Math 101 Section 003 – Midterm 1 Extra Extra Problem Solutions

Below are the solutions to the extra extra problems provided. Graphs have been omitted. Please see me if you have trouble with graphs.

1. Let \( f(x) = x^2 - 2x + 3 \).

   (a) Notice that \( f(x) = x^2 - 2x + 3 = (x^2 - 2x + 1) + 2 = (x - 1)^2 + 2 \).
   
   (b) Domain(\(f\)): \((−∞, ∞)\).
   
   Range(\(f\)): \([2, ∞)\).
   
   (c) \([-1, ∞)\).
   
   (d) \(f^{-1}(x) = \sqrt{x - 2} + 1\).


   (a) \( f(x) = -\sin(2x) + 2 \). Domain(\(f\)): \((−∞, ∞)\). Range(\(f\)): \([1, 3]\). This is not 1-1 on entire domain. It is 1-1 on \([-\pi/4, \pi/4]\).
   
   On this interval, \( f^{-1}(x) = \frac{1}{2} \arcsin(-x + 2) \). Domain(\(f^{-1}\)): \([1, 3]\). Range(\(f^{-1}\)): \([-\pi/4, \pi/4]\).

   (b) \( g(x) = \ln(2x) - 4 \). Domain(\(g\)): \((0, ∞)\). Range(\(g\)): \((−∞, ∞)\). This is 1-1.
   
   \( g^{-1}(x) = \frac{e^{x+4}}{2} \). Domain(\(g^{-1}\)): \((−∞, ∞)\). Range(\(g^{-1}\)): \((0, ∞)\).

   (c) \( h(x) = 2\sqrt{x + 1} \). Domain(\(h\)): \([-1, ∞)\). Range(\(f\)): \([0, ∞)\). This is 1-1.
   
   \( h^{-1}(x) = \frac{1}{4} x^2 - 1, x ≥ 0 \). Domain(\(h^{-1}\)): \([0, ∞)\). Range(\(h^{-1}\)): \([-1, ∞)\).

3. \( f(x) = \begin{cases} \sqrt{-x - 2}, & \text{if } x < -2, \\ 5 - x, & \text{if } -2 ≤ x < 1, \\ (x - 3)^2, & \text{if } x > 1. \end{cases} \)

   (a) \( i. \lim_{x→-2^+} f(x) = 7 \)
   
   \( ii. \lim_{x→-2^-} f(x) = 0 \)
   
   \( iii. \lim_{x→-2} f(x) \text{ DNE since} \)

   \[ \lim_{x→-2^+} f(x) = 7 ≠ \lim_{x→-2^-} f(x) = 0 \]

   (b) \( \text{iv. \lim}_{x→-1^+} f(x) = 4 \)
   
   \( \text{v. \lim}_{x→-1^-} f(x) = 4 \)
   
   \( \text{vi. \lim}_{x→-1} f(x) = 4 \)

   (b) \( f \) is discontinuous at \( x = 2 \). It is continuous on \((-∞, -2) \cup (-2, ∞)\).

   (c) Graph.

   (d) \( f \) is differentiable on \((-∞, -2) \cup (-2, 1) \cup (1, ∞)\) since these are our basic differentiable functions on these intervals. It is not differentiable at \( x = -2 \) since here it is not continuous. It is not differentiable at \( x = 1 \) since

   \[ \lim_{x→-1^+} \frac{f(x) - f(1)}{x - 1} ≠ \lim_{x→-1^-} \frac{f(x) - f(1)}{x - 1} \]

   (i.e. There is a kink in the graph).

4. (a) Domain(\(h\)): \((−∞, ∞)\).

   Notice that \( \cos(x) \) is continuous on \((-∞, ∞)\) and it’s range is \([-1, 1]\). Since \( e^x \) is continuous on \([-1, 1]\), it follows from theorem 9 on page 125 that \( e^{\cos(x)} \) is continuous on \((-∞, ∞)\).

   Furthermore, \( x^3 \) is continuous on \((-∞, ∞)\) and thus \( h(x) = x^3 e^{\cos(x)} \), being the product of these, is continuous on \((-∞, ∞)\) by theorem 4 on page 122.
(b) Domain(g): \([-2, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, 2]\). Call this \(D\).

Notice that \(x^2 - 4\) is continuous on \((-\infty, \infty)\) and is therefore continuous on \(D\). \(\sqrt{x}\) is continuous on \([0, \infty]\) and \(x^2 - 4 \geq 0\) when \(-2 \leq x \leq 2\). This means \(\sqrt{x^2 - 4}\) is continuous on \([-2, 2]\) by theorem 9 on page 125. It is therefore continuous on \(D\). Now, \(x^2 - 3\) is continuous on \((-\infty, \infty)\) and is therefore continuous on \([-2, 2]\) but \(x^2 - 3 = 0\) when \(x = \sqrt{3}, -\sqrt{3}\). Then, by theorem 4 on page 122, \(g\) is continuous on \(D\).

5. (a) No, \(f_1\) is NOT differentiable at \(x = 1\), since

\[
\lim_{x \to 1^+} \frac{f_1(x) - f_1(1)}{x - 1} = 1 \neq \lim_{x \to 1^-} \frac{f_1(x) - f_1(1)}{x - 1} = -1
\]

(b) No, \(f_2\) is NOT differentiable at \(x = 1\), since \(f_2(1)\) is undefined.

(c) No, \(f_3\) is NOT differentiable at \(x = 1\), since the following limit does not exist.

\[
\lim_{x \to 1^-} \frac{f_3(x) - f_3(1)}{x - 1}
\]

(d) Yes. \(f_4\) is differentiable at \(x = 1\).