hence

$$N'(x) = \frac{x^2 \cdot |\mathcal{S}|}{M^2} \prod_{p \nmid M} \left(1 - \frac{\rho(p^2)}{p^4}\right) + O\left(\frac{x^2}{\xi}\right) + O\left(x \cdot \sum_{\ell \le x^{2/3}} \frac{\rho(\ell^2)}{\ell^2}\right)$$

By Lemma 3.2.1, we have

$$O\left(x \cdot \sum_{\ell \le x^{2/3}} \frac{\rho(\ell^2)}{\ell^2}\right) = O\left(x \cdot \sum_{\ell \le x^{2/3}} d_k(\ell)\right) = O(x \cdot x^{2/3} \log^{k-1} x),$$

where we have used the well-known fact that

$$\sum_{n \le x} d_k(n) = O(x \log^{k-1} x);$$

see, for example, [IK04, (1.80)]. Since $\xi = (1/3)\log x$, it follows that

$$\begin{aligned} N'(x) &= \frac{x^2 \cdot |\mathcal{S}|}{M^2} \prod_{p \nmid M} \left(1 - \frac{\rho(p^2)}{p^4}\right) + O\left(\frac{x^2}{\xi}\right) + O(x \cdot x^{2/3} \log^{k-1} x) \\ &= \frac{x^2 \cdot |\mathcal{S}|}{M^2} \prod_{p \nmid M} \left(1 - \frac{\rho(p^2)}{p^4}\right) + O\left(\frac{x^2}{\log x}\right). \end{aligned}$$

□

3.2.1 Making sure that C does not vanish

In this section we explore the possibility that the constant C for the principal term of $N(x)$ is zero. This will depend on the particular binary form $F(m,n)$, the integer M and the set $\mathcal{S}$. For any prime $p \nmid M$, let

$$C_p = \left(1 - \frac{\rho(p^2)}{p^4}\right),$$

so that $C = \frac{|\mathcal{S}|}{M^2} \prod_{p \nmid M} C_p$. For $p \nmid M$ we know that $\rho(p^2) = O(p^2)$ (see Lemma 3.2.1); hence C vanishes if and only if either $\mathcal{S} = \emptyset$, or one of the factors C_p vanishes.

Lemma 3.2.6. *With notation as above, if $p \nmid M$ and $p \ge \deg F$, then $C_p \ne 0$.*

Proof. If $p \nmid M$ then $C_p = 0$ if and only if $\rho(p^2) = p^4$, which happens if and only if all pairs of integers (m,n) modulo $\mathbb{Z}/p^2\mathbb{Z}$ are solutions to $F(m,n) \equiv 0 \bmod p^2$. But then *all* pairs of integers (m,n) give solutions to the given congruence equation. By Remark 1.5.2, this can happen only if $p < \deg(F)$. □

3.2.2 An application of the modified sieve

Corollary 3.2.7 (Pseudo-squarefree sieve). *Let $F(m,n) \in \mathbb{Z}[m,n]$ be a homogeneous binary form of degree d. Assume that no square of a nonunit in $\mathbb{Z}[m,n]$ divides $F(m,n)$, and that no irreducible factor of F has degree greater than 6. Fix*

- *a sequence* $S = (p_1, \ldots, p_r)$ *of distinct prime numbers and*
- *a sequence* $T = (t_1, \ldots, t_r)$ *of nonnegative integers.*

Let M *be an integer such that* $p^2 \mid M$ *for all primes* $p < \deg F$ *and* $p_1^{t_1+1} \cdots p_r^{t_r+1} \mid M$*. Suppose that there exist integers* a, b *such that*

$$F(a,b) \not\equiv 0 \bmod p^2 \quad \textit{whenever } p \mid M \textit{ and } p \neq p_i \textit{ for any } i, \tag{3.12}$$

and such that

$$v_{p_i}(F(a,b)) = t_i \quad \textit{for all } i. \tag{3.13}$$

Then there are infinitely many pairs of integers (m, n) *such that*

$$m \equiv a \bmod M, \qquad n \equiv b \bmod M, \tag{3.14}$$

and

$$F(m,n) = p_1^{t_1} \cdots p_r^{t_r} \cdot \ell,$$

where ℓ *is squarefree and* $v_{p_i}(\ell) = 0$ *for all* i*.*

Proof. Let $\mathcal{S} = \{(a,b)\}$. By Theorem 3.2.2, there are infinitely many pairs of integers (m,n) such that

$$m \equiv a \bmod M, \quad n \equiv b \bmod M, \quad \text{and } F(m,n) \not\equiv 0 \bmod p^2 \quad \text{whenever } p \nmid M,$$

(Note that $|\mathcal{S}| = 1$ and $C \neq 0$ by Lemma 3.2.6.) The condition (3.12) then guarantees that $F(m,n)$ is not divisible by the square of any prime outside the sequence S. We also have

$$m \equiv a \bmod p_i^{t_i+1}, \quad n \equiv b \bmod p_i^{t_i+1}, \quad \text{for all } i,$$

because $p_i^{t_i+1} \mid M$ for all i, and hence

$$F(m,n) = F(a,b) \bmod p_i^{t_i+1} \quad \text{for all } i.$$

Using condition (3.13), we conclude that

$$v_{p_i}(F(m,n)) = t_i. \qquad \square$$

3.3 Proof of Theorems 1.5.3 and 1.5.4

For a finite extension L/k of number fields, we let $S(L/k)$ denote the set of unramified prime ideals of k that have a degree 1 prime over k in L. Given two sets A and B, we write $A \doteq B$ if A and B differ by finitely many elements and $A \sqsubseteq B$ if $x \in A \implies x \in B$ with finitely many exceptions.

Proposition 3.3.1 (Bauer, [Neu99, p. 548]). *Let k be a number field, N/k a Galois extension of k and M/k an arbitrary finite extension of k. Then*

$$S(M/k) \sqsubseteq S(N/k) \iff M \supseteq N.$$

Lemma 3.3.2. *Let $f(t) \in \mathbb{Z}[t]$ be an irreducible nonconstant polynomial, and let $N = \mathbb{Q}[t]/f(t)$. Let μ_3 denote the group of third roots of unity, and suppose that $\mathbb{Q}(\mu_3) \not\subseteq N$. Then there are infinitely many rational primes p such that $p \equiv 2 \pmod 3$ and there exists a degree 1 prime $\mathfrak{p} \subseteq N$ over p lying over it.*

Proof. Since $\mathbb{F}_p^\times$ contains an element of order 3 if and only if $3|(p-1)$, it follows that

$$S(\mathbb{Q}(\mu_3)/\mathbb{Q}) \doteq \{p \in \mathbb{Z} : p \text{ prime and } p \equiv 1 \bmod 3\}.$$

Suppose that the following implication holds (with possibly finitely many exceptions):

$$p \in \mathbb{Z} \text{ has a degree 1 prime in } N \implies p \equiv 1 \bmod 3.$$

Then

$$S(N/\mathbb{Q}) \sqsubseteq S(\mathbb{Q}(\mu_3)/\mathbb{Q}).$$

It follows from Proposition 3.3.1 that $\mathbb{Q}(\mu_3) \subseteq N$, a contradiction. $\square$

A similar argument proves the following entirely analogous lemma.

Lemma 3.3.3. *Let $g(t) \in \mathbb{Z}[t]$ be an irreducible nonconstant polynomial, and let $N = \mathbb{Q}[t]/g(t)$. Let μ_4 denote the group of fourth roots of unity, and suppose that $\mathbb{Q}(\mu_4) \not\subseteq N$. Then there are infinitely rational primes p such that $p \equiv 2 \pmod 3$ and there exists a degree 1 prime $\mathfrak{p} \subseteq N$ over p lying over it.* $\square$

Proof of Theorem 1.5.3. First, one checks that a surface in $\mathbb{P}_{\mathbb{Q}}(1,1,2,3)$ given by an equation of the form (1.3) is smooth if and only if F_1 is a squarefree binary form of degree

6. Blowing up the anticanonical point $[0:0:1:1]$ of X we obtain an elliptic surface $\rho\colon \mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ whose fiber above $[m:n] \in \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$ is isomorphic to a curve in $\mathbb{P}^2_{\mathbb{Q}}$ whose affine equation is given by

$$y^2 = x^3 + F(m,n) \tag{3.15}$$

(see §2.2.4). This is an elliptic curve for almost all $[m:n] \in \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$.

Write $c = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where the p_i are distinct primes. Let $S = (p_1, \dots, p_r)$, $T = (0, \dots, 0)$ and let

$$M = (2 \cdot 3 \cdot 5)^3 \cdot (p_1 \cdots p_r).$$

Since $F_1(m,n)$ has no fixed prime divisors, we know that for each prime $p \mid M$ with $p \neq p_i$ for all i there exist congruence classes a_p, b_p modulo p^2 such that

$$F_1(a_p, b_p) \not\equiv 0 \bmod p^2.$$

Similarly, for a prime p_i in the sequence S there exist congruence classes a_{p_i}, b_{p_i} modulo p_i such that

$$F_1(a_{p_i}, b_{p_i}) \not\equiv 0 \bmod p_i;$$

in other words, $v_{p_i}(F_1(a_{p_i}, b_{p_i})) = 0$. By the Chinese remainder theorem there exist congruence classes a, b modulo M such that

$$\begin{cases} a \equiv a_p \bmod p^2 & \text{for all primes } p \text{ such that } p \mid M,\ p \neq p_i \text{ for any } i, \\ a \equiv a_{p_i} \bmod p_i & \text{for all primes } p_i \text{ in the sequence } S, \end{cases} \tag{3.16}$$

and

$$\begin{cases} b \equiv b_p \bmod p^2 & \text{for all primes } p \text{ such that } p \mid M,\ p \neq p_i \text{ for all } i, \\ b \equiv b_{p_i} \bmod p_i & \text{for all primes } p_i \text{ in the sequence } S, \end{cases} \tag{3.17}$$

By Corollary 3.2.7, applied to F_1, S, T, M, a and b as above, there is an infinite set $\mathscr{F}_1$ of pairs $(m,n) \in \mathbb{Z}^2$ such that

$$F_1(m,n) = \ell,$$

where ℓ is a squarefree integer with $\gcd(c, \ell) = 1$, by our choice of S and T. Note that the elements m, n of each pair must be coprime since $F_1(m,n)$ is squarefree. Furthermore, the congruence class of ℓ modulo $2^3 \cdot 3^3$ is fixed (by our choice of M) and nonzero (because ℓ is squarefree). Thus, for $(m,n) \in \mathscr{F}_1$ we have

$$F(m,n) = c\ell \qquad \gcd(c, \ell) = 1,$$

and the congruence class of $c\ell/2^{v_2(c\ell)}3^{v_3(c\ell)}$ modulo $2^2 \cdot 3^2$ is fixed and nonzero.

By Lemma 3.3.2, applied to a number field $N := \mathbb{Q}[t]/f_i(t,1)$ such that (1.4) holds, there is a rational prime $q \equiv 2 \bmod 3$ and a prime $\mathfrak{q}$ in N lying over q of degree 1. In fact, we may choose q so that $q > 5$, $\gcd(q,c) = 1$, and so that it does not divide the discriminant of $f_i(t,1)$.

We apply Corollary 3.2.7 again to $F_1(m,n)$. This time we let $S = (p_1, \ldots, p_r, q)$ and $T = (0, \ldots, 0, 2+6k)$, where k is a large positive integer[1]. Let

$$M = (2 \cdot 3 \cdot 5)^3 \cdot (p_1 \cdots p_r) \cdot q^{3+6k}.$$

We claim that there exist integers m_q, n_q such that

$$v_q(F_1(m_q, n_q)) = 2 + 6k.$$

Indeed, since q has a prime $\mathfrak{q}$ of degree 1 in N and it does not divide the discriminant of $f_i(t,1)$, the equation

$$f_i(t,1) = 0$$

has a simple root in $\mathbb{F}_q$. By Hensel's lemma, this solution lifts to a root in $\mathbb{Q}_q$. Hence $F_1(t,1) = 0$ has a root in $\mathbb{Q}_q$. Approximating this solution by a rational number $r_q = m_q/n_q$ we can control $v_q(F_1(r_q,1))$ modulo 6; i.e., there exists a pair $(m_q, n_q) \in \mathbb{Z}^2$ of coprime integers such that $v_q(F_1(m_q,n_q)) = 2 + 6k$ for some (possibly very large) positive integer k. By the Chinese remainder theorem, there exists a pair of integers (a,b) simultaneously satisfying (3.16), (3.17) and

$$a \equiv m_q \bmod q^{3+6k}, \quad \text{and } b \equiv n_q \bmod q^{3+6k}. \tag{3.18}$$

By Corollary 3.2.7, applied to F_1, S, T, M, a and b as above, there is an infinite set $\mathscr{F}_2$ of pairs $(m,n) \in \mathbb{Z}^2$ such that

$$F_1(m,n) = q^{2+6k}\eta,$$

for some squarefree integer η with $\gcd(c, q\eta) = \gcd(q, \eta) = 1$, by our choice of S and T. Suppose that $(m,n) \in \mathscr{F}_2$. Then

$$F(m,n) = cq^{2+6k}\eta \qquad \gcd(c,\eta) = \gcd(q, c\eta) = 1.$$

[1]We will pick k large enough to ensure that $C \neq 0$ upon application of the pseudo squarefree sieve.

Furthermore, we claim that $\gcd(m,n) = 1$. To see this, note that since η is squarefree and F_1 is homogeneous of degree 6, then $\gcd(m,n)$ is some power of q; by (3.14), (3.18), and because $\gcd(m_q,n_q) = 1$, this power of q must be 1. As before, the congruence class of $cq^{2+6k}\eta/2^{v_2(c\eta)}3^{v_3(c\eta)} \bmod 2^2 \cdot 3^2$ is fixed, nonzero, and equal to that of $F_1(m,n)$ for $(m,n) \in \mathscr{F}_1$ (by our choice of a and b).

Whenever (3.15) is smooth, we write $W(F(m,n))$ for its root number. By Corollary 3.1.6, if $(m_1,n_1) \in \mathscr{F}_1$ and $(m_2,n_2) \in \mathscr{F}_2$ then

$$W(F(m_1,n_1)) = -W(F(m_2,n_2)).$$

Zariski density of rational points on X now follows by arguing as in Remark 3.1.7. □

The proof of Theorem 1.5.4 is quite similar; we give enough details so that the interested reader can reconstruct it from the proof of Theorem 1.5.3.

Proof of Theorem 1.5.4. The surface in $\mathbb{P}_{\mathbb{Q}}(1,1,2,3)$ given by an equation of the form (1.5) is smooth if and only if G_1 is a squarefree binary form of degree 4. Blowing up the anticanonical point $[0:0:1:1]$ of X we obtain an elliptic surface $\rho\colon \mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ whose fiber above $[m:n] \in \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$ is isomorphic to a curve in $\mathbb{P}^2_{\mathbb{Q}}$ whose affine equation is given by

$$y^2 = x^3 + G(m,n)x \tag{3.19}$$

(see §2.2.4). This is an elliptic curve for almost all $[m:n] \in \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$.

We apply Corollary 3.2.7 twice, as in the proof of Theorem 1.5.3. First, we apply it to $G_1(m,n)$ by taking $S = (p_1,\dots,p_r)$, $T = (0,\dots,0)$, where $c = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, and the p_i are distinct primes. We use

$$M = \left(2^2\cdot 3\right)^3 \cdot (p_1\cdots p_r).$$

This way we obtain an infinite set $\mathscr{F}_1$ of coprime pairs of integers (m,n) such that

$$G(m,n) = c\ell \qquad \gcd(c,\ell) = 1,$$

and the congruence class of $c\ell/2^{v_2(c\ell)}3^{v_3(c\ell)}$ modulo $2^4\cdot 3^2$ is fixed and nonzero.

By Lemma 3.3.3, applied to a number field $N := \mathbb{Q}[t]/g_i(t,1)$ such that (1.6) holds, there is a rational prime $q \equiv 3 \bmod 4$ and a prime $\mathfrak{q}$ in N lying over q of degree 1. In fact, we may choose q so that $q > 5$, $\gcd(q,c) = 1$, and so that it does not divide the discriminant of $g_i(t,1)$.

We apply Corollary 3.2.7 again to $G_1(m,n)$ with $S = (p_1, \ldots, p_r, q)$ and $T = (0, \ldots, 0, 2+4k)$, where k is a large positive integer, and

$$M = (2^2 \cdot 3)^3 \cdot (p_1 \cdots p_r) \cdot q^{3+4k}$$

Using Hensel's lemma as in the proof of Theorem 1.5.3, we obtain a different infinite set $\mathscr{F}_2$ of coprime pairs integers (m,n) such that

$$G(m,n) = cq^{2+4k}\eta \qquad \gcd(c,\eta) = \gcd(q, c\eta) = 1,$$

where η is a squarefree integer. As before, the congruence class of $cq^{2+4k}\eta/2^{v_2(c\eta)}3^{v_3(c\eta)}$ modulo $2^4 \cdot 3^2$ is fixed, nonzero, and equal to that of $G_1(m,n)$ for $(m,n) \in \mathscr{F}_1$ (by our choice of a and b).

Whenever (3.19) is smooth, we write $W(G(m,n))$ for its root number. By Corollary 3.1.10, if $(m_1,n_1) \in \mathscr{F}_1$ and $(m_2,n_2) \in \mathscr{F}_2$ then

$$W(G(m_1,n_1)) = -W(G(m_2,n_2)).$$

Zariski density of rational points on X now follows by arguing as in Remark 3.1.11. $\square$

3.4 Diagonal del Pezzo surfaces of degree 1

We begin this section with two examples of del Pezzo surfaces of degree 1 that show how the sieving technique used in the proof of Theorems 1.5.3 and 1.5.4 can fail. In one case, however, we can show that rational points are Zariski dense, by exhibiting explicit nontorsion sections of the associated elliptic surfaces.

Example 3.4.1. Consider the del Pezzo surface of degree 1 given by

$$w^2 = z^3 + 27x^6 + 16y^6$$

in $\mathbb{P}_{\mathbb{Q}}(1,1,2,3)$ Let $\rho\colon \mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ be its associated elliptic fibration and let $U \subseteq \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$ be the set of points whose fibers are elliptic curves. The elliptic curve $E_{m,n}$ above the point $[m:n] \in U$ is given by

$$E_{m,n}\colon \quad y^2 = x^3 + 27m^6 + 16n^6.$$

We claim that $W(E_{m,n}) = +1$ *for all* $[m:n] \in U$. We may assume that $\gcd(m,n) = 1$. Let $\alpha = 27m^6 + 16n^6$. Suppose that $p \geq 5$ divides α (in particular, $p \nmid m$). Then

$$-3 \equiv (4n^3/3m^3)^2 \bmod p,$$

and thus $\left(\frac{-3}{p}\right) = 1$; hence the product over $p^2 \mid \alpha$ in (3.6) is equal to 1. Using the notation of Proposition 3.1.5, it remains to see that $R(\alpha) = -1$. Since $\gcd(m,n) = 1$, we have $v_2(\alpha) = 4$ or 0, according to whether $2 \mid m$ or not. In either case, using Lemma 3.1.1, we see that

$$W_2(\alpha) \cdot \left(\frac{-1}{\alpha_2}\right) = 1 \quad \text{for all } \alpha.$$

Similarly, $v_3(\alpha) = 0$ or 3 according to whether $3 \nmid n$ or not. By Lemma 3.1.1 it also follows that

$$W_3(\alpha) \cdot (-1)^{v_3(\alpha)} = -1 \quad \text{for all } \alpha,$$

and hence $R(\alpha) = -1$, as desired.

The flipping technique of Corollary 3.1.6 thus cannot possibly work! Furthermore, assuming the parity conjecture, it follows that $E_{m,n}$ has even Mordell-Weil rank for all $[m:n] \in U$. In fact, we claim that *all but finitely many* fibers have even rank ≥ 2. To see this note the family contains the points

$$(-3m^2, 4n^3) \qquad \text{and} \qquad \left(\frac{9m^4}{4n^2}, \frac{27m^6}{8n^3} + 4n^3\right).$$

We can check that these points are independent on the fiber $[m:n] = [1:1]$, and thus they are independent as points on the generic fiber of $\mathcal{E}$. Then Silverman's Specialization Theorem [Sil94, Theorem 11.4] shows that the points are independent for all but finitely many pairs (m,n). Hence, rational points are Zariski dense on the original del Pezzo surface[2].

One might naively hope that whenever the sieving/flipping technique fails to reveal an infinite family of elliptic curves with negative root number one can find sections as in Example 3.4.1, which show that rational points are nevertheless Zariski dense on a diagonal del Pezzo surface of degree 1. We offer the following example as a challenge to that hope.

Example 3.4.2. Consider the del Pezzo surface of degree 1 given by

$$w^2 = z^3 + 6(27x^6 + y^6)$$

in $\mathbb{P}_{\mathbb{Q}}(1,1,2,3)$. The elliptic curve $E_{m,n}$ above a point $[m:n] \subseteq \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$ of the associated elliptic surface $\mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ is given by

$$E_{m,n}\colon \quad y^2 = x^3 + 6(27m^6 + n^6).$$

[2]In fact, this surface is not minimal. The two non-torsion sections of $\mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ correspond to exceptional curves on X that are defined over $\mathbb{Q}$. Contracting these curves gives a del Pezzo surface of degree 3 with a rational point. This surface is unirational by the Segre-Manin Theorem.

As in Example 3.4.1 we can show that whenever $E_{m,n}$ is smooth, $W(E_{m,n}) = +1$. However, we cannot find readily available sections.

The key point behind both of examples above is that condition (1.4) on the form $F_1(m,n)$ fails. The following lemma gives a necessary condition for the failure of (1.4) to occur, and suggests how to find the above examples.

Lemma 3.4.3. *Let $F_1(m,n) = Am^6 + Bn^6 \in \mathbb{Z}[m,n]$, and assume that $\gcd(A,B) = 1$. Write $F_1 = \prod_i f_i$, where the $f_i \in \mathbb{Z}[m,n]$ are irreducible homogeneous forms. Let μ_3 denote the group of third roots of unity. Then*

$$\mu_3 \subseteq \mathbb{Q}[t]/f_i(t,1) \text{ for all } i \implies 3A/B \text{ is a rational square.} \tag{3.20}$$

Proof. The proof is an exercise in Galois theory. We will prove the case where F_1 is irreducible to illustrate the method. Choose a sixth root ξ of $-B/A$ and an isomorphism $\mathbb{Q}[t]/(At^6 + B) \xrightarrow{\sim} \mathbb{Q}(\xi)$. Suppose that $\mathbb{Q}(\mu_3) \subseteq \mathbb{Q}(\xi)$, so that $\mathbb{Q}(\xi)/\mathbb{Q}$ is a Galois extension of degree 6. Its unique quadratic subextension is $\mathbb{Q}(\mu_3) = \mathbb{Q}(\sqrt{-3})$, hence

$$\xi^3 = a + b\sqrt{-3} \quad \text{for some } a, b \in \mathbb{Q}.$$

Squaring both sides of the above equation and rearranging we obtain

$$-B/A - a^2 + 3b^2 = 2ab\sqrt{-3}$$

so that $ab = 0$. Since $\xi^3 \notin \mathbb{Q}$, it follows that $a = 0$ and $B/A = 3b^2$. □

If $3A/B$ is a rational square, it is often the case that not all fibers of the associated elliptic surface have positive root number: the 2-adic and 3-adic part of Am^6+Bn^6 may vary enough to guarantee the existence of infinitely many fibers with root number -1. This idea, together with Theorem 1.5.3, are the necessary ingredients in the proof of Corollary 1.5.5.

Proof of Corollary 1.5.5. Let $F(x,y) = Ax^6+By^6$ and put $c = \gcd(A,B)$. Write $F_1(x,y) = A_1x^6 + B_1y^6$, where $cA_1 = A$ and $cB_1 = B$. One easily checks that F_1 has no fixed prime factors. Write $F_1 = \prod_i f_i$, where the $f_i \in \mathbb{Z}[x,y]$ are irreducible homogeneous forms. If $3A/B$ is not a rational square then it follows from Lemma 3.4.3 that

$$\mu_3 \nsubseteq \mathbb{Q}[t]/f_i(t,1) \qquad \text{for some } i,$$

so by Theorem 1.5.3, $X(\mathbb{Q})$ is Zariski dense in X.

If, on the other hand, $3A/B$ is a rational square, then by assumption $c = 1$ and $9 \nmid AB$. After possibly interchanging A and B, we may write $A = 3a^2$ and $B = b^2$ for some relatively prime $a, b \in \mathbb{Z}$ not divisible by 3. A smooth fiber above $[m : n] \in \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$ of the elliptic surface $\mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ associated to X is the plane curve

$$E_\alpha\colon \quad y^2 = x^3 + \alpha,$$

where $\alpha = 3a^2m^6 + b^2n^6$. Arguing as in Example 3.4.1 we see that the product over $p^2 \mid \alpha$ in (3.6) is equal to 1.

To conclude the proof, it suffices to show that there are infinitely many pairs (m, n) of relatively prime integers such that $R(\alpha) = 1$ (see Proposition 3.1.5 for the definition of $R(\alpha)$). Indeed, if there are infinitely many such pairs, then (subject to the hypothesis that Tate-Shafarevich groups are finite) it follows that for such pairs (m, n), the curve E_α has odd Mordell-Weil rank and hence rational points on X are Zariski dense. To construct such pairs (m, n), first suppose that $3 \mid n$ (whence $3 \nmid m$). Then $v_3(\alpha) = 1$ and $\alpha_3 \equiv 1 \bmod 3$, so by Lemma 3.1.1

$$W_3(\alpha) \cdot (-1)^{v_3(\alpha)} = (-1) \cdot (-1) = 1.$$

Next, we compute the product

$$w_2 := W_2(\alpha) \left(\frac{-1}{\alpha_2} \right).$$

We proceed by analyzing two cases, according to the 2-adic valuation of b, which we may assume is either $0, 1$ or 2. We use Lemma 3.1.1 to compute the local root number at 2:

1. $v_2(b) = 0$: choose n even. Then, regardless of the value of $v_2(a)$ (which we may also assume is $0, 1$ or 2), we obtain $v_2(\alpha)$ even and $\alpha_2 \equiv 3 \bmod 4$, whence $w_2 = 1$.

2. $v_2(b) = 1$ or 2: choose m odd, so that $v_2(\alpha) = 0$ and $\alpha_2 \equiv 3 \bmod 4$, whence $w_2 = 1$.

In any case, there are infinitely many pairs $(m, n) \in \mathbb{Z}^2$ such that $R(3a^2m^6 + b^2n^6) = 1$, as desired. □

Remark 3.4.4. If $3A/B$ is a rational square, and either $\gcd(A, B) \neq 1$ or if $9 \mid AB$ then it can happen that all the elliptic curves which are fibers of the rational surface associated to X have root number $+1$ (see Examples 3.4.1 and 3.4.2). Even when $9 \mid AB$ there are examples of surfaces, such as

$$w^2 = z^3 + 3^5x^6 + 2^4y^6,$$

where we were not able to find nontorsion sections.

3.5 Towards weak-weak approximation

It is currently not known whether del Pezzo surfaces of degree 1 are unirational. Theorem 1.4.5 gives us some hope that this might be the case. If so, then Conjecture 2.3.14 predicts that these surfaces (over global fields) satisfy weak-weak approximation. Following a suggestion of Colliot-Thélène, we use our sieving method to show that the surfaces of Theorems 1.5.3 and 1.5.4 satisfy a property that would be implied by weak-weak approximation.

Let X be a del Pezzo surface of degree 1 over a number field k. Let $\rho\colon \mathcal{E} \to \mathbb{P}^1_k$ be the elliptic surface obtained by blowing up the anticanonical point. Let $\mathcal{R}$ be the set of points $x \in \mathbb{P}^1_k(k)$ such that the fiber $\mathcal{E}_x = \rho^{-1}(x)$ is an elliptic curve of positive Mordell-Weil rank. As a surrogate for weak-weak approximation of X we might ask if there exists a finite set $P_0 \subseteq \Omega_k$ such that for every finite set $P \subseteq \Omega_k$ with $P \cap P_0 = \emptyset$ the image of the embedding

$$\mathcal{R} \hookrightarrow \prod_{v \in P} \mathbb{P}^1(k_v)$$

is dense for the product topology of the v-adic topologies. Problems of this nature are considered for general elliptic surfaces in [CTSSD98].

Theorem 3.5.1. *Let X be a del Pezzo surface of degree 1 over $\mathbb{Q}$ of the kind considered in either Theorem 1.5.3 or 1.5.4. Let $\rho\colon \mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ be the elliptic surface obtained by blowing up the anticanonical point of X, and let $\mathcal{R}$ be the set of points $x \in \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$ such that the fiber $\mathcal{E}_x = \rho^{-1}(x)$ an elliptic curve of positive Mordell-Weil rank. Finally, assume that Tate-Shafarevich groups of elliptic curves over $\mathbb{Q}$ with j-invariant 0 or 1728 are finite. Then there exists a finite set of primes P_0, containing the infinite prime, such that for every finite set of primes P with $P \cap P_0 = \emptyset$, the image of the embedding*

$$\mathcal{R} \hookrightarrow \prod_{p \in P} \mathbb{P}^1(\mathbb{Q}_p)$$

is dense for the product topology of the p-adic topologies.

Proof. We carry out the details for the case of a surface X as in Theorem 1.5.3, the other case being similar. The fiber of ρ above $[m : n] \in \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$ is isomorphic to the plane curve

$$y^2 = x^3 + F(m, n) \tag{3.21}$$

which is an elliptic curve for almost all $[m : n]$. As in Theorem 1.5.3, we write c for the content of F and $F_1(m,n) := (1/c)F(m,n)$. By Lemma 3.3.2, applied to a number field $N := \mathbb{Q}[t]/f_i(t,1)$ such that (1.4) holds, there is a rational prime $q \equiv 2 \bmod 3$ and a prime $\mathfrak{q}$ in N lying over q of degree 1 over $\mathbb{Q}$. We may assume that $q > 5$, $\gcd(c,q) = 1$, and that q does not divide the discriminant of $f_i(t,1)$. Write $c = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where the p_i are distinct primes. Let $P_0 = \{2, 3, 5, p_1, \cdots, p_r, q, \infty\}$.

Fix a finite set of distinct primes $P = \{q_1 \ldots, q_s\}$ such that $P \cap P_0 = \emptyset$, as well as a point $[m_p : n_p] \in \mathbb{P}^1(\mathbb{Q}_p)$ for each $p \in P$. We may assume that $m_p, n_p \in \mathbb{Z}_p$, and without loss of generality[3] we will further assume that $n_p \in \mathbb{Z}_p^\times$ for every $p \in P$. Let $\epsilon > 0$ be given and choose an integer N large so that

$$1/p^N < \epsilon \quad \text{and} \quad v_p(F_1(m_p, n_p)) < N \quad \text{for every } p \in P. \tag{3.22}$$

Let

$$S = (p_1, \ldots, p_r, q_1, \ldots, q_s), \quad T = \big(0, \ldots, 0, v_{q_1}(F_1(m_{q_1}, n_{q_1})), \ldots, v_{q_s}(F_1(m_{q_s}, n_{q_s}))\big),$$

and let

$$M = (2 \cdot 3 \cdot 5)^3 \cdot (p_1 \cdots p_r) \cdot (q_1 \cdots q_s)^N.$$

Since $F_1(m,n)$ has no fixed prime factors, for any prime $p \,|\, M$ such that $p \neq p_i$ for all i and $p \notin P$, there exist congruence classes a_p, b_p modulo p^2 such that

$$F_1(a_p, b_p) \not\equiv 0 \bmod p^2.$$

Similarly, for a prime p_i with $1 \leq i \leq r$, there exist congruence classes a_{p_i}, b_{p_i} modulo p_i such that

$$F_1(a_{p_i}, b_{p_i}) \not\equiv 0 \bmod p_i.$$

By the Chinese remainder theorem there exist congruence classes a, b modulo M such that

$$\begin{cases} a \equiv a_p \bmod p^2 & \text{for primes } p \text{ such that } p \,|\, M,\ p \neq p_i \text{ for all } i \text{ and } p \notin P, \\ a \equiv a_{p_i} \bmod p_i & \text{for primes } p_i \text{ with } 1 \leq i \leq r, \\ a \equiv m_p \bmod p^N & \text{for primes } p \in P. \end{cases} \tag{3.23}$$

[3] In fact, we may only really assume that either $m_p \in \mathbb{Z}_p^\times$ or $n_p \in \mathbb{Z}_p^\times$. We can interchange the roles of m_p and n_p in any one step of the proof without much difficulty, so the assumption that $n_p \in \mathbb{Z}_p^\times$ is an artifact to clean up the details of the proof.

and

$$\begin{cases} b \equiv b_p \bmod p^2 & \text{for primes } p \text{ such that } p \mid M,\ p \neq p_i \text{ for all } i \text{ and } p \notin P, \\ b \equiv b_{p_i} \bmod p_i & \text{for primes } p_i \text{ with } 1 \leq i \leq r, \\ b \equiv n_p \bmod p^N & \text{for primes } p \in P. \end{cases} \tag{3.24}$$

By construction,

$$F_1(a,b) \equiv F_1(m_p,n_p) \bmod p^N \quad \text{for all } p \in P.$$

It follows from (3.22) that

$$v_p(F_1(a,b)) = v_p(F_1(m_p,n_p)) \quad \text{for all } p \in P.$$

By Corollary 3.2.7, applied to F_1, S, T, M, a, b as above, there is an infinite set $\mathscr{F}_1$ of pairs $(m,n) \in \mathbb{Z}^2$ such that

$$F_1(m,n) = \ell,$$

where ℓ is a squarefree integer with $\gcd(c,\ell) = 1$, by our choice of S and T. Furthermore, the congruence class of ℓ modulo $2^3 \cdot 3^3$ is fixed (by our choice of M) and nonzero (because ℓ is squarefree). Thus, for $(m,n) \in \mathscr{F}_1$ we have

$$F(m,n) = c\ell \qquad \gcd(c,\ell) = 1,$$

and the congruence class of $c\ell/2^{v_2(c\ell)}3^{v_3(c\ell)}$ modulo $2^2 \cdot 3^2$ is fixed and nonzero.

We apply Corollary 3.2.7 again to $F_1(m,n)$. This time we let

$$S = (p_1, \ldots, p_r, q_1, \ldots, q_s, q), \;\; T = \big(0, \ldots, 0, v_{q_1}(F_1(m_{q_1}, n_{q_1})), \ldots, v_{q_s}(F_1(m_{q_s}, n_{q_s})), 2{+}6k\big),$$

where k is a large positive integer (large enough to ensure that $C \neq 0$ upon application of the sieve), and we let

$$M = (2 \cdot 3 \cdot 5)^3 \cdot (p_1 \cdots p_r) \cdot (q_1 \cdots q_s)^N \cdot q^{3+6k}.$$

Arguing as in the proof of Theorem 1.5.3, using Hensel's lemma and Lemma 3.3.2, we can show that there exist integers a_q, b_q such that

$$v_q(F_1(a_q,b_q)) = 2 + 6k$$

for some large positive integer k. By the Chinese remainder theorem, there exist congruence classes a, b modulo M such that (3.23), (3.24) hold, and in addition

$$a \equiv a_q \bmod q^{3+6k}, \quad \text{and } b \equiv b_q \bmod q^{3+6k}.$$

By Corollary 3.2.7 there is an infinite set $\mathscr{F}_2$ of pairs $(m, n) \in \mathbb{Z}^2$ such that

$$F_1(m, n) = q^{2+6k}\eta, \tag{3.25}$$

where η is a squarefree integer such that $\gcd(c, \eta) = \gcd(q, c\eta) = 1$ (by the choice of S and T). In summary, for $(m, n) \in \mathscr{F}_2$, we have

$$F(m, n) = cq^{2+6k}\eta \qquad \gcd(c, \eta) = \gcd(q, c\eta) = 1,$$

and the congruence class of $cq^{2+6k}\eta/2^{v_2(c\eta)} \bmod 2^2 \cdot 3^2$ is fixed, nonzero, and equal to that of $F_1(m, n)$ for $(m, n) \in \mathscr{F}_1$.

Whenever (3.21) is smooth, we write $W(F(m, n))$ for its root number. By Corollary 3.1.6, if $(m_1, n_1) \in \mathscr{F}_1$ and $(m_2, n_2) \in \mathscr{F}_2$, then

$$W(F(m_1, n_1)) = -W(F(m_2, n_2)).$$

Hence, there exists a pair $(m_0, n_0) \in \mathscr{F}_1 \cup \mathscr{F}_2$ such that $W(F(m_0, n_0)) = -1$. By the assumption that Tate-Shafarevich groups are finite we conclude that the fiber of ρ above $[m_0 : n_0]$ has positive Mordell-Weil rank, i.e., $[m_0 : n_0] \in \mathcal{R}$. By construction, $n_0 \neq 0$, and

$$m_0 \equiv m_p \bmod p^N, \quad \text{and} \quad n_0 \equiv n_p \bmod p^N \quad \text{for all } p \in P.$$

Hence

$$\left|\frac{m_p}{n_p} - \frac{m_0}{n_0}\right|_p = |m_p n_0 - m_0 n_p|_p \leq \frac{1}{p^N} < \epsilon \qquad \text{for all } p \in P,$$

and $[m_0 : n_0]$ is arbitrarily close to $[m_p : n_p]$ for all $p \in P$. This concludes the proof of the theorem. $\square$

Chapter 4

Weak approximation on del Pezzo surfaces of degree 1

In this chapter we study the question of weak approximation on diagonal del Pezzo surfaces of degree 1 over global fields. Our goal is to prove Theorem 1.5.7. In order to do this, we must understand the geometry of these del Pezzo surfaces in an explicit way. More precisely, we need equations for the 240 exceptional curves on a del Pezzo surface X of degree 1, and a concrete description of the Galois action on these curves. This is the subject matter of the first three sections. In §4.4, we explain how to use this knowledge to track down Azumaya algebras in $\operatorname{Br} X/\operatorname{Br} k$ that are amenable to computation, and we use these algebras to construct the counterexamples given in §4.5.

All computer calculations in this chapter were carried out using `Magma`; see [BCP97].

4.1 Exceptional curves on del Pezzo surfaces of degree 1

4.1.1 The Bertini involution

Let X be a del Pezzo surface of degree 1 given as a smooth sextic $V(f)$ in $\mathbb{P}_k(1,1,2,3)$. Write $f(x,y,z,w) = w^2 - aw - b$, where $a, b \in k[x,y,z]$ have degrees 3 and 6, respectively. If $\operatorname{char} k \neq 2$, then we may (and do) assume that $a = 0$ by making the change of variables $w \mapsto w + a/2$. The map

$$\psi\colon \mathbb{P}_k(1,1,2,3) \to \mathbb{P}_k(1,1,2,3), \quad [x:y:z:w] \mapsto [x:y:z:-w+a]$$

restricts to an automorphism of X called the Bertini involution; see [Dem80, p. 68].

Assume that k is algebraically closed, and let

$$\Gamma := V(z - Q(x,y), w - C(x,y)) \subseteq \mathbb{P}_k(1,1,2,3) = \operatorname{Proj} k[x,y,z,w],$$

where $Q(x,y)$ and $C(x,y)$ are homogeneous forms of degrees 2 and 3, respectively, in $k[x,y]$. Define Γ' as the image of Γ under ψ.

Lemma 4.1.1. *Let X be a del Pezzo surface of degree 1, given as a sextic hypersurface in $\mathbb{P}_k(1,1,2,3)$. If Γ is a divisor on X then so is Γ'; in this case $\Gamma \cap \Gamma'$ is finite, and $(\Gamma, \Gamma')_X = 3$.*

Proof. Assume that Γ is a divisor on X, in which case it is clear that Γ' is also a divisor on X. Assume first that $\operatorname{char} k \neq 2$. The defining ideal of the scheme $\Gamma \cap \Gamma'$ is

$$(z - Q(x,y), w - C(x,y), w + C(x,y)) = (z - Q(x,y), w, C(x,y));$$

it is now easy to see that $\Gamma \cap \Gamma'$ is finite. Now note that $(\Gamma, \Gamma')_X$ is equal to the degree of $\Gamma \cap \Gamma'$. We compute

$$\deg(\operatorname{Proj} k[x,y,z,w]/(z - Q, w, C)) = \deg(\operatorname{Proj} k[x,y]/(C)) = 3.$$

When $\operatorname{char} k = 2$, the ideal of $\Gamma \cap \Gamma'$ is $(z + Q(x,y), w + C(x,y), a)$. A calculation similar to the one above shows that $(\Gamma, \Gamma')_X = 3$. □

4.1.2 The bianticanonical map

Let X be a del Pezzo surface of degree 1 over k. The map

$$\phi_2 \colon X \to \mathbb{P}\big(\mathrm{H}^0\big(X, -2K_X\big)^*\big) = \mathbb{P}^3_k$$

is known as the bianticanonical map. If $X = V(f(x,y,z,w)) \subseteq \mathbb{P}_k(1,1,2,3)$, then the basis elements x^2, xy, y^2, z for $\mathrm{H}^0\big(X, -2K_X\big)$ are homogeneous coordinates for ϕ_2 (see §2.2.3). Let $T_0, \dots, T_3$ be coordinates for $\mathbb{P}^3_k$. The map ϕ_2 is 2-to-1 onto the quadric cone $\mathcal{Q} = V(T_0T_2 - T_1^2)$. This cone is in turn isomorphic to the space $\mathbb{P}_k(1,1,2)$ via the map

$$j \colon \mathbb{P}_k(1,1,2) \to \mathcal{Q}, \qquad [x:y:z] \mapsto [x^2 : xy : y^2 : z]. \tag{4.1}$$

The composition $j^{-1} \circ \phi_2 \colon X \to \mathbb{P}_k(1,1,2)$ is just the restriction to X of the natural projection $\mathbb{P}_k(1,1,2,3) \dashrightarrow \mathbb{P}_k(1,1,2)$. We fix the notation $\pi_2 := j^{-1} \circ \phi_2$ for future reference.

Lemma 4.1.2 ([CO99, Proposition 2.6(ii)]). *Let V denote the vertex of the cone $\mathcal{Q}$, and let Γ be an exceptional curve on X. Then $\phi_2|_\Gamma\colon \Gamma \to \phi_2(\Gamma)$ is 1-to-1 and $\phi_2(\Gamma)$ is a smooth conic, the intersection of $\mathcal{Q}$ with a hyperplane H that misses V.* $\square$

Remark 4.1.3. The image of the anticanonical point under ϕ_2 is $V \in \mathcal{Q}$. By Lemma 4.1.2, the anticanonical point does not lie on any exceptional curve of X.

4.1.3 Proof of Theorem 1.5.9

We may assume k is separably closed, since the exceptional curves of X are defined over k^s, by the results of §2.1. First, we show that any Γ as in the theorem is an exceptional curve by proving that it is irreducible and that $(\Gamma, K_X)_X = (\Gamma,\Gamma)_X = -1$ (the adjunction formula then shows that Γ has arithmetic genus 0; see [Ser88, IV.8, Proposition 5]). Note that $V(x) \in |-K_X|$. Hence

$$(\Gamma, -K_X)_X = \deg(\operatorname{Proj} k[x,y,z,w]/(z-Q, w-C, x)) = \deg(\operatorname{Proj} k[y]) = 1,$$

so Γ is irreducible, because $-K_X$ is ample. Let $D = V(z - Q(x,y)) \subseteq \mathbb{P}_k(1,1,2)$. Since D is isomorphic under (4.1) to a hyperplane section of the cone $\mathcal{Q}$, we have $\pi_2^*(D) \in |-2K_X|$, so $(\pi_2^*(D), \pi_2^*(D))_X = 4$. Define Γ' as the image of Γ under the Bertini involution.

By Lemma 4.1.1, Γ' is a divisor on X and $\Gamma \cap \Gamma'$ is finite; thus $\Gamma \neq \Gamma'$, and the divisor $\pi_2^*(D) - \Gamma - \Gamma'$ is effective. As above, we may show that $(\Gamma', -K_X) = 1$. Thus

$$(\pi_2^*(D) - \Gamma - \Gamma', -K_X)_X = (-2K_X - \Gamma - \Gamma', -K_X)_X = 2 - 1 - 1 = 0,$$

from which it follows that $\Gamma + \Gamma' = \pi_2^*(D)$, because $-K_X$ is ample; see [Deb01, Theorem 1.27]. By Lemma 4.1.1 and symmetry of the intersection form on X we know that $(\Gamma, \Gamma')_X = (\Gamma', \Gamma)_X = 3$. Thus

$$\begin{aligned} 4 = (\pi_2^*(D), \pi_2^*(D))_X &= (\Gamma + \Gamma', \Gamma + \Gamma')_X \\ &= (\Gamma,\Gamma)_X + 2(\Gamma,\Gamma')_X + (\Gamma',\Gamma')_X \\ &= (\Gamma,\Gamma)_X + (\Gamma',\Gamma')_X + 6. \end{aligned}$$

Since the Bertini involution preserves the intersection form on X (see §2.2.2) and interchanges Γ and Γ', we conclude that $(\Gamma,\Gamma)_X = -1$, and thus that Γ is an exceptional curve.

Now we prove the converse. Let Γ be an exceptional curve on X. By Lemma 4.1.2 we know that $\phi_2(\Gamma)$ is a smooth conic. It is isomorphic under the map j to the curve $\pi_2(\Gamma)$

in $\mathbb{P}_k(1,1,2)$. The equation for the conic in $\mathbb{P}_k(1,1,2)$ can be written as $z = Q(x,y)$, where $Q(x,y)$ is homogenous of degree 2 in $k[x,y]$ (the coefficient of z is nonzero because $\phi_2(\Gamma)$ misses the vertex V of the cone $\mathcal{Q}$).

Let $D = V(z - Q(x,y)) \subseteq \mathbb{P}_k(1,1,2)$, as before. We have shown that $\Gamma \subseteq \pi_2^*(D)$. Since $\pi_2^*(D) \in |-2K_X|$ as above, we have $(\pi_2^*(D), \Gamma)_X = 2$. If $\pi_2^*(D) = m\Gamma$ for some $m \geq 1$ then

$$2 = (\pi_2^*(D), \Gamma)_X = m(\Gamma, \Gamma)_X = -m,$$

a contradiction. Hence $\pi_2^*(D)$ is reducible, and $\pi_2^*(D) = \Gamma + \Gamma_1$ for some divisor $\Gamma_1 \neq \Gamma$. Note that

$$\begin{aligned}(\Gamma_1, \Gamma_1)_X &= (\pi_2^*(D) - \Gamma, \pi_2^*(D) - \Gamma)_X = (-2K_X - \Gamma, -2K_X - \Gamma)_X \\ &= 4(K_X, K_X)_X + 4(K_X, \Gamma)_X + (\Gamma, \Gamma)_X = 4 - 4 - 1 = -1.\end{aligned}$$

and similarly

$$\begin{aligned}(\Gamma_1, -K_X)_X &= (\pi_2^*(D) - \Gamma, -K_X)_X = (-2K_X - \Gamma, -K_X)_X \\ &= 2(K_X, K_X)_X + (\Gamma, K_X)_X = 2 - 1 = 1,\end{aligned}$$

so Γ_1 is an exceptional curve of X. We have

$$\pi_2^*(D) = V(f(x,y,z,w), z - Q(x,y)).$$

On the affine open subset where $x \neq 0$, the coordinate ring of $\pi_2^*(D)$ is

$$k[y,z,w]/(f(1,y,z,w), z - Q(1,y)) \cong k[y,w]/(f(1,y,Q(1,y),w)).$$

Since $\pi_2^*(D)$ is reducible, the polynomial $f(1,y,Q(1,y),w)$ must factor, and degree considerations force a factorization of the following form:

$$(w - C(1,y))(w - C'(1,y)),$$

where $C(x,y)$ and $C'(x,y)$ are homogeneous forms of degree 3. Hence Γ has the form we claimed. This concludes the proof of Theorem 1.5.9.

Remark 4.1.4. The divisor Γ_1 in the proof above is the image of Γ under the Bertini involution.

Remark 4.1.5. We have used several ideas from the proof of [CO99, Key-lemma 2.7] to prove Theorem 1.5.9. The theorem can also be deduced from the work of Shioda on rational elliptic surfaces $S \to \mathbb{P}^1$ (see [Shi90, Theorem 10.10]). Shioda shows that rational elliptic surfaces have at most 240 sections $\mathbb{P}^1 \to S$ of a particular form, whose description bears a striking resemblance to the divisors of the form Γ above. A rational elliptic surface (over an algebraically closed field) with exactly 240 of these special sections corresponds to the blow up of a del Pezzo surface X of degree 1 with center at the anticanonical point; the special sections of the elliptic surface are in one to one correspondence with the exceptional curves of X. Under this correspondence, Shioda's explicit description of the 240 sections becomes the explicit description of the exceptional curves of Theorem 1.5.9. Cragnolini and Oliverio have a somewhat different description of the exceptional curves on a del Pezzo surface of degree 1 [CO99, Key-lemma 2.7] (see also [Dem80, p. 68]).

Remark 4.1.6. Suppose that k is not separably closed. The Bertini involution interchanges Γ and Γ'; since it is defined over k we conclude that

$$^{\sigma}(\Gamma') = (^{\sigma}\Gamma)' \quad \text{for all } \sigma \in \mathrm{Gal}(k^s/k).$$

We will therefore use the unambiguous notation $^{\sigma}\Gamma'$ for this divisor.

4.2 Exceptional curves on diagonal surfaces

We begin by studying the particular surface Y given by the sextic $w^2 = z^3 + x^6 + y^6$ in $\mathbb{P}_k(1,1,2,3)$. Suppose first that $k = \overline{\mathbb{Q}}$. By Theorem 1.5.9, the exceptional curves on Y are given as $V(w - C(x,y), z - Q(x,y))$, where

$$C(x,y)^2 = Q(x,y)^3 + x^6 + y^6. \tag{4.2}$$

To compute the curves explicitly, let

$$\begin{aligned} Q(x,y) &= ax^2 + bxy + cy^2, \\ C(x,y) &= rx^3 + sx^2y + txy^2 + uy^3, \end{aligned}$$

but consider a, b, c, r, s, t and u as indeterminates. Substituting into (4.2) and comparing coefficients of the monomials $x^i y^j$ we find that a, b, c, r, s, t and u satisfy the system of equations:

$$\begin{aligned}
u^2 - c^3 - 1 &= 0\\
2tu - 3c^2b &= 0\\
2su + t^2 - 3ac^2 - 3cb^2 &= 0\\
2ru + 2st - 6acb - b^3 &= 0\\
2rt + s^2 - 3a^2c - 3ab^2 &= 0\\
2rs - 3a^2b &= 0\\
r^2 - a^3 - 1 &= 0
\end{aligned}$$

Let I be the ideal over the polynomial ring $\mathbb{Q}[a, b, c, r, s, t, u]$ corresponding to the above system of equations. Fix the lexicographic order $r > s > t > u > a > b > c$. Then, using the computer software package `Magma`, we find a Gröbner basis for I. The last element of this basis is the polynomial

$$c(c-2)(c+1)(c^2-c+1)(c^2+2c+4)(c^3+4)(c^3+6c^2+4)(c^6-6c^5+36c^4+8c^3-24c^2+16) \quad (4.3)$$

The roots of this polynomial are the possible values for c as a coefficient of y^2 in $Q(x, y)$. Each value of c can be substituted into the elements of the Gröbner basis for I to determine the corresponding possible values of b, a, u, t, s and r. An easy way to do this is to append one of the factors of (4.3) to I and recalculate the Gröbner basis for the appended ideal. For example, appending $c^3 + 4$ to I we obtain the Gröbner basis

$$\{6r + ub^3, 2s + uac^2, 2t + ubc^2, u^2 + 3, 12a^3 + b^6 + 12, 6ab + b^3c^2, b^7 + 108b, c^3 + 4\}$$

This gives us a few of the possible values for r, s, t, u, a, b and c. After repeating for all the factors of (4.3) we count exactly 240 tuples (r, s, t, u, a, b, c), as predicted by Theorem 1.5.9.

To do arithmetic on the surface Y we need to know the subfield of $\overline{\mathbb{Q}}$ over which the 240 exceptional curves are defined. The splitting field of the polynomial (4.3) is $\mathbb{Q}(\zeta, \sqrt[3]{2})$, where ζ is a primitive cube root of unity. It is somewhat surprising that *every root to every expression* in the ideal I is in fact contained in $\mathbb{Q}(\zeta, \sqrt[3]{2})$, This is easily checked using `Magma`. Consequently the image of the homomorphism ϕ_X of §2.2.2 is isomorphic to a subgroup of order 6 of $W(E_8)$. This will keep many subsequent computations reasonable.

If k is algebraically closed of characteristic 0 then the equations for the exceptional curves we calculated over $\overline{\mathbb{Q}}$ give exceptional curves over k via an embedding $\iota\colon \overline{\mathbb{Q}} \hookrightarrow k$.

Now suppose that k is algebraically closed of characteristic $p > 3$. Let $W(k)$ be the ring of Witt vectors of k, and let F be its field of fractions. Let $\mathcal{X}$ be the del Pezzo surface over $W(k)$ given by the equation $w^2 = z^3 + x^6 + y^6$ in $\mathbb{P}_{W(k)}(1, 1, 2, 3)$. The generic fiber of $\mathcal{X}$ is a del Pezzo surface over F. We may write down its 240 exceptional curves as above: even though F is not algebraically closed, we may embed $\mathbb{Q}(\zeta, \sqrt[3]{2})$ in it, and this is enough to write down equations for all the exceptional curves.

The usual specialization map $\theta\colon \operatorname{Pic} \mathcal{X}_F \to \operatorname{Pic} \mathcal{X}_k$ is a homomorphism, and it preserves the intersection pairings on $\operatorname{Pic} \mathcal{X}_F$ and $\operatorname{Pic} \mathcal{X}_k$; see [Ful98, §20.3]. It is injective because the pairing on $\operatorname{Pic} \mathcal{X}_F$ is nondegenerate. A standard computation shows that $\theta(K_{\mathcal{X}_F}) = K_{\mathcal{X}_k}$; see [Ful98, §20.3.1]. Hence θ maps exceptional curves to exceptional curves. The injectivity of θ, together with the fact that distinct exceptional curves have distinct classes in $\operatorname{Pic} \mathcal{X}_F$, shows that the 240 exceptional curves on $\mathcal{X}_F$ specialize to 240 *distinct* exceptional curves.

Let us drop the assumption that k is separably closed. We turn to the general diagonal surface X over k, given by $w^2 = z^3 + Ax^6 + By^6$. Fix a sixth root α of A, a sixth root β of B in k^s, and let $s = \sqrt[3]{2}$. If $\Gamma = V(z - Q(x, y), w - C(x, y))$ is an exceptional curve on $w^2 = z^3 + x^6 + y^6$, then $V(z - Q(\alpha x, \beta y), w - C(\alpha x, \beta y))$ is an exceptional curve on X, and vice versa. We deduce that the splitting field of X is contained in $k(\zeta, \sqrt[3]{2}, \alpha, \beta)$.

Proposition 4.2.1. *Let k be a field with* $\operatorname{char} p \neq 2, 3$*. Let X be the del Pezzo surface of degree 1 given by*

$$w^2 = z^3 + Ax^6 + By^6,$$

in $\mathbb{P}_k(1, 1, 2, 3)$. Then the splitting field of X is $K := k(\zeta, \sqrt[3]{2}, \alpha, \beta)$, and therefore $[K : K]$ divides 216.

Proof. Let L denote the splitting field of X. The above discussion shows that $L \subseteq K$. By Theorem 1.5.9, the subschemes of $\mathbb{P}_K(1, 1, 2, 3)$ given by

$$\begin{aligned}
&V(z - s\alpha\beta xy, w - \alpha^3x^3 - \beta^3y^3),\\
&V(z + s\zeta\alpha\beta xy, w - \alpha^3x^3 - \beta^3y^3),\\
&V(z + \alpha^2x^2 - s^2\zeta\beta^2y^2, w - s(\zeta + 1)\alpha^2\beta x^2y + (2\zeta - 1)\beta^3y^3) \text{ and}\\
&V(z - s^2\zeta\alpha^2x^2 + \beta^2y^2, w - (2\zeta - 1)\alpha^3x^3 + s(\zeta + 1)\alpha\beta^2xy^2)
\end{aligned}$$

are exceptional curves on X. By definition of L, we find that

$$S := \{s\alpha\beta, s\zeta\alpha\beta, s(\zeta+1)\alpha\beta^2, s(\zeta+1)\alpha^2\beta\} \subseteq L.$$

Taking the quotient of the second element of S by the first shows that $\zeta \in L$. Adding the first two elements of S we see that $s(\zeta+1)\alpha\beta \in L$, which shows that $s(\zeta+1)\alpha\beta^2/s(\zeta+1)\alpha\beta = \beta \in L$. Similarly $s(\zeta+1)\alpha^2\beta/s(\zeta+1)\alpha\beta = \alpha \in L$. Finally, we deduce that $s \in L$. This shows that $K \subseteq L$. $\square$

To end our discussion on exceptional curves on diagonal surfaces, we give generators for $\operatorname{Pic} X_{k^s}$ in terms of these curves. Consider the following exceptional curves on X:

$$\begin{aligned}
\Gamma_1 &= V(z + \alpha^2x^2, w - \beta^3y^3),\\
\Gamma_2 &= V(z - (-\zeta+1)\alpha^2x^2, w + \beta^3y^3),\\
\Gamma_3 &= V(z - \zeta\alpha^2x^2 + s^2\beta^2y^2, w - (s\zeta - 2s)\alpha^2\beta x^2y - (-2\zeta+1)\beta^3y^3),\\
\Gamma_4 &= V(z + 2\zeta\alpha^2x^2 - (2s\zeta - s)\alpha\beta xy - (-s^2\zeta + s^2)\beta^2y^2,\\
&\qquad w - 3\alpha^3x^3 - (-2s\zeta - 2s)\alpha^2\beta x^2y - 3s^2\zeta\alpha\beta^2xy^2 - (-2\zeta+1)\beta^3y^3),\\
\Gamma_5 &= V(z + 2\zeta\alpha^2x^2 - (s\zeta - 2s)\alpha\beta xy - s^2\zeta\beta^2y^2\\
&\qquad w + 3\alpha^3x^3 - (4s\zeta - 2s)\alpha^2\beta x^2y - 3s^2\alpha\beta^2xy^2 - (-2\zeta+1)\beta^3y^3),\\
\Gamma_6 &= V(z - (-s^2\zeta + s^2 - 2s + 2\zeta)\alpha^2x^2 - (2s^2\zeta - 2s^2 + 3s - 4\zeta)\alpha\beta xy\\
&\qquad\quad - (-s^2\zeta + s^2 - 2s + 2\zeta)\beta^2y^2,\\
&\qquad w - (2s^2\zeta - 4s^2 + 2s\zeta + 2s - 6\zeta + 3)\alpha^3x^3\\
&\qquad\quad - (-5s^2\zeta + 10s^2 - 6s\zeta - 6s + 16\zeta - 8)\alpha^2\beta x^2y\\
&\qquad\quad - (5s^2\zeta - 10s^2 + 6s\zeta + 6s - 16\zeta + 8)\alpha\beta^2xy^2\\
&\qquad\quad - (-2s^2\zeta + 4s^2 - 2s\zeta - 2s + 6\zeta - 3)\beta^3y^3),\\
\Gamma_7 &= V(z - (-s^2 - 2s\zeta + 2s + 2\zeta)\alpha^2x^2 - (-2s^2\zeta + 3s + 4\zeta - 4)\alpha\beta xy\\
&\qquad\quad - (-s^2\zeta + s^2 + 2s\zeta - 2)\beta^2y^2,\\
&\qquad w - (2s^2\zeta + 2s^2 + 2s\zeta - 4s - 6\zeta + 3)\alpha^3x^3\\
&\qquad\quad - (10s^2\zeta - 5s^2 - 6s\zeta - 6s - 8\zeta + 16)\alpha^2\beta x^2y\\
&\qquad\quad - (5s^2\zeta - 10s^2 - 12s\zeta + 6s + 8\zeta + 8)\alpha\beta xy^2\\
&\qquad\quad - (-2s^2\zeta - 2s^2 - 2s\zeta + 4s + 6\zeta - 3)\beta^3y^3),
\end{aligned}$$

$$\begin{aligned}\Gamma_8 = V(z &- (s^2\zeta + 2s\zeta + 2\zeta)\alpha^2x^2 - (2s^2 + 3s + 4)\alpha\beta xy \\ &- (-s^2\zeta + s^2 - 2s\zeta + 2s - 2\zeta + 2)\beta^2y^2, \\ w &- (-4s^2\zeta + 2s^2 - 4s\zeta + 2s - 6\zeta + 3)\alpha^3x^3 \\ &- (-5s^2\zeta - 5s^2 - 6s\zeta - 6s - 8\zeta - 8)\alpha^2\beta x^2y \\ &- (5s^2\zeta - 10s^2 + 6s\zeta - 12s + 8\zeta - 16)\alpha\beta^2xy^2 \\ &- (4s^2\zeta - 2s^2 + 4s\zeta - 2s + 6\zeta - 3)\beta^3y^3).\end{aligned}$$

A calculation shows that the above exceptional curves are all skew, that is, $(\Gamma_i, \Gamma_j)_X = 0$ for $i \neq j$; note that it is enough to do this calculation for $A = B = 1$. We will also need the exceptional curve

$$\Gamma_9 = V(z - st\alpha\beta xy, w - \alpha^3x^3 + \beta^3y^3).$$

The curve Γ_9 intersects each of Γ_1 and Γ_2 at exactly one point and is skew to all the other Γ_i.

Proposition 4.2.2. *Let X be the del Pezzo surface over k defined by*

$$w^2 = z^3 + Ax^6 + By^6,$$

in $\mathbb{P}_k(1,1,2,3)$, and let $K = k(\sqrt[3]{2}, \zeta, \alpha, \beta)$. Then $\operatorname{Pic} X_{k^s} = \operatorname{Pic} X_K$ is the free abelian group with the classes of Γ_i for $1 \leq i \leq 8$ and $\Gamma_9 + \Gamma_1 + \Gamma_2$ as a basis.

Proof. By Proposition 4.2.1 we know K is the splitting field of X. The classes of Γ_i for $1 \leq i \leq 8$ and $\Gamma_9 + \Gamma_1 + \Gamma_2$ generate a *unimodular* sublattice of $\operatorname{Pic} X_K$ of rank 9. Hence they span the whole lattice. □

4.3 Galois action on $\operatorname{Pic} X_K$

Suppose that $\sqrt[3]{2}, \zeta \notin k$; let $K = k(\zeta, \sqrt[3]{2}, \alpha, \beta)$, as above, and assume that $[K : k] = 216$. The action of $\operatorname{Gal}(k^s/k)$ on $\operatorname{Pic} X_{k^s}$ factors through the finite quotient $\operatorname{Gal}(K/k)$, which acts on the coefficients of the equations defining generators of $\operatorname{Pic} X_K$ (cf. §2.2.2). The group $\operatorname{Gal}(K/k)$ has 4 generators, which we will denote $\sigma, \tau, \iota_A, \iota_B$, whose action on the elements $\zeta, \sqrt[3]{2}, \alpha$ and β is recorded in Table 4.1. If $[K : k] < 216$ and if $\sqrt[3]{2} \in k$ (resp. $\zeta \in k$), then we do not need the generator σ (resp. τ).

Using the basis for $\operatorname{Pic} X_K$ of Proposition 4.2.2 we can write σ, τ, ι_A and ι_B as 9×9 matrices with integer entries. This 9-dimensional faithful representation is useful

	σ	τ	ι_A	ι_B
$\sqrt[3]{2}$	$-\zeta\sqrt[3]{2}$	$\sqrt[3]{2}$	$\sqrt[3]{2}$	$\sqrt[3]{2}$
ζ	ζ	ζ^{-1}	ζ	ζ
α	α	α	$\zeta\alpha$	α
β	β	β	β	$\zeta\beta$

Table 4.1: Action of the generators of $\mathrm{Gal}(K/k)$, assuming $\sqrt[3]{2}, \zeta \notin k$.

because the action of $\mathrm{Gal}(K/k)$ on $\mathrm{Pic}\, X_K$ becomes right matrix multiplication on the space of row vectors $\mathbb{Z}^9$. This description allows us to calculate the birational invariant $\mathrm{H}^1\big(\mathrm{Gal}(k^s/k), \mathrm{Pic}\, X_{k^s}\big)$, as follows.

Proof of Theorem 1.5.10. Assume first that $\sqrt[3]{2}, \zeta \notin k$. Then $G_0 := \langle \sigma, \tau, \iota_A, \iota_B \rangle \subseteq GL_9(\mathbb{Z})$ is isomorphic to the generic image of $\mathrm{Gal}(k^s/k)$ in $\mathrm{Aut}(\mathrm{Pic}\, X_{k^s})$ for a diagonal del Pezzo surface of degree 1. For a particular surface, a choice of sixth roots α and β of A and B, respectively, and a primitive third root of unity ζ gives a realization of $G := \mathrm{Gal}(K/k)$ as a subgroup of G_0, where $K = k(\zeta, \sqrt[3]{2}, \alpha, \beta)$.

We turn this idea around by focusing on the subgroup lattice of G_0. We use `Magma` to compute the first group cohomology (with coefficients in $\mathrm{Pic}\, X_K$) of subgroups in this lattice. We note there is no need to compute this cohomology group for every subgroup in the lattice. For example, any two subgroups of G_0 conjugate in $W(E_8)$ give rise to isomorphic cohomology groups. There are 448 conjugacy classes of subgroups of G_0 in $W(E_8)$.

We also note that in order for a subgroup $G \subseteq G_0$ to correspond to at least one diagonal del Pezzo surface of degree 1, it is necessary that the natural map $G \to G_0/\langle \iota_A, \iota_B \rangle$ be surjective because $k(\zeta, \sqrt[3]{2}) \subseteq K$. This cuts the number of conjugacy classes for which we need to compute group cohomology to 242.

Fix a subgroup $G \subseteq G_0$. For each exceptional curve Γ (given as a row vector in $\mathbb{Z}^9$, using Proposition 4.2.2) we may compute the orbit of Γ under the action of G. If there is a G-stable set of *skew* exceptional curves, then any surface X that has G for its image of $\mathrm{Gal}(k^s/k)$ in $\mathrm{Aut}(\mathrm{Pic}\, X_{k^s})$ is not minimal. Hence, we discard any such G. This way we get rid of 58 conjugacy classes of subgroups of G_0 and guarantee that surfaces we deal with in the rest of this chapter are minimal.

The above reductions cut the number of candidate groups for G to 184. The results of our computations are summarized in Table 4.2. For each abstract group $\operatorname{Br} X/\operatorname{Br} k$ we list the number $C(G)$ of conjugacy classes of subgroups of G_0 that give the listed cohomology group. We also give an example of a subgroup $G \subseteq G_0$ that has the given cohomology group, and a pair of elements $A, B \in k^*$ such that the surface X of the form (1.8) realizes G as a Galois group acting on $\operatorname{Pic} X_{\bar{k}}$. The elements A, B are defined in terms of any a, b, c and $d \in k^*$ satisfying the restrictions in the last column of the table. This shows all the possible cohomology groups *do* occur.

If $\sqrt[3]{2} \in k$ but $\zeta \notin k$ then we may repeat the above process starting with $G_0 = \langle \tau, \iota_A, \iota_B \rangle$. If $\zeta \in k$ but $\sqrt[3]{2} \notin k$ then we use $G_0 = \langle \sigma, \iota_A, \iota_B \rangle$. Finally, if $\zeta, \sqrt[3]{2} \in k$ then we use $G_0 = \langle \iota_A, \iota_B \rangle$. The results in these three cases are summarized in Table 4.2. $\square$

Remark 4.3.1. In [Cor07, Theorem 4.1] Corn determines the possible groups

$$\operatorname{Br} X/\operatorname{Br} k \cong \mathrm{H}^1\big(\operatorname{Gal}(\bar{k}/k), \operatorname{Pic} X_{\bar{k}}\big)$$

for all del Pezzo surfaces X over a number field k. In particular, Corn shows that the only primes that divide the order of this group are 2, 3 and 5, and the latter can occur only when X is of degree 1. Unfortunately, *diagonal* surfaces of degree 1 cannot be used to give examples of 5-torsion in $\operatorname{Br} X/\operatorname{Br} k$. This follows either from Theorem 1.5.10 or, more easily, from the isomorphism (4.6): the group $\mathrm{H}^1\big(\operatorname{Gal}(K/k), \operatorname{Pic} X_K\big)$ is annihilated by $[K : k]$, which divides 216, by Proposition 4.2.1.

4.3.1 An observation

Looking through our computations we observe that $\mathrm{H}^1\big(G_0, \operatorname{Pic} X_K\big) = 0$, regardless of whether the elements $\sqrt[3]{2}$ and ζ belong to k or not. This means that *generically there is no Brauer–Manin obstruction to weak approximation* on diagonal del Pezzo surfaces of degree 1 over a number field. If Conjecture 2.3.13 holds (even in some of the milder formulations discussed in §2.3.3), then our observation shows that, generically, diagonal del Pezzo surfaces of degree 1 over a number field satisfy weak approximation.

	$\operatorname{Br} X/\operatorname{Br} k$	$C(G)$	Example of G	A, B	Restrictions
$\sqrt[3]{2} \notin k$,	$\{1\}$	65	$\langle \sigma\iota_B^4, \tau, \iota_A^2 \rangle$	$a^2c^6, \pm 4d^6$	$a \notin \langle 2, k^{*3} \rangle$
$\zeta \notin k$	$\mathbb{Z}/2\mathbb{Z}$	18	$\langle \sigma\iota_A, \tau, \iota_B^3 \rangle$	$4a^3c^6, b^3d^6$	$a, b \notin \langle 2, -3, k^{*2} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^2$	9	$\langle \sigma, \tau, \iota_A^3\iota_B^3 \rangle$	a^3c^6, a^3d^6	$a \notin \langle 2, -3, k^{*2} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^3$	4	$\langle \sigma\iota_A^2, \tau, \iota_A^3\iota_B^3 \rangle$	$16a^3c^6, a^3d^6$	$a \notin \langle 2, -3, k^{*2} \rangle$
	$\mathbb{Z}/3\mathbb{Z}$	56	$\langle \sigma\iota_A\iota_B^2, \iota_A^3, \tau \rangle$	$4a^3c^6, \pm 16d^6$	$a \notin \langle -3, k^{*2} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^2$	26	$\langle \tau, \sigma\iota_A^2\iota_B^2 \rangle$	ac^6, ad^6	$a \in \pm 16k^{*6}$
	$\mathbb{Z}/6\mathbb{Z}$	6	$\langle \sigma\iota_A, \iota_A^3, \tau\iota_B \rangle$	$4a^3c^6, -3d^6$	$a \notin \langle 3, k^{*2} \rangle$
$\sqrt[3]{2} \in k$,	$\{1\}$	11	$\langle \tau, \iota_A\iota_B \rangle$	ac^6, ad^6	$a \notin \langle 3, k^{*2}, k^{*3} \rangle$
$\zeta \notin k$	$\mathbb{Z}/2\mathbb{Z}$	7	$\langle \tau, \iota_A\iota_B^3 \rangle$	ac^6, a^3d^6	$a \notin \langle 3, k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^2$	2	$\langle \tau, \iota_A\iota_B^5 \rangle$	ac^6, a^5d^6	$a \notin \langle 3, k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^3$	1	$\langle \tau, \iota_A^3, \iota_B^3 \rangle$	a^3c^6, b^3d^6	$a, b \notin \langle -3, k^{*2} \rangle$; $a \neq b$
	$(\mathbb{Z}/2\mathbb{Z})^4$	2	$\langle \tau, \iota_A^3\iota_B^3 \rangle$	a^3c^6, a^3d^6	$a \notin \langle -3, k^{*2} \rangle$
	$\mathbb{Z}/3\mathbb{Z}$	8	$\langle \tau, \iota_A^2\iota_B^5 \rangle$	a^2c^6, a^5d^6	$a \notin \langle 3, k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^2$	5	$\langle \tau, \iota_A^2\iota_B^2 \rangle$	a^2c^6, a^2d^6	$a \notin \langle 3, k^{*3} \rangle$
	$\mathbb{Z}/6\mathbb{Z}$	4	$\langle \tau, \iota_A \rangle$	ac^6, d^6	$a \notin \langle 3, k^{*2}, k^{*3} \rangle$
$\sqrt[3]{2} \notin k$,	$\{1\}$	26	$\langle \sigma\iota_A^2\iota_B^2, \iota_A^3, \iota_B^3 \rangle$	$16a^3c^6, 16b^3d^6$	$a, b \notin \langle 2, k^{*2} \rangle$; $a \neq b$
$\zeta \in k$	$(\mathbb{Z}/2\mathbb{Z})^2$	10	$\langle \sigma\iota_B^4, \iota_A^3\iota_B^3 \rangle$	$a^3c^6, 4a^3d^6$	$a \notin \langle 2, k^{*2} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^4$	6	$\langle \sigma, \iota_A^3\iota_B^3 \rangle$	a^3c^6, a^3d^6	$a \notin \langle 2, k^{*2} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^6$	2	$\langle \sigma\iota_B^2, \iota_A^3\iota_B^3 \rangle$	$a^3c^6, 16a^3d^6$	$a \notin \langle 2, k^{*2} \rangle$
	$\mathbb{Z}/3\mathbb{Z}$	16	$\langle \sigma\iota_A^2, \iota_A^5\iota_B^2 \rangle$	$16a^5c^6, a^2d^6$	$a \notin \langle 2, k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^2$	16	$\langle \sigma\iota_A\iota_B^2 \rangle$	$4a^3c^6, 16d^6$	$a \notin \langle 2, k^{*2} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^3$	4	$\langle \sigma\iota_B^2, \iota_A^2\iota_B^2 \rangle$	$a^2c^6, 16a^2d^6$	$a \notin \langle 2, k^{*3} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^4$	3	$\langle \sigma\iota_A^2\iota_B^2 \rangle$	$16c^6, 16d^6$	—
	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	2	$\langle \sigma, \iota_A \rangle$	a, d^6	$a \notin \langle 2, k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/6\mathbb{Z})^2$	2	$\langle \sigma\iota_B \rangle$	$c^6, 4b^3d^6$	$b \notin \langle 2, k^{*3} \rangle$
$\sqrt[3]{2} \in k$,	$\{1\}$	5	$\langle \iota_A\iota_B \rangle$	ac^6, ad^6	$a \notin \langle k^{*2}, k^{*3} \rangle$
$\zeta \in k$	$(\mathbb{Z}/2\mathbb{Z})^2$	5	$\langle \iota_A^3\iota_B \rangle$	a^3c^6, ad^6	$a \notin \langle k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^4$	1	$\langle \iota_A\iota_B^5 \rangle$	ac^6, a^5d^6	$a \notin \langle k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^6$	1	$\langle \iota_A^3, \iota_B^3 \rangle$	a^3b^6, b^3d^6	$a, b \notin k^{*2}$; $a \neq b$
	$(\mathbb{Z}/2\mathbb{Z})^8$	1	$\langle \iota_A^3\iota_B^3 \rangle$	a^3c^6, a^3d^6	$a \notin k^{*2}$
	$\mathbb{Z}/3\mathbb{Z}$	2	$\langle \iota_A, \iota_B^2 \rangle$	ac^6, b^2d^6	$a \notin \langle k^{*2}, k^{*3} \rangle$; $b \notin k^{*3}$
	$(\mathbb{Z}/3\mathbb{Z})^2$	3	$\langle \iota_A^5\iota_B^2 \rangle$	a^5c^6, a^2d^6	$a \notin \langle k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^4$	1	$\langle \iota_A^2\iota_B^2 \rangle$	a^2c^6, a^2d^6	$a \notin k^{*3}$
	$(\mathbb{Z}/6\mathbb{Z})^2$	2	$\langle \iota_B \rangle$	c^6, bd^6	$a \notin \langle k^{*2}, k^{*3} \rangle$

Table 4.2: Possible groups $\mathrm{H}^1(G, \operatorname{Pic} X)$. See the proof of Theorem 1.5.10 for an explanation.

4.4 Finding cyclic algebras in $\operatorname{Br} X$

Having computed the group $\mathrm{H}^1\big(\operatorname{Gal}(k^s/k, \operatorname{Pic} X_{k^s})\big)$ for diagonal del Pezzo surfaces of degree 1, it will be important for us to invert, as best we can, the isomorphism

$$\operatorname{Br} X/\operatorname{Br} k \xrightarrow{\sim} \mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big)$$

furnished by the Hochschild-Serre spectral sequence; see §2.3.4. This problem is quite hard in general. Instead, we present a simple strategy to search for cohomology classes in $\mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big)$ which correspond to *cyclic algebras* in the image of the natural map

$$\operatorname{Br} X/\operatorname{Br} k \to \operatorname{Br} k(X)/\operatorname{Br} k, \tag{4.4}$$

where X is a locally soluble smooth geometrically integral variety over a number field k. We are interested in cyclic algebras because their local invariants are relatively easy to compute. We note that the map 4.4 is injective whenever X is a regular, integral, quasi-compact scheme and $X(\mathbb{A}_k) \neq \emptyset$; see [Mil80, III.2.22].

4.4.1 Review of cyclic algebras

Let L/k be a finite cyclic degree-n extension of fields. Fix a generator σ of $\operatorname{Gal}(L/k)$. Let $L[x]_\sigma$ be the "twisted" polynomial ring, where $\ell x = x^\sigma(\ell)$ for all $\ell \in L$. Given $b \in k^*$, we construct the central simple k-algebra $L[x]_\sigma/(x^n - b)$. This algebra is usually denoted $(L/k, b)$; it depends on the choice of σ, though the notation does not show this. If X is a geometrically integral k-variety, then the cyclic algebra $(k(X_L)/k(X), f)$ is also denoted $(L/k, f)$; this should not cause confusion because $\operatorname{Gal}(k(X_L)/k(X)) \cong \operatorname{Gal}(L/k)$.

For a smooth variety X over a global field k, and a Galois extension L/k, we write $N_{L/k}\colon \operatorname{Div} X_L \to \operatorname{Div} X_k$ and $\overline{N}_{L/k}\colon \operatorname{Pic} X_L \to \operatorname{Pic} X_k$ for the usual norm maps, respectively. The following is a criterion for testing whether or not a cyclic algebra is in the image of the map $\operatorname{Br} X \to \operatorname{Br} k(X)$. For a proof, see [Cor05, Prop. 2.2.3] or [Bri02, Prop. 4.17].

Proposition 4.4.1. *Let X be a smooth, geometrically integral variety over a field k. Let L/k a finite cyclic extension and $f \in k(X)^*$. Then the cyclic algebra $(L/k, f)$ is in the image of the natural map $\operatorname{Br}(X) \to \operatorname{Br} k(X)$ if and only if $(f) = N_{L/k}(D)$, for some $D \in \operatorname{Div} X_L$. If, furthermore, X is locally soluble, then $(L/k, f)$ comes from $\operatorname{Br} k$ if and only if we can take D to be principal.* □

4.4.2 Cyclic Azumaya algebras

Let X be a smooth geometrically integral variety over a global field k. Assume that X is locally soluble, and let L/k be a cyclic extension. Define the set

$$\operatorname{Br}_{\mathrm{cyc}}(X, L) := \left\{ \begin{matrix} \text{classes } [(L/k, f)] \text{ in the image of the} \\ \text{map } \operatorname{Br} X/\operatorname{Br} k \to \operatorname{Br} k(X)/\operatorname{Br} k \end{matrix} \right\}$$

Lemma 4.4.2. *Viewing $\Delta := 1 - \sigma$ as an endomorphism of $\operatorname{Div} X_L$, we have $\ker N_{L/k} = \operatorname{im} \Delta$.*

Proof. By Tate cohomology we know that $\mathrm{H}^1\big(\operatorname{Gal}(L/k), \operatorname{Div} X_L\big) \cong \ker N_{L/k}/\operatorname{im}\Delta$. On the other hand, this cohomology group is trivial: $\operatorname{Div} X_L$ is a permutation module, so the result follows from Shapiro's Lemma. $\square$

The seeds behind the following theorem can already be found in [Bri02, §4.3.2, especially Lemma 4.18].

Theorem 4.4.3. *Let X be a k-variety as above. Let L/k be a cyclic degree-n extension, generated by σ, and view $\Delta = 1 - \sigma$ as an endomorphism of $\operatorname{Pic} X_L$. The map*

$$\psi\colon\ \ker \overline{N}_{L/k}/\operatorname{im}\Delta \to \operatorname{Br}_{\mathrm{cyc}}(X, L) \qquad [D] \mapsto [(L/k, f)],$$

where $f \in k(X)^*$ is any function such that $N_{L/k}(D) = (f)$, is a group isomorphism.

Proof. First we check that ψ is well-defined by showing that

(i) the class $[(L/k, f)]$ is independent of the choice of f: if $N_{L/k}(D) = (f) = (g)$, then $g = af$ for some $a \in k^*$. Since $(L/k, a) \in \operatorname{Br} k$, we obtain $[(L/k, f)] = [(L/k, g)]$.

(ii) if D and D' are linearly equivalent divisors in $\ker \overline{N}_{L/k}$, with $N_{L/k}(D) = (f)$ and $N_{L/k}(D') = (f')$, then $[(L/k, f)] = [(L/k, f')]$: Suppose that $D = D' + (h)$. Then $(f/f') = N_{L/k}((h))$, and by Proposition 4.4.1, we have $(L/k, f/f') \in \operatorname{Br}(k)$.

(iii) an element in $\operatorname{im}\Delta$ maps to zero: by (ii) it suffices to assume that D is of the form $E - {}^{\sigma}E$. Then $N_{L/k}(D) = 0$, so we can take $f \in k^*$ in the definition of ψ, in which case $(L/k, f) \in \operatorname{Br}(k)$.

If $N_{L/k}(D) = (f)$ and $N_{L/k}(D') = (g)$ then

$$\begin{aligned} \psi([D] + [D']) &= \psi([D + D']) = [(L/k, fg)] \\ &= [(L/k, f)] + [(L/k, g)] = \psi([D]) + \psi([D']), \end{aligned} \tag{4.5}$$

so ψ is a homomorphism. The map ψ is injective: if $\psi([D]) = [(L/k, f)]$ is 0 in $\operatorname{Br} k(X)/\operatorname{Br} k$, then by Proposition 4.4.1 there exists an $h \in k(X_L)^*$ such that $(f) = N_{L/k}((h))$. Hence $D - (h) \in \ker N_{L/k} = \operatorname{im}\Delta$ (see Lemma 4.4.2). Surjectivity also follows from Proposition 4.4.1: given a class $[(L/k, f)]$, take any divisor D such that $N_{L/k}(D) = (f)$; then $\psi([D]) = [(L/k, f)]$. $\square$

4.4.3 Cyclic algebras on rational surfaces

Let X be a nice locally soluble rational surface over a global field k, and let K be the splitting field of X. We saw in §2.3.4 that there exist isomorphisms

$$\mathrm{H}^1\big(\operatorname{Gal}(K/k), \operatorname{Pic} X_K\big) \xrightarrow{\text{inf}} \mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big) \xleftarrow{\sim} \operatorname{Br} X/\operatorname{Br} k, \tag{4.6}$$

where the map on the right comes from the Hochschild-Serre spectral sequence.

Let $G = \operatorname{Gal}(K/k)$ and suppose that H is a normal subgroup of G such that G/H is cyclic. Let L be the fixed field of H. Since $(\operatorname{Pic} X_K)^H \cong \operatorname{Pic} X_L$ (by (2.14)), we obtain an injection

$$\mathrm{H}^1\big(\operatorname{Gal}(L/k), \operatorname{Pic} X_L\big) \xrightarrow{\text{inf}} \mathrm{H}^1\big(G, \operatorname{Pic} X_K\big) \cong \operatorname{Br} X/\operatorname{Br} k. \tag{4.7}$$

On the other hand, by Tate cohomology we know that

$$\mathrm{H}^1\big(\operatorname{Gal}(L/k), \operatorname{Pic} X_L\big) \cong \ker \overline{N}_{L/k}/\operatorname{im}\Delta.$$

We can thus use Theorem 4.4.3 to write down cyclic algebras $(L/k, f)$ in the image of the injection $\operatorname{Br} X/\operatorname{Br} k \to \operatorname{Br} k(X)/\operatorname{Br} k$. Indeed, since G is finite, we may search through its subgroup lattice to find subgroups H as above, and hence write down all cyclic algebras coming from cyclic extensions of k (contained in K) in $\operatorname{Br} X/\operatorname{Br} k$. In summary, for each H we consider the diagram

$$\begin{tikzcd}
\operatorname{Br} X/\operatorname{Br} k \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{H}^1\big(\operatorname{Gal}(k^s/k), \operatorname{Pic} X_{k^s}\big) \\
\operatorname{Br} k(X)/\operatorname{Br} k & \mathrm{H}^1\big(\operatorname{Gal}(K/k), \operatorname{Pic} X_K\big) \arrow[u, "\text{inf}", "\sim"'] \\
 & \mathrm{H}^1\big(\operatorname{Gal}(L/k), \operatorname{Pic} X_L\big) \arrow[u, hook, "\text{inf}"] \arrow[d, "\sim"] \\
\operatorname{Br}_{\text{cyc}}(X, L) \arrow[uuu, hook] & \ker \overline{N}_{L/k}/\operatorname{im}\Delta \arrow[l, "\psi", "\sim"']
\end{tikzcd}$$

and we construct elements of $\operatorname{Br} X / \operatorname{Br} k$ by making the map ψ explicit. We believe the above diagram commutes (perhaps up to sign), but have not checked this, and we do not require to know this.

Remark 4.4.4. Finding Brauer-Manin obstructions to the *Hasse principle* on del Pezzo surfaces of degree greater than 1 may require the injection (4.7) to be an isomorphism. (This will be the case, for example, if $\mathrm{H}^1\big(H, \operatorname{Pic} X_K\big) = 0$). We may need representative Azumaya algebras for *every* class in $\operatorname{Br} X / \operatorname{Br} k$ to detect a Brauer-Manin obstruction (for example, see [Cor07, 9.4]). Obstructions to weak approximation require only one Azumaya algebra.

Remark 4.4.5. Let X be a diagonal del Pezzo surface of degree 1 over $\mathbb{Q}$ such that the order of $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$ is divisible by 3. Let K be the splitting field of X. Then an exhaustive computer search reveals that there does not exist a normal subgroup H of $G := \operatorname{Gal}(K/\mathbb{Q})$ such that $|G/H|$ is divisible by 3. This means that any counterexamples to weak approximation over $\mathbb{Q}$ we find using the above strategy will always arise from 2-torsion Azumaya algebras.

Remark 4.4.6. Not all Brauer-Manin obstructions on del Pezzo surfaces arise from cyclic algebras: for example, see [KT04, Example 8].

4.5 Counterexamples to Weak Approximation

4.5.1 A warm-up example

Let ζ be a primitive third root of unity. We begin with an counterexample to weak approximation over $k = \mathbb{Q}(\zeta)$ for which we do not need to use the descent procedure described in §2.3.5, and for which $\operatorname{Gal}(K/k)$ is small. The presence of an obstruction to weak approximation on it cannot be explained by a standard conic bundle structure (see Remark 4.5.2).

Proposition 4.5.1. *Let X be the del Pezzo surface of degree 1 over $k = \mathbb{Q}(\zeta)$ given by*

$$w^2 = z^3 + 16x^6 + 16y^6$$

in $\mathbb{P}_k(1,1,2,3)$. Then X is k-minimal and there is a Brauer-Manin obstruction to weak approximation on X. The obstruction arises from a cyclic algebra class in $\operatorname{Br} X / \operatorname{Br} k$.

Proof. Let $\alpha = \beta = \sqrt[3]{4}$. By Proposition 4.2.1, the exceptional curves of X are defined over $K := k(\sqrt[3]{2})$, and in the notation of §4.3 we have $G := \mathrm{Gal}(K/k) = \langle \rho \rangle$, where $\rho = \sigma \iota_A^2 \iota_B^2$. Since G is cyclic, we may apply the strategy of §4.4.3 by taking H to be the trivial subgroup (so $L = K$). Using the basis for $\mathrm{Pic}\, X_K \cong \mathbb{Z}^9$ of Proposition 4.2.2 we compute

$$\ker \overline{N}_{L/k} / \operatorname{im} \Delta \cong (\mathbb{Z}/3\mathbb{Z})^4;$$

see Table 4.2. The classes

$$\mathfrak{h}_1 = [(0,1,0,0,0,0,0,2,-1)], \qquad \mathfrak{h}_2 = [(0,0,0,0,1,0,0,2,-1)],$$
$$\mathfrak{h}_3 = [(0,0,0,0,0,0,1,2,-1)], \qquad \mathfrak{h}_4 = [(0,0,0,0,0,0,0,3,-1)]$$

of $\mathrm{Pic}\, X_K$ determine generators for this group.

Consider the divisor class $\mathfrak{h}_1 - \mathfrak{h}_2 = [\Gamma_2 - \Gamma_5] \in \mathrm{Pic}\, X_K$. By Theorem 4.4.3, this class gives a cyclic algebra $(K/k, f)$ in the image of the map $\mathrm{Br}\, X/\mathrm{Br}\, k \to \mathrm{Br}\, k(X)/\mathrm{Br}\, k$, where $f \in k(X)^*$ is any function such that $N_{K/k}(\Gamma_2 - \Gamma_5) = (f)$, that is, a function with zeroes along $\Gamma_2 + {}^{\rho}\Gamma_2 + {}^{\rho^2}\Gamma_2$ and poles along $\Gamma_5 + {}^{\rho}\Gamma_5 + {}^{\rho^2}\Gamma_5$. Using the explicit equations for Γ_2 in §4.2 we see that the polynomial $w + 4y^3$ vanishes along $\Gamma_2 + {}^{\rho}\Gamma_2 + {}^{\rho^2}\Gamma_2$.

Let I be the ideal of functions that vanish on $\Gamma_5, {}^{\rho}\Gamma_5$ and ${}^{\rho^2}\Gamma_5$. Explicitly,

$$I = (z - Q_5, w - C_5) \cap (z - {}^{\rho}Q_5, w - {}^{\rho}C_5) \cap (z - {}^{\rho^2}Q_5, w - {}^{\rho^2}C_5),$$

where Q_5 and C_5 are the quadratic and cubic forms, respectively, corresponding to Γ_5, and, for example, ${}^{\rho}Q_5$ is the result of applying ρ to the coefficients of Q_5. We compute a Gröbner basis for I (under the lexicographic order $w > z > y > x$) and find the polynomial $w + (2\zeta + 2)zy + (-8\zeta + 4)y^3 + 12x^3$ in this basis. Hence

$$f := \frac{w + 4y^3}{w + (2\zeta + 2)zy + (-8\zeta + 4)y^3 + 12x^3}$$

has the required zeroes and poles.

Consider the following rational points of X:

$$P_1 = [1 : 0 : 0 : 4] \quad \text{and} \quad P_2 = [0 : 1 : 0 : 4].$$

Let $\mathscr{A}$ be the Azumaya algebra of X corresponding to $(K/k, f)$. Specializing the algebra $\mathscr{A}$ at P_1 we obtain the cyclic algebra $\mathscr{A}(P_1) = (K/k, 1/4)$ over k. On the other hand, specializing at P_2 we compute $\mathscr{A}(P_2) = (K/k, 1/(1-\zeta)) = (K/k, \zeta)$.

Let $\mathfrak{p}$ be the unique prime above 3 in k. To compute the invariants we observe that

$$\mathrm{inv}_{\mathfrak{p}}(\mathscr{A}(P_i)) = \frac{1}{3}[f(P_i), 2]_{\mathfrak{p}} \in \mathbb{Q}/\mathbb{Z},$$

where $[f(P_i)_{\mathfrak{p}}, 2]_{\mathfrak{p}} \in \mathbb{Z}/3\mathbb{Z}$ is the (additive) norm residue symbol. We compute $[1/4, 2]_{\mathfrak{p}} \equiv 0 \bmod 3$ (using [CTKS87, (77)]) and $[\zeta, 2]_{\mathfrak{p}} \equiv 1 \bmod 3$ (using biadditivity of the norm residue symbol and [CTKS87, (75)] with $\theta = -\zeta$, $a = 1$). Let $P \in X(\mathbb{A}_k)$ be the point that is equal to P_1 at all places except $\mathfrak{p}$, and is P_2 at $\mathfrak{p}$. Then

$$\sum_v \mathrm{inv}_v(\mathscr{A}(P_v)) = 1/3,$$

so $P \in X(\mathbb{A}_k) \setminus X(\mathbb{A}_k)^{\mathrm{Br}}$ and X is a counterexample to weak approximation.

To see that X is k-minimal, see the proof of Theorem 1.5.10: the surface X appears as the example in the twelfth line from the bottom of Table 4.2. □

Remark 4.5.2. The surface X of Proposition 4.5.1 is not birational to a conic bundle C, since the birational invariant $\mathrm{Br}\, X/\,\mathrm{Br}\, k$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^4$, while $\mathrm{Br}\, C/\,\mathrm{Br}\, k$ is always 2-torsion. In particular, the failure of weak approximation cannot be accounted for by the presence of a rational conic bundle structure.

4.5.2 Main Counterexamples

Proof of Theorem 1.5.7. Let $\alpha = \beta = \sqrt{p}$. By Proposition 4.2.1, the exceptional curves of X are defined over $K := \mathbb{Q}(\zeta, \sqrt[3]{2}, \sqrt{p})$, and in the notation of §4.3 we have $G := \mathrm{Gal}(K/\mathbb{Q}) = \langle \sigma, \tau, \iota_A^3 \iota_B^3 \rangle$. One easily checks that the element $\rho := \iota_A^3 \iota_B^3$ acts on exceptional curves as the Bertini involution of the surface (see §4.1.1).

The subgroup $H := \langle \sigma, \tau \rangle$ of G has index 2; hence it is normal and G/H is cyclic. Thus, we are in the situation described in §4.4.3, that is,

$$H^1(\mathrm{Gal}(L/\mathbb{Q}), \mathrm{Pic}\, X_L) \hookrightarrow \mathrm{Br}\, X/\,\mathrm{Br}\, \mathbb{Q},$$

where $L = K^H$ is $\mathbb{Q}(\sqrt{p})$ in this case. The injection is in fact an isomorphism because $H^1(H, \mathrm{Pic}\, X_K) = 0$, though we will not use this fact[1]. Using the basis for $\mathrm{Pic}\, X_K \cong \mathbb{Z}^9$ of Proposition 4.2.2 we compute

$$\ker \overline{N}_{L/k}/\,\mathrm{im}\, \Delta \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

[1]The isomorphism follows from the inflation-restriction exact sequence and (4.7).

The classes

$$\mathfrak{h}_1 = [(2,1,1,1,1,0,1,2,-3)] \qquad \text{and} \qquad \mathfrak{h}_2 = [(0,0,0,0,0,1,0,-1,0)] \tag{4.8}$$

of $\operatorname{Pic} X_L$ generate this group. In fact, $\operatorname{Pic} X_L \cong \mathbb{Z}^3$, generated by the classes (4.8) and the anticanonical class, and ρ interchanges the classes (4.8). It follows that $(\operatorname{Pic} X_L)^{\operatorname{Gal}(L/\mathbb{Q})} = \mathbb{Z}$, generated by the anticanonical class. By (2.14), we obtain $\operatorname{Pic} X \cong \mathbb{Z}$, and thus X is minimal.

Next, we apply the procedure of §2.3.5 to descend the line bundle $\mathscr{O}_{X_K}(\Gamma_6 - \Gamma_8)$ in the class of $\mathfrak{h}_2$ to a line bundle defined over $\mathbb{Q}(\sqrt{p})$. We must give isomorphisms

$$f_h : \mathscr{O}_{X_K}(\Gamma_6 - \Gamma_8) \to \mathscr{O}_{X_K}({}^h\Gamma_6 - {}^h\Gamma_8),$$

one for each $h \in H$, satisfying the cocycle condition. In this case H is isomorphic to the symmetric group on 3 elements, with presentation

$$H = \langle \sigma, \tau \,|\, \sigma^3 = \tau^2 = 1, \sigma\tau = \tau\sigma^2 \rangle,$$

so it is enough to find isomorphisms f_σ and f_τ as above such that

$$\begin{aligned} {}^{\sigma^2}f_\sigma \circ {}^{\sigma}f_\sigma \circ f_\sigma &= id, \\ {}^{\tau}f_\tau \circ f_\tau &= id, \\ {}^{\sigma}f_\tau \circ f_\sigma &= {}^{\tau\sigma}f_\sigma \circ {}^{\tau}f_\sigma \circ f_\tau. \end{aligned}$$

For example, the map f_σ is just multiplication by a function having zeroes at Γ_6 and ${}^\sigma\Gamma_8$ and poles at Γ_8 and ${}^\sigma\Gamma_6$. We also denote this function f_σ, and find it as follows. First, take a function that vanishes on Γ_8, ${}^\sigma\Gamma_6$, and possibly some extra lines. For example, recall that

$$\Gamma_6 = V(z - Q_6(x,y), w - C_6(x,y)), \quad \text{and} \quad \Gamma_8 = V(z - Q_8(x,y), w - C_8(x,y)),$$

where Q_6 and Q_8 (resp C_6 and C_8) are the quadratic (resp. cubic) forms in x and y, corresponding to Γ_6 and Γ_8 given in §4.2. Let ${}^\sigma Q_6$ denote the result of applying σ to the coefficients of Q_6, and similarly for the other binary and cubic forms. The function

$$g_1 = (z - {}^\sigma Q_6(x,y))(z - Q_8(x,y))$$

vanishes on the exceptional curves[2] ${}^\sigma\Gamma_6$, ${}^\sigma\Gamma_6'$, Γ_8 and Γ_8'. Let I be the ideal of functions that vanish on Γ_6, ${}^\sigma\Gamma_8$, ${}^\sigma\Gamma_6'$ and Γ_8'. Explicitly,

$$I = (z - Q_6, w - C_6) \cap (z - {}^\sigma Q_8, w - {}^\sigma C_8) \cap (z - {}^\sigma Q_6, w + {}^\sigma C_6) \cap (z - Q_8, w + C_8).$$

[2]The notation ${}^\sigma\Gamma_6'$ is unambiguous (cf. Remark 4.1.6).

We compute a Gröbner basis for I (under the lexicographic order $w > z > y > x$) and find the following degree 4 polynomial in the basis:

$$\begin{aligned} f_1 = {} & 6\sqrt{p}wy + 3\sqrt{p}(\zeta-1)(s^2+2)wx + (-2\zeta+1)sz^2 + 2p(2\zeta-1)(s^2+s+1)zy^2 \\ & + p(-\zeta-1)(3s^2+2s+2)zyx + 2p(-\zeta+2)(s^2+s+1)zx^2 + 2p^2(2\zeta-1)(s^2+1)y^4 \\ & + p^2(-\zeta-1)(3s^2+2s+2)y^3x + 2p^2(-\zeta+2)(s^2+s+1)y^2x^2 \\ & + 2p^2(2\zeta-1)(s+1)yx^3 + p^2(\zeta+1)(s^2-2)x^4. \end{aligned}$$

The function f_1/g_1 has the right zeroes and poles to be f_σ. We set

$$f_\sigma := \frac{1}{(-2\zeta+1)s} \cdot \frac{f_1}{g_1}.$$

The constant in front of f_1/g_1 is a normalization factor, making $f_\sigma([0:0:1:1]) = 1$.

Similarly, f_τ denotes a function with zeroes at Γ_6 and ${}^\tau\Gamma_8$ and poles at Γ_8 and ${}^\tau\Gamma_6$. Let

$$\begin{aligned} g_2 = {} & (z - {}^\tau Q_6(x,y))(z - Q_8(x,y)), \\ f_2 = {} & 6\sqrt{p}wy - \sqrt{p}wx + (-2\zeta+1)kz^2 + 2p(2\zeta-1)(k^2+k+1)zy^2 \\ & + 2p(2\zeta-1)(k+1)zyx + 2p(2\zeta-1)(k^2+k+1)zx^2 + 2p^2(2\zeta-1)(k^2+1)y^4 \\ & + 2p^2(2\zeta-1)(k+1)y^3x + 2p^2(2\zeta-1)(k^2+k+1)y^2x^2 + 2p^2(2\zeta-1)(k+1)yx^3 \\ & + 2p^2(2\zeta-1)(k^2+1)x^4. \end{aligned}$$

Then the function

$$f_\tau := \frac{1}{(-2\zeta+1)k} \cdot \frac{f_2}{g_2}$$

has zeroes at Γ_6 and ${}^\tau\Gamma_8$ and poles at Γ_8 and ${}^\tau\Gamma_6$. Because of the normalization, f_τ and f_σ satisfy the cocycle condition. Thus $\mathscr{O}_{X_K}(\Gamma_6 - \Gamma_8)$ descends to a line bundle $\mathscr{F}$ over L, as we expected. It remains to find a divisor over L in the class of $\mathscr{F}$. To this end, we average the rational section 1 of $\mathscr{O}_{X_K}(\Gamma_6 - \Gamma_8)$ over the group H to obtain a rational section

$$\mathfrak{s} = \sum_{h \in H} {}^{h^{-1}}(f_h) = 1 + {}^{\sigma^2}f_\sigma + {}^\tau f_\tau + {}^{\sigma^2}f_\sigma \cdot {}^\sigma f_\sigma + {}^{\sigma^2}f_\sigma \cdot {}^\sigma f_\sigma \cdot {}^{\sigma\tau}f_\tau + {}^{\sigma^2}f_\sigma \cdot {}^{\tau\sigma}f_\tau$$

of $\mathscr{F}$. The common denominator of $\mathfrak{s}$ is

$${}^{\sigma^2}g_1 \cdot {}^\tau g_2 \cdot {}^\sigma g_1 \cdot {}^{\sigma\tau}g_2 \cdot {}^{\tau\sigma}g_2.$$

By definition of g_1 and g_2 this denominator vanishes along the divisor

$$\begin{aligned}2\Gamma_6 + 2\Gamma_6' + {}^{\sigma^2}\Gamma_8 + {}^{\sigma^2}\Gamma_8' + {}^{\tau}\Gamma_8 + {}^{\tau}\Gamma_8' + 2\big({}^{\sigma^2}\Gamma_6\big) + 2\big({}^{\sigma^2}\Gamma_6'\big)\\ + {}^{\sigma}\Gamma_8 + {}^{\sigma}\Gamma_8' + {}^{\sigma\tau}\Gamma_8 + {}^{\sigma\tau}\Gamma_8' + {}^{\sigma}\Gamma_6 + {}^{\sigma}\Gamma_6' + {}^{\tau\sigma}\Gamma_8 + {}^{\tau\sigma}\Gamma_8'.\end{aligned}$$

Here $2\Gamma_6$ means, for example, that the denominator vanishes on this curve to order 2. The numerator of $\mathfrak{s}$ vanishes along the divisor

$$\Gamma_6 + 2\Gamma_6' + {}^{\sigma^2}\Gamma_8' + {}^{\tau}\Gamma_8' + 2\big({}^{\sigma^2}\Gamma_6\big) + 2\big({}^{\sigma^2}\Gamma_6'\big) + {}^{\sigma}\Gamma_8' + {}^{\sigma\tau}\Gamma_8' + {}^{\sigma}\Gamma_6 + {}^{\sigma}\Gamma_6' + {}^{\tau\sigma}\Gamma_8' + Z,$$

where Z is some curve on X_L. Thus, the divisor of $\mathfrak{s}$ as a rational function is $Z - P$, where

$$P := \Gamma_6 + {}^{\sigma^2}\Gamma_8 + {}^{\tau}\Gamma_8 + {}^{\sigma\tau}\Gamma_8 + {}^{\sigma}\Gamma_8 + {}^{\tau\sigma}\Gamma_8$$

The divisor of $\mathfrak{s}$ as a *rational section* of $\mathscr{O}_{X_K}(\Gamma_6 - \Gamma_8)$, is $(Z - P) - (\Gamma_6 - \Gamma_8)$. Let

$$P' := \Gamma_8 + {}^{\sigma^2}\Gamma_8 + {}^{\tau}\Gamma_8 + {}^{\sigma\tau}\Gamma_8 + {}^{\sigma}\Gamma_8 + {}^{\tau\sigma}\Gamma_8 = \sum_{h\in H} {}^{h}\Gamma_8;$$

the divisor $Z - P' \in \operatorname{Div} X_L$ represents the class of $\Gamma_6 - \Gamma_8$.

Let $\bar{\rho}$ be the image of ρ in $\operatorname{Gal}(L/\mathbb{Q})$. By Theorem 4.4.3, the class $[Z - P']$ gives a cyclic algebra $(\mathbb{Q}(\sqrt{p})/\mathbb{Q}, f)$ in $\operatorname{Br} X/\operatorname{Br}\mathbb{Q}$, where $f \in \mathbb{Q}(X)^*$ is any function such that

$$N_{\mathbb{Q}(\sqrt{p})/\mathbb{Q}}(Z - P') = Z + {}^{\bar{\rho}}Z - (P' + {}^{\bar{\rho}}P') = (f).$$

We find an explicit f. The numerator of $\mathfrak{s}$ (after cancelling out common divisors) is a polynomial of degree 12 in $\mathbb{Q}(\sqrt[3]{2}, \zeta, \sqrt{p})[x, y, z, w]$. We may express it as

$$p_1 + \sqrt[3]{2}p_2 + \sqrt[3]{4}p_3 + \zeta p_4 + \zeta\sqrt[3]{2}p_5 + \zeta\sqrt[3]{4}p_6,$$

where $p_i \in \mathbb{Q}(\sqrt{p})[x, y, z, w]$ for $i = 1, \dots, 6$. Then $Z = V(p_1, \dots, p_6)$. We find constants $b_i \in \mathbb{Q}(\sqrt{p})$ such that the polynomial $q = \sum_i b_i p_i$ belongs to $\mathbb{Q}[x, y, z, w]$; then the polynomial q vanishes on $Z \cup {}^{\bar{\rho}}Z$ and is a suitable numerator for f. A little linear algebra reveals that

$$\begin{aligned}q = 12z^6 &- 72pz^5y^2 - 192pz^5yx - 48pz^5x^2 + 300p^2z^4y^4 + 600p^2z^4y^3x + 576p^2z^4y^2x^2\\&+ 408p^2z^4yx^3 + 156p^2z^4x^4 - 288p^3z^3y^6 - 720p^3z^3y^5x - 888p^3z^3y^4x^2 - 768p^3z^3y^3x^3\\&- 756p^3z^3y^2x^4 - 264p^3z^3yx^5 - 204p^3z^3x^6 + 144p^4z^2y^8 + 456p^4z^2y^7x\\&+ 1032p^4z^2y^6x^2 + 1080p^4z^2y^5x^3 + 756p^4z^2y^4x^4 + 864p^4z^2y^3x^5 + 684p^4z^2y^2x^6\\&+ 456p^4z^2yx^7 - 48p^4z^2x^8 + 192p^5zy^{10} - 48p^5zy^9x - 720p^5zy^8x^2 - 1104p^5zy^7x^3\\&- 600p^5zy^6x^4 - 216p^5zy^5x^5 - 240p^5zy^4x^6 - 480p^5zy^3x^7 - 504p^5zy^2x^8 - 24p^5zyx^9\\&+ 48p^5zx^{10} - 192p^6y^{12} - 288p^6y^{11}x + 192p^6y^{10}x^2 + 528p^6y^9x^3 + 432p^6y^8x^4\\&+ 168p^6y^7x^5 - 192p^6y^6x^6 - 288p^6y^5x^7 + 192p^6y^4x^8 + 312p^6y^3x^9 - 48p^6yx^{11}\end{aligned}$$

works. Now we look for a polynomial r of the same degree as q vanishing on $P' + {}^{\bar\rho}P'$. Since ρ acts as the Bertini involution $\Gamma \mapsto \Gamma'$ on exceptional curves, we have

$$P' + {}^{\bar\rho}P' = \sum_{h\in H} {}^h(\Gamma_8 + \Gamma_8').$$

The polynomial $z - Q_8(x, y)$ vanishes on $\Gamma_8 + \Gamma_8'$. Hence we may take

$$r = \prod_{h\in H} (z - {}^hQ_8(x, y)),$$

and obtain

$$\begin{aligned}r = z^6 &- 6pz^5y^2 - 24pz^5yx - 6pz^5x^2 + 36p^2z^4y^4 + 78p^2z^4y^3x + 132p^2z^4y^2x^2\\&+ 78p^2z^4yx^3 + 36p^2z^4x^4 + 8p^3z^3y^6 - 60p^3z^3y^5x - 168p^3z^3y^4x^2 - 276p^3z^3y^3x^3\\&- 168p^3z^3y^2x^4 - 60p^3z^3yx^5 + 8p^3z^3x^6 - 24p^4z^2y^8 - 24p^4z^2y^7x + 156p^4z^2y^6x^2\\&+ 396p^4z^2y^5x^3 + 540p^4z^2y^4x^4 + 396p^4z^2y^3x^5 + 156p^4z^2y^2x^6 - 24p^4z^2yx^7\\&- 24p^4z^2x^8 + 24p^5zy^9x + 24p^5zy^8x^2 - 120p^5zy^7x^3 - 324p^5zy^6x^4 - 432p^5zy^5x^5\\&- 324p^5zy^4x^6 - 120p^5zy^3x^7 + 24p^5zy^2x^8 + 24p^5zyx^9 + 16p^6y^{12} + 48p^6y^{11}x\\&+ 48p^6y^{10}x^2 + 48p^6y^9x^3 + 120p^6y^8x^4 + 192p^6y^7x^5 + 212p^6y^6x^6 + 192p^6y^5x^7\\&+ 120p^6y^4x^8 + 48p^6y^3x^9 + 48p^6y^2x^{10} + 48p^6yx^{11} + 16p^6x^{12}.\end{aligned}$$

Let $f = q/r$ and let $\mathscr{A}$ denote the Azumaya algebra on X corresponding to $(L/\mathbb{Q}, f)$. There are two obvious rational points on the surface X other than the anticanonical point, namely,

$$P_1 = [1 : 0 : -p : 0] \quad \text{and} \quad P_2 = [0 : 1 : -p : 0].$$

Specializing the algebra $\mathscr{A}$ at P_1 we obtain the quaternion algebra $(p, 12) \cong (p, 3)$ over $\mathbb{Q}$. The invariant of this algebra at a prime q is calculated using the Hilbert symbol $[\,\cdot\,,\cdot\,]_q \in \{\pm 1\}$ of the quaternion algebra: the invariant is 0 if the Hilbert symbol is $+1$ and $1/2$ if the Hilbert symbol is -1. Using the formulas for the Hilbert symbol in [Ser73], we find that

$$[p,3]_q = \begin{cases} (-1)^{(p-1)/2} & \text{if } q = 2, \\ \left(\dfrac{p}{3}\right) & \text{if } q = 3, \\ \left(\dfrac{3}{p}\right) & \text{if } q = p, \\ 1 & \text{otherwise,} \end{cases}$$

where $\left(\dfrac{p}{q}\right)$ is the usual Legendre symbol. On the other hand, specializing $\mathscr{A}$ at P_2 we obtain the quaternion algebra $(p, 16) \cong (p, 1)$ over $\mathbb{Q}$. We find that $[p,1]_q = 1$ for all primes q. Hence

$$\mathrm{inv}_3(p,3) \neq \mathrm{inv}_3(p,1) \text{ if } p \equiv 5 \bmod 6 \quad \text{and} \quad \mathrm{inv}_2(p,3) \neq \mathrm{inv}_2(p,1) \text{ if } p \equiv 3 \bmod 4. \tag{4.9}$$

Let $P \in X(\mathbb{A}_\mathbb{Q})$ be the point that is equal to P_1 at all places except p, and is P_2 at p. Then by (4.9) it follows that if $p \equiv 5 \bmod 6$ then

$$\sum_v \mathrm{inv}_v(\mathscr{A}(P_v)) = 1/2.$$

Similarly, if $P' \in X(\mathbb{A}_\mathbb{Q})$ is the point that is equal to P_1 at all places except 2, and is P_2 at 2, then by (4.9) we find that the sum of invariants is again $1/2$ when $p \equiv 3 \bmod 4$.

If $p > 3$ is a prime such that $p \not\equiv 1 \bmod 12$, then either $p \equiv 3 \bmod 4$ or $p \equiv 5 \bmod 6$, and by our computations above it follows that $X(\mathbb{A}_\mathbb{Q}) \neq X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}$. We conclude that X does not satisfy weak approximation. $\square$

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