Math 101 Section 003 – Midterm 1 Extra Extra Problem Solutions

Below are the solutions to the extra extra problems provided. Graphs have been omitted. Please see me if you have trouble with graphs.

1. Let $f(x) = x^2 - 2x + 3$.
   (a) Notice that $f(x) = x^2 - 2x + 3 = (x^2 - 2x + 1) + 2 = (x-1)^2 + 2$.
   (b) Domain(f): $(-\infty, \infty)$.
   Range(f): $[2, \infty)$.
   (c) $[-1, \infty)$.
   (d) $f^{-1}(x) = \sqrt{x-2} + 1$.

2. Find domain, range. Is it 1-1? Largest interval on which it is 1-1? Find inverse. State domain, range of inverse.
   (a) $f(x) = -\sin(2x) + 2$. Domain(f): $(-\infty, \infty)$. Range(f): $[1, 3]$. This is not 1-1 on entire domain. It is 1-1 on $[-\pi/4.\pi/4]$.
   On this interval, $f^{-1}(x) = \frac{1}{2}\arcsin(-x+2)$. Domain($f^{-1}$): $[1, 3]$. Range(f^{-1}): $[-\pi/4, \pi/4]$.
   (b) $g(x) = ln(2x) - 4$. Domain(g): $(0, \infty)$. Range(g): $(-\infty, \infty)$. This is 1-1.
   $g^{-1}(x) = \frac{1}{2}e^{x+4}$. Domain($g^{-1}$): $(-\infty, \infty)$. Range(g^{-1}): $(0, \infty)$.
   (c) $h(x) = 2\sqrt{x+1}$. Domain(h): $[-1, \infty)$. Range(f): $[0, \infty)$. This is 1-1.
   $h^{-1}(x) = \frac{1}{4}x^2 - 1, x \geq 0$. Domain($h^{-1}$): $[0, \infty)$. Range(h^{-1}): $[-1, \infty)$.

3.
$$f(x) = \begin{cases} \sqrt{-x-2}, & \text{if } x < -2. \\ 5 - x, & \text{if } -2 \leq x < 1. \\ (x-3)^2, & \text{if } x > 1. \end{cases}$$

   (a) i. $\lim_{x \to -2^+} f(x) = 7$
   ii. $\lim_{x \to -2^-} f(x) = 0$
   iii. $\lim_{x \to -2} f(x)$. DNE since
$$\lim_{x \to -2^+} f(x) = 7 \neq \lim_{x \to -2^-} f(x) = 0$$
   iv. $\lim_{x \to 1^+} f(x) = 4$
   v. $\lim_{x \to 1^-} f(x) = 4$
   vi. $\lim_{x \to 1} f(x) = 4$
   (b) f is discontinuous at $x = 2$. It is continuous on $(-\infty, -2) \cup (-2, \infty)$.
   (c) Graph.
   (d) f is differentiable on $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$ since these are our basic differentiable functions on these intervals. It is not differentiable at $x = -2$ since here it is not continuous. It is not differentiable at $x = 1$ since
$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} \neq \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$
   (i.e. There is a kink in the graph).

4. (a) Domain(h): $(-\infty, \infty)$.
   Notice that $\cos(x)$ is continuous on $(-\infty, \infty)$ and it's range is $[-1, 1]$. Since e^x is continuous on $[-1, 1]$, it follows from theorem 9 on page 125 that $e^{cos(x)}$ is continuous on $(-\infty, \infty)$. Furthermore, x^3 is continuous on $(-\infty, \infty)$ and thus $h(x) = x^3 e^{cos(x)}$, being the product of these, is continuous on $(-\infty, \infty)$ by theorem 4 on page 122.

(b) Domain(g): $[-2,-\sqrt{3}) \cup (-\sqrt{3},\sqrt{3}) \cup (\sqrt{3},2]$. Call this D.
Notice that x^2-4 is continuous on $(-\infty,\infty)$ and is therefore continuous on D. $\sqrt{x}$ is continuos on $[0,\infty]$ and $x^2-4 \geq 0$ when $-2 \leq x \leq 2$. This means $\sqrt{x^2-4}$ is continuous on $[-2.2]$ by theorem 9 on page 125. It is therefore continuous on D. Now, x^2-3 is continuous on $(-\infty,\infty)$ and is therefore continuous on $[-2,2]$ but $x^2-3=0$ when $x=\sqrt{3},-\sqrt{3}$. Then, by theorem 4 on page 122, g is continuous on D.

5. (a) No, f_1 is NOT differentiable at $x=1$, since

$$\lim_{x\to 1^+} \frac{f_1(x)-f_1(1)}{x-1} = 1 \neq \lim_{x\to 1^-} \frac{f_1(x)-f_1(1)}{x-1} = -1$$

(b) No, f_2 is NOT differentiable at $x=1$, since $f_2(1)$ is undefined.

(c) No, f_3 is NOT differentiable at $x=1$, since the following limit does not exist.

$$\lim_{x\to 1^-} \frac{f_3(x)-f_3(1)}{x-1}$$

(d) Yes. f_4 is differentiable at $x=1$.