Math 101 Fall 2004 Final Exam **Solutions**

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **three hours**. Do all 12 problems. Please do all your work on the paper provided.
Please print you name clearly here.

Print name: ________________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

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Grader's use only:

1. ________ /25

2. ________ /15

3. ________ /20

4. ________ /20

5. ________ /10

6. ________ /15

7. ________ /10

8. ________ /10

9. ________ /10

10. ________ /15

11. ________ /10

12. ________ /15

1. [25 points] Evaluate the derivatives of the following functions

(a) $f(x) = \ln\left(\frac{e^x}{1+e^x}\right)$

Simplifying $f(x) = x - \ln(1+e^x)$ so

$$f'(x) = 1 - \frac{e^x}{1+e^x} = \frac{1}{1+e^x},$$

or without simplifying

$$\begin{aligned} f'(x) &= \frac{1}{\frac{e^x}{1+e^x}} \cdot \frac{(1+e^x)\frac{de^x}{dx} - e^x\frac{d(1+e^x)}{dx}}{(1+e^x)^2} = \frac{1+e^x}{e^x}\frac{e^x(1+e^x) - e^x(e^x)}{(1+e^x)^2} \\ &= \frac{1}{1+e^x}. \end{aligned}$$

(b) $y = \sqrt{1+\tan(t^2)}$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{1+\tan(t^2)}}\frac{d(1+\tan(t^2))}{dt} = \frac{2t\sec^2(t^2)}{2\sqrt{1+\tan(t^2)}} = \frac{t\sec^2(t^2)}{\sqrt{1+\tan(t^2)}}.$$

(c) $g(x) = \arctan\left(\frac{1}{x^2}\right) + \arccos(3x^3)$

$$g'(x) = \frac{1}{1+(x^{-2})^2} \cdot \frac{-2}{x^3} - \frac{1}{\sqrt{1-(3x^2)^2}} \cdot (9x^2) = \frac{-2x}{x^4+1} - \frac{9x^2}{\sqrt{1-9x^4}}.$$

(d) $f(t) = \int_{-\pi}^{\sqrt{t}} \cos x \, dx$

$$f'(t) = \cos(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} = \frac{\cos(\sqrt{t})}{2\sqrt{t}}.$$

(e) $h(z) = (4z+3)^4(z+1)^{-3}$

$$\begin{aligned} h'(z) &= 4(4z+3)^3 \cdot (4)(z+1)^{-3} - 3(4z+3)^4(z+1)^{-4} \cdot (1) \\ &= 16(4z+3)^3(z+1)^{-3} - 3(4z+3)^4(z+1)^{-4}. \end{aligned}$$

2. [15 points] Evaluate the following limits, if they exist.

(a) $\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2}$

The limit is indeterminate 0/0 so L'Hôpital gives

$$\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2} = \lim_{x\to 2} \frac{\frac{2x}{2\sqrt{x^2+12}}}{1} = \lim_{x\to 2} \frac{x}{\sqrt{x^2+12}} = \frac{2}{4} = \frac{1}{2}.$$

(b) $\lim_{\theta\to +\infty} \frac{\theta}{2} \sin\left(\frac{5}{\theta}\right)$

The limit is indeterminate $0\cdot\infty$, so rewriting and applying L'Hôpital gives

$$\begin{aligned}\lim_{\theta\to +\infty} \frac{\theta}{2} \sin\left(\frac{5}{\theta}\right) &= \lim_{\theta\to +\infty} \frac{\sin\left(\frac{5}{\theta}\right)}{(2/\theta)} \\ &= \lim_{\theta\to +\infty} \frac{\frac{-5}{\theta^2}\cos\left(\frac{5}{\theta}\right)}{-\frac{2}{\theta^2}} \\ &= \lim_{\theta\to +\infty} \frac{5}{2}\cos\left(\frac{5}{\theta}\right) = \frac{5}{2}\cos(0) = \frac{5}{2}.\end{aligned}$$

(c) $\lim_{t\to 0^+} \left(\frac{1}{t} - \frac{1}{\sqrt{t}}\right)$

The limit is indeterminate $\infty - \infty$, so rewriting gives

$$\lim_{t\to 0^+} \left(\frac{1}{t} - \frac{1}{\sqrt{t}}\right) = \lim_{t\to 0^+} \frac{1-\sqrt{t}}{t}.$$

This limit is 1/0, which is not indeterminate. The numerator is near 1, hence positive and the denominator is a small positive number so

$$\lim_{t\to 0^+} \left(\frac{1}{t} - \frac{1}{\sqrt{t}}\right) = \infty.$$

3. [20 points] Evaluate the following integrals

(a) $\int_0^{\pi/2} \frac{\sin x}{(5+3\cos x)^3}\, dx$

Let $u = 5{+}3\cos x$, $du = -3\sin x\, dx$, $x = 0 \Rightarrow u = 8$, and $x = \pi/2 \Rightarrow u = 5$ so

$$\int_0^{\pi/2} \frac{\sin x}{(5+3\cos x)^3}\, dx = -\frac{1}{3}\int_8^5 \frac{1}{u^3}\, du = \left.\frac{1}{6u^2}\right|_8^5 = \frac{1}{150} - \frac{1}{384} = \frac{13}{3200}.$$

(b) $\int \frac{x}{\sqrt{1-4x^4}}\, dx$

Let $u = 2x^2$, $du = 4x\, dx$, so

$$\int \frac{x}{\sqrt{1-4x^4}}\, dx = \frac{1}{4}\int \frac{1}{\sqrt{1-u^2}}\, du = \frac{1}{4}\arcsin u + C = \frac{1}{4}\arcsin(2x^2) + C.$$

(c) $\int_0^1 x^2\sqrt{3-x^3}\, dx$

Let $u = 3 - x^3$, $du = -3x^2\, dx$, $x = 0 \Rightarrow u = 3$, $x = 1 \Rightarrow u = 2$, so

$$\int_0^1 x^2\sqrt{3-x^3}\, dx = -\frac{1}{3}\int_3^2 u^{1/2}\, du = \left.-\frac{2}{9}u^{3/2}\right|_3^2 = \frac{2}{9}\left(3^{3/2} - 2^{3/2}\right).$$

(d) $\int \frac{\operatorname{sech}^2(\ln x)}{x}\, dx$

Let $u = \ln x$, $du = \frac{1}{x}dx$, so

$$\int \frac{\operatorname{sech}^2(\ln x)}{x}\, dx = \int \operatorname{sech}^2 u\, du = \tanh u + C = \tanh(\ln x) + C.$$

4. [20 points] For the function

$$f(x) = \frac{x^3 - x^2 - 12x}{x^2 - 16},$$

the first two derivatives are

$$f'(x) = \frac{x^2 + 8x + 12}{(x+4)^2} \quad \text{and} \quad f''(x) = \frac{8}{(x+4)^3}.$$

YOU NEED NOT VERIFY THESE FORMULAS.

(a) Find all discontinuities of the function $f(x)$ and classify each as a jump, removable (point), or infinite discontinuity. For each discontinuity compute both the left and right hand limits of $f(x)$.

The discontinuities occur where the denominator is zero, i.e., at $x = \pm 4$. At $x = 4$, the numerator is also zero and we compute

$$\lim_{x \to 4} \frac{x^3 - x^2 - 12x}{x^2 - 16} = \lim_{x \to 4} \frac{3x^2 - 2x - 12}{2x} = \frac{28}{8} = \frac{7}{2}.$$

Hence $x = 4$ is a removable discontinuity. At $x = -4$, the numerator is -32 and we have an infinite discontinuity (an asymptote). Since the numerator is negative and $x^2 - 16$ is positive for $x \to -4^-$ and negative for $x \to -4^+$, we see

$$\lim_{x \to -4^-} \frac{x^3 - x^2 - 12x}{x^2 - 16} = -\infty,$$

$$\lim_{x \to -4^+} \frac{x^3 - x^2 - 12x}{x^2 - 16} = \infty.$$

(b) Find the intervals on which $f(x)$ is increasing and those on which it is decreasing.

The denominator of $f'(x)$ is positive for $x \neq -4$. The numerator factors as $(x+2)(x+6)$. For $x > -2$, both factors are positive and hence $f'(x) > 0$. For $-4 < x < -2$ and for $-6 < x < -4$, $x + 6$ is positive and $x + 2$ is negative, hence $f'(x) < 0$. For $x < -6$, both $x + 2$ and $x + 6$ are negative, hence $f'(x) > 0$. Thus f is increasing on $(-\infty, -6)$, f is decreasing on $(-6, -4)$ and on $(-4, -2)$, and f is increasing on $(-2, \infty)$.

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(c) Find the critical points of $f(x)$ and classify them as local maxima, local minima or neither.

Since $f'(x)$ exists except at $x = -4$ (which is an asymptote), the only critical points are where $f'(x) = 0$, i.e., at $x = -6$ and $x = -2$. Since $f'(x)$ changes from positive to negative at $x = -6$, the first derivative test shows $x = -6$ is a local maximum. Alternately, since $f''(-6) = -1 < 0$, the second derivative test shows $x = -6$ is a local maximum. Since $f'(x)$ changes from negative to positive at $x = -2$, the first derivative test shows $x = -2$ is a local minimum. Alternately, since $f''(-2) = 1 > 0$, the second derivative test shows $x = -2$ is a local minimum.

(d) Find the intervals on which $f(x)$ is concave upward and those on which it is concave downward.

For $x > -4$, $x + 4$ is positive and $f''(x) = 8(x + 4)^{-3}$ is positive. For $x < -4$, $x + 4$ is negative and $f''(x) = 8(x + 4)^{-3}$ is negative. Hence f is concave up on $(-4, \infty)$ and concave down on $(-\infty, -4)$.

5. [10 points] Compute the first three derivatives of the following function

$$g(x) = \sin(x^2)$$

$$\begin{aligned}
g'(x) &= 2x\cos(x^2) \\
g''(x) &= 2\cos(x^2) - 4x^2\sin(x^2) \\
g'''(x) &= -4x\sin(x^2) - 8x\sin(x^2) - 8x^3\cos(x^2) \\
&= -12x\sin(x^2) - 8x^3\cos(x^2)
\end{aligned}$$

6. [15 points] A billboard will cost \$1 per square foot of area plus \$10 per foot of width for the base. Given that the cost of a billboard was \$400, what is the minimum possible perimeter of the billboard? Be sure to say all the words required to justify that your answer is really the minimum.

Let the width of the base be x feet and the height y feet. The quantity we want to minimize is $P = 2x + 2y$. To relate x and y, we note that the cost is $10x + xy = 400$ dollars, so $y = (400/x) - 10$. Since they are distances $x > 0$ and $(400/x) - 10 = y > 0$ or $x < 40$. Hence we want to minimize

$$P(x) = 2x + \frac{800}{x} - 20 \text{ on } 0 < x < 40.$$

We compute

$$P'(x) = 2 - \frac{800}{x^2} = \frac{2(x^2 - 400)}{x^2}.$$

Hence the only critical point in the domain is at $x = 20$ which gives $y = 10$ and $P = 60$.

To see that this really is the minimum we either note that $P'(x) < 0$ for $0 < x < 20$ and $P'(x) > 0$ for $x > 20$ and apply the first derivative test to conclude $x = 20$ is a global minimum or we compute that

$$P''(x) = \frac{1600}{x^3} > 0$$

for all $x > 0$ and use the second derivative test to conclude $x = 20$ is a global minimum.

7. [10 points] A train track runs west-east through a town and a road runs north-south through the town. A train is going east at 80 mph and a car is driving north at 50 mph. How fast is the distance between the train and the car changing when the train is 30 miles east of town and the car is 40 miles south of town?

Let x be the number of miles east of town the train is and let y be the number of miles south of town the car is. Then we are told that

$$\frac{dx}{dt} = 80 \text{ and } \frac{dy}{dt} = -50.$$

(Note the minus sign on $\frac{dy}{dt}$ since the car is travelling north.) Let z be the distance between the train and the car, so that we are asked for $\frac{dz}{dt}$ when $x = 30$ and $y = 40$. The Pythagorean Theorem gives

$$z^2 = x^2 + y^2.$$

Hence

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

or

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} + \frac{y}{z}\frac{dy}{dt}.$$

When $x = 30$ and $y = 40$, we compute $z = 50$ so

$$\left.\frac{dz}{dt}\right|_{x=30,\, y=40} = \frac{30}{50} \cdot 80 + \frac{40}{50} \cdot (-50) = 48 - 40 = 8,$$

Thus the distance between the train and the car is increasing at 8 mph.

8. [10 points] Find the area of the region in the plane bounded by

$$y = x^4 - 2x^2 \quad \text{and} \quad y = 2x^2.$$

The two curves intersect when $x^4 - 2x^2 = 2x^2$ or $0 = x^4 - 4x^2 = x^2(x^2 - 4)$, hence at $x = -2$, $x = 0$ and $x = 2$. Hence the area computation has potentially two regions $-2 \leq x \leq 0$ and $0 \leq x \leq 2$. At $x = -1$, the height of the quartic is $y = (-1)^4 - 2(-1)^2 = -1$ and the height of the parabola is $y = 2(-1)^2 = 2$. At $x = 1$, the height of the quartic is $y = (1)^4 - 2(1)^2 = -1$ and the height of the parabola is $y = 2(1)^2 = 2$. Hence in both regions $y = 2x^2 = f(x)$ is the higher curve and $y = x^4 - 2x^2 = g(x)$ is the lower curve (and we can do the area with a single integral from -2 to 2 rather than splitting it). (One can also use symmetry.) Hence

$$\begin{aligned} A &= \int_{-2}^{2} (2x^2 - (x^4 - 2x^2))dx = \int_{-2}^{2} (4x^2 - x^4)dx \\ &= \left(\frac{4x^3}{3} - \frac{x^5}{5} \right) \Bigg|_{-2}^{2} = \left(\frac{32}{3} - \frac{32}{5} \right) - \left(\frac{-32}{3} - \frac{-32}{5} \right) \\ &= \frac{64}{3} - \frac{64}{5} = \frac{128}{15}. \end{aligned}$$

9. [10 points] A ball is dropped from the top of a 144 ft building. Air resistance is neglected.

(a) Solve the following initial value problem to determine the velocity function $v(t)$ and the position function $x(t)$ of the ball.

$$a(t) = -32 \text{ ft/s}^2, \quad v(0) = 0 \text{ ft/s}, \quad x(0) = 144 \text{ ft}$$

Since $v(t)$ is an antiderivative of $a(t)$, $v(t) = -32t + C$. Plugging in $t = 0$ gives $C = 0$ and $v(t) = -32t$. Since $x(t)$ is an antiderivative of $v(t)$, $x(t) = -16t^2 + C$. Plugging in $t = 0$ gives $C = 144$ and $x(t) = 144 - 16t^2$.

(b) How long does it take for the ball to reach the ground? With what velocity does the ball strike the ground?

The ball strikes the ground when $x(t) = 0$, hence $144 - 16t^2 = 0$ or $t^2 = 9$ or $t = 3$. When $t = 3$, the velocity is $v(3) = -32(3) = -96$ ft/sec.

10. [15 points] Let R be the region in the plane bounded by the curve $y = x^2$ and the line $y = 3x$. Let S be the solid that results from revolving R about the y-axis. Express the volume of S as a definite integral in TWO ways, using the method of cross-sections and the method of shells. Evaluate ONE of these two integrals (your choice).

Setting the y coordinates equal, we see that the two curves intersect when $x^2 = 3x$ or $x(x-3) = 0$, hence at $x = 0$ and $x = 3$. The intersection points are $(0,0)$ and $(3,9)$.

cross-sections:

Since S is obtained by revolving R about the y-axis, to use cross-sections we need the region R described in terms of y. The line is just $x = y/3$. The parabola is two graphs $x = \sqrt{y}$ and $x = -\sqrt{y}$. The intersections both occurred for $x \geq 0$, so only the first of these matters. The lowest y value is $y = 0 = c$ and the highest is $y = 9 = d$ (at the two intersections). Plugging in $y = 1$, we see that the parabola is farther to the right ($x = 1$ vs. $x = 1/3$ for the line). Hence $x = \sqrt{y} = k(y)$ and $x = y/3 = h(y)$. Hence the volume is

$$\begin{aligned} V &= \int_c^d \pi(k(y)^2 - h(y)^2)\,dy = \int_0^9 \pi((\sqrt{y})^2 - (y/3)^2)dy \\ &= \pi \int_0^9 \left(y - \frac{y^2}{9}\right) dy = \pi \left(\frac{y^2}{2} - \frac{y^3}{27}\right)\Bigg|_0^9 \\ &= \pi\left(\frac{81}{2} - \frac{729}{27}\right) - 0 = \frac{27\pi}{2}. \end{aligned}$$

shells:

Since S is obtained by revolving R about the y-axis, to use shells we need the region R described in terms of x. The lowest x value is $x = 0 = a$ and the highest x value is $x = 3 = b$ (at the two intersections). Plugging in $x = 1$, we see that the higher curve is $y = 3x$ ($y = 3$ vs. $y = 1$ for the parabola). Hence $y = 3x = f(x)$ and $y = x^2 = g(x)$. Hence the volume is

$$\begin{aligned} V &= \int_a^b 2\pi x(f(x) - g(x))\,dx = \int_0^3 2\pi x(3x - x^2)\,dx \\ &= 2\pi \int_0^3 (3x^2 - x^3)\,dx = 2\pi\left(x^3 - \frac{x^4}{4}\right)\Bigg|_0^3 \\ &= 2\pi\left(27 - \frac{81}{4}\right) - 0 = 2\pi\frac{27}{4} = \frac{27\pi}{2}. \end{aligned}$$

11. [10 points] Consider the curve C given by $y = x^{3/2} - \frac{1}{3}\sqrt{x}$ for $1 \le x \le 4$.

(a) Find the length of the curve C.

The curve is $y = f(x) = x^{3/2} - \frac{1}{3}x^{1/2}$, so we compute

$$f'(x) = \frac{3}{2}x^{1/2} - \frac{1}{6}x^{-1/2}$$

and

$$(f'(x))^2 = \frac{9}{4}x - \frac{1}{2} + \frac{1}{36}x^{-1}.$$

Hence

$$1 + (f'(x))^2 = \frac{9}{4}x + \frac{1}{2} + \frac{1}{36}x^{-1} = \left(\frac{3}{2}x^{1/2} + \frac{1}{6}x^{-1/2}\right)^2.$$

Thus the arc length is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_1^4 \left(\frac{3}{2}x^{1/2} + \frac{1}{6}x^{-1/2}\right) dx \\ &= \left(x^{3/2} + \frac{1}{3}x^{1/2}\right)\Big|_1^4 \\ &= \left(8 + \frac{2}{3}\right) - \left(1 + \frac{1}{3}\right) = \frac{22}{3} \end{aligned}$$

(b) Find the area of the surface that results from revolving C about the y-axis.

Since the curve is revolved about the y-axis the radius is x and the surface area is

$$\begin{aligned} S &= \int_a^b 2\pi x\sqrt{1 + (f'(x))^2}\, dx = \int_1^4 2\pi x \left(\frac{3}{2}x^{1/2} + \frac{1}{6}x^{-1/2}\right) dx \\ &= 2\pi \int_1^4 \left(\frac{3}{2}x^{3/2} + \frac{1}{6}x^{1/2}\right) dx = 2\pi \left(\frac{3}{5}x^{5/2} + \frac{1}{9}x^{3/2}\right)\Big|_1^4 \\ &= 2\pi\left(\frac{96}{5} + \frac{8}{9}\right) - 2\pi\left(\frac{3}{5} + \frac{1}{9}\right) \\ &= 2\pi\left(\frac{93}{5} + \frac{7}{9}\right) = \frac{1744\pi}{45}. \end{aligned}$$

12. [15 points] Consider the "triangular" region R in the first quadrant between the circle $x^2 + y^2 = 9$ and the lines $x = 3$ and $y = 3$.

(a) Use geometry to compute the area of the region R.

The region is the region left of a square of side length 3 (hence area 9) when we remove 1/4 of a circle of radius 3. Hence the area removed is $\frac{1}{4}(\pi 3^2) = \frac{9\pi}{4}$ and the area left is

$$A = 4 - \frac{9\pi}{4} = \frac{16 - 9\pi}{4}.$$

(b) Let the centroid of the region R be denoted by the coordinates $(\bar{x}, \bar{y})$. Express $\bar{x}$ and $\bar{y}$ as definite integrals.

The region R is below $y = f(x) = 3$ and above $y = g(x) = \sqrt{9 - x^2}$ for $0 \leq x \leq 3$, so

$$\bar{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))dx = \frac{4}{16 - 9\pi}\int_0^3 x(3 - \sqrt{9 - x^2})\,dx,$$

$$\begin{aligned}\bar{y} &= \frac{1}{2A}\int_a^b (f(x)^2 - g(x)^2)dx = \frac{2}{16 - 9\pi}\int_0^3 (3^2 - (\sqrt{9 - x^2})^2)\,dx \\ &= \frac{2}{16 - 9\pi}\int_0^3 (9 - (9 - x^2))dx = \frac{2}{16 - 9\pi}\int_0^3 x^2\,dx\end{aligned}$$

(c) Compute the centroid of the region R. Hint: The region R is symmetric about the line $y = x$.

By symmetry $\bar{x} = \bar{y}$, so we need only do one of the two integrals from (b). The easier one is

$$\bar{y} = \frac{2}{16 - 9\pi}\int_0^3 x^2\,dx = \frac{2}{16 - 9\pi}\left.\frac{x^3}{3}\right|_0^3 = \frac{18}{16 - 9\pi}$$

so the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{18}{16 - 9\pi}, \frac{18}{16 - 9\pi}\right).$$