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| Rice University | Math 101 - Fall 2009 |
| Department of Mathematics | Midterm Exam 1 |
| Solutions | |

1. Let

$$f(x) = 2\sin 3x - 1.$$

   (a) What is the domain of $f$? *[2 points]*
   $(-\infty, \infty)$

   (b) Draw the graph of $f$. *[6 points]*
   See attached page.

   (c) What is the range of $f$? *[2 points]*
   $[-3, 1]$

   (d) State the largest interval containing 0 for which $f(x)$ is one-to-one. *[2 points]*
   $[-\frac{\pi}{6}, \frac{\pi}{6}]$

   (e) What is $f^{-1}(0)$? *[3 points]*

$$2\sin 3x - 1 = 0 \Rightarrow \sin 3x = \frac{1}{2} \Rightarrow 3x = \frac{\pi}{6} \Rightarrow x = \frac{\pi}{18}$$

   *Note*: Although there are several points $x$ such that $f(x) = 0$, we restrict the domain to $[-\pi/6, \pi/6]$ in order to define the inverse function $f^{-1}$. There is only one $x$, namely $x = \pi/18$, in this restricted domain which satisfies $f(x) = 0$ and hence $f^{-1}(0) = \pi/18$.

2. Compute the following limits or show that they do not exist. *[5 points each]*

(a)

$$\lim_{x\to -3} \frac{x^2-9}{x^2+2x-3}$$

$\lim_{x\to-3} \frac{x^2-9}{x^2+2x-3} = \lim_{x\to-3} \frac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to-3} \frac{x-3}{x-1} = \frac{(-3)-3}{(-3)-1} = \frac{3}{2}$

(b)

$$\lim_{x\to -\infty} \frac{\sqrt{x^2-9}}{2x-6}$$

$\lim_{x\to-\infty} \frac{\sqrt{x^2-9}}{2x-6} = \lim_{x\to-\infty} \frac{|x|\sqrt{1-\frac{9}{x^2}}}{x(2-\frac{6}{x})} = \lim_{x\to-\infty} \frac{|x|}{x} * \frac{\sqrt{1-\frac{9}{x^2}}}{2-\frac{6}{x}}$

$= \lim_{x\to-\infty} \frac{|x|}{x} * \lim_{x\to-\infty} \frac{\sqrt{1-\frac{9}{x^2}}}{2-\frac{6}{x}} = (-1)(\frac{\sqrt{1-0}}{2-0}) = -1/2$

(c)

$$\lim_{\theta\to 0} \frac{\tan(6\theta)}{\sin(2\theta)}$$

$\lim_{\theta\to0} \frac{\tan(6\theta)}{\sin(2\theta)} = \lim_{\theta\to0} \frac{\frac{\sin(6\theta)}{\cos(6\theta)}}{\sin(2\theta)} = \lim_{\theta\to0} \frac{\sin(6\theta)}{\cos(6\theta)\sin(2\theta)} = \lim_{\theta\to0}(\frac{\sin(6\theta)}{6\theta})(\frac{6\theta}{2\theta\cos(x\theta)})(\frac{2\theta}{\sin(2\theta)})$

$= (\lim_{\theta\to0} \frac{\sin(6\theta)}{6\theta})(\lim_{\theta\to0} \frac{3}{\cos(6\theta)})(\lim_{\theta\to0} \frac{1}{\frac{\sin(2\theta)}{2\theta)}}) = (1)(\frac{3}{1})(\frac{1}{1}) = 3$

3. Use the intermediate value property for continuous functions to show that the equation $e^{-x^2} = x$ has at least one solution in the interval [0,1]. *[8 points]*
We want to find solutions to the equation $e^{-x^2} - x = 0$. At $x = 0$, we have $e^{-(0)^2} - (0) = 1 < 0$. At $x = 1$, we have $e^{-(1)^2} - (1) = 1/e - 1 < 0$. But $f(x) = e^{-x^2} - x$ is a function which is continuous along its entire domain, so $f(0) > 0 > f(1)$ means there must be some $x = c$ in the interval $[0, 1]$ such that $f(c) = e^{-c^2} - c = 0$ by the intermediate value property.

4. (a) Write the formal, mathematical, definition of the derivative a function $f(x)$. *[5 points]*
$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$ OR $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

(b) Let

$$f(x) = \frac{2}{1-3x}.$$

Compute the derivative of $f(x)$ using the definition. No credit will be given if the derivative is found by other means. *[5 points]*
$f'(x) = \lim_{h\to0} \frac{\frac{2}{1-3(x+h)} - \frac{2}{1-3x}}{h} = \lim_{h\to0}(\frac{2(1-3x)-2(1-3(x+h))}{(1-3x)(1-3(x+h))})(\frac{1}{h})$
$= \lim_{x\to0} \frac{6h}{(1-3x)(1-3(x+h)} \frac{1}{h} = lim_{h\to0} \frac{6}{(1-3x)(1-3(x+h)} = \frac{6}{(1-3x)^2}$

(c) Find the equation of the line tangent to the graph of $f$ at the point (0,2). *[5 points]*
$f'(0) = \frac{6}{(1-3(0))^2} = 6$
$y - 2 = 6(x - 0) \Rightarrow y = 6x + 2$

5. Compute the derivatives of the following functions. *[5 points each]*

(a)

$$y = \sqrt{x} + \frac{1}{\sqrt[3]{x^4}}$$

$y = \sqrt{x} + \frac{1}{\sqrt[3]{x^4}} = x^{1/2} + x^{-4/3} \Rightarrow y' = (\frac{1}{2})x^{(\frac{1}{2}-1)} + (-\frac{4}{3})x^{(-\frac{4}{3}-1)} = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3}$

(b)

$$f(x) = \frac{e^x}{\sin x + x^2}$$

Use Quotient Rule. $f'(x) = \frac{(\sin x+x^2)(e^x)-(e^x)(\cos x+2x)}{(\sin x+x^2)^2}$.

(c)

$$g(\theta) = (\theta^2 - 2\theta + 2)\tan\theta$$

Use Product Rule.
$f'(x) = (\theta^2 - 2\theta + 2)'(\tan\theta) + (\theta^2 - 2\theta + 2)(\tan\theta)' = (2\theta - 2)(\tan\theta) + (\theta^2 - 2\theta + 2)(\sec^2\theta)$.

6. Let $f$ be the function defined by

$$f(x) = \begin{cases} x^2 \cos\frac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Show that

$$\lim_{x\to 0} f(x) = 0.$$

(Hint: Squeeze Theorem) *[4 points]*
Notice that $-1 \le \cos(\frac{1}{x^2}) \le 1$. Thus, $-x^2 \le x^2\cos\frac{1}{x^2} \le 1$ and $\lim_{x\to 0} -x^2 = lim_{x\to 0}x^2 = 0$. The squeeze theorem then says that this means $\lim_{x\to 0} x^2\cos\frac{1}{x^2} = 0$.

(b) Use the fact in (a) to argue that $f$ is everywhere continuous. *[4 points]*
When $x \neq 0$, $f(x) = x^2\cos\frac{1}{x^2}$ is a the product and composition of continuous functions and is therefore continuous. In part (a), we showed that $\lim_{x\to 0} f(x) = 0 = f(0)$ so $f(x)$ is continuous at $x = 0$ as well and is therefore continuous everywhere.

(c) Is $f$ differentiable at $x = 0$? Justify your answer. *[4 points]*
$f$ is both defined and continous at $x = 0$ so we must lastly check that $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}$. $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{(x^2\cos\frac{1}{x^2})-(0)}{x-0} = \lim_{x\to 0^+} x\cos\frac{1}{x^2}$. $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{(x^2\cos\frac{1}{x^2})-(0)}{x-0} = \lim_{x\to 0^-} x\cos\frac{1}{x^2}$. These are equal and exist, (use squeeze theorem exist lim $= 0$ for both) and so $f$ is differentiable at $x = 0$.