54. (a) For $f(x) = \ln(2 + \ln x)$, we must have $2 + \ln x > 0 \quad \Rightarrow \quad \ln x > -2 \quad \Rightarrow \quad x > e^{-2}$. Thus, the domain of f is (e^{-2}, ∞).

(b) $y = f(x) = \ln(2 + \ln x) \quad \Rightarrow \quad e^y = 2 + \ln x \quad \Rightarrow \quad \ln x = e^y - 2 \quad \Rightarrow \quad x = e^{e^y - 2}$. Interchange x and y: $y = e^{e^x - 2}$. So $f^{-1}(x) = e^{e^x - 2}$. The domain of f^{-1}, as well as the range of f, is $\mathbb{R}$.

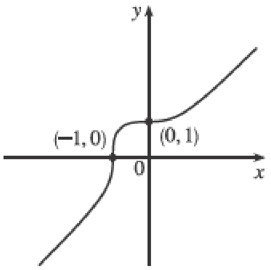
25. $x^2 + xy + y^2 = 3 \quad \Rightarrow \quad 2x + xy' + y \cdot 1 + 2yy' = 0 \quad \Rightarrow \quad xy' + 2yy' = -2x - y \quad \Rightarrow \quad y'(x + 2y) = -2x - y \quad \Rightarrow$ $y' = \dfrac{-2x - y}{x + 2y}$. When $x = 1$ and $y = 1$, we have $y' = \dfrac{-2 - 1}{1 + 2 \cdot 1} = \dfrac{-3}{3} = -1$, so an equation of the tangent line is $y - 1 = -1(x - 1)$ or $y = -x + 2$.

16. $\dfrac{dT}{dt} = k(T - 20)$. Let $y = T - 20$. Then $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 95 - 20 = 75$, so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\dfrac{dT}{dt} = -1°\text{C/min}$. Equivalently, $\dfrac{dy}{dt} = -1$ when $y(t) = 50$. Thus, $-1 = \dfrac{dy}{dt} = ky(t) = 50k$ and $50 = y(t) = 75e^{kt}$. The first relation implies $k = -1/50$, so the second relation says $50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \quad \Rightarrow \quad -t/50 = \ln\left(\frac{2}{3}\right) \quad \Rightarrow \quad t = -50\ln\left(\frac{2}{3}\right) \approx 20.27$ min.

5. $V = \pi r^2 h = \pi(5)^2 h = 25\pi h \quad \Rightarrow \quad \dfrac{dV}{dt} = 25\pi \dfrac{dh}{dt} \quad \Rightarrow \quad 3 = 25\pi \dfrac{dh}{dt} \quad \Rightarrow \quad \dfrac{dh}{dt} = \dfrac{3}{25\pi}$ m/min.

30. $y = f(x) = \sqrt[3]{x^3 + 1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0 \quad \Leftrightarrow \quad x^3 + 1 = 0 \quad \Rightarrow \quad x = -1$

C. No symmetry **D.** No asymptote **E.** $f'(x) = \frac{1}{3}(x^3 + 1)^{-2/3}(3x^2) = \dfrac{x^2}{\sqrt[3]{(x^3 + 1)^2}}$. $f'(x) > 0$ if $x < -1$, $-1 < x < 0$, and $x > 0$, so f is increasing on $\mathbb{R}$. **F.** No local extrema **H.**



G. $$f''(x) = \frac{(x^3 + 1)^{2/3}(2x) - x^2 \cdot \frac{2}{3}(x^3 + 1)^{-1/3}(3x^2)}{[(x^3 + 1)^{2/3}]^2}$$

$$= \frac{x(x^3 + 1)^{-1/3}[2(x^3 + 1) - 2x^3]}{(x^3 + 1)^{4/3}} = \frac{2x}{(x^3 + 1)^{5/3}}$$

$f''(x) > 0 \quad \Leftrightarrow \quad x < -1$ or $x > 0$ and $f''(x) < 0 \quad \Leftrightarrow \quad -1 < x < 0$, so f is CU on $(-\infty, -1)$ and $(0, \infty)$ and CD on $(-1, 0)$. IP at $(-1, 0)$ and $(0, 1)$

66. $$\lim_{n \to \infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right) = \lim_{n \to \infty} \frac{1 - 0}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x}\, dx = \left[\frac{2x^{3/2}}{3}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$$

25. Let $u = x^2 + 4x$. Then $du = (2x + 4)\,dx = 2(x + 2)\,dx$, so

$$\int \frac{x + 2}{\sqrt{x^2 + 4x}}\,dx = \int u^{-1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2} \cdot 2u^{1/2} + C = \sqrt{u} + C = \sqrt{x^2 + 4x} + C.$$

35. Let $u = 1 + \sec\theta$. Then $du = \sec\theta\,\tan\theta\,d\theta$, so

$$\int \frac{\sec\theta\,\tan\theta}{1+\sec\theta}\,d\theta = \int \frac{1}{1+\sec\theta}(\sec\theta\,\tan\theta\,d\theta) = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+\sec\theta| + C.$$

38. Since $\sqrt{x} - 1 < 0$ for $0 \le x < 1$ and $\sqrt{x} - 1 > 0$ for $1 < x \le 4$, we have $\left|\sqrt{x}-1\right| = -\left(\sqrt{x}-1\right) = 1 - \sqrt{x}$ for $0 \le x < 1$ and $\left|\sqrt{x}-1\right| = \sqrt{x} - 1$ for $1 < x \le 4$. Thus,

$$\begin{aligned}\int_0^4 \left|\sqrt{x}-1\right|dx &= \int_0^1 \left(1-\sqrt{x}\right)dx + \int_1^4 \left(\sqrt{x}-1\right)dx = \left[x - \tfrac{2}{3}x^{3/2}\right]_0^1 + \left[\tfrac{2}{3}x^{3/2} - x\right]_1^4 \\ &= \left(1-\tfrac{2}{3}\right) - 0 + \left(\tfrac{16}{3} - 4\right) - \left(\tfrac{2}{3} - 1\right) = \tfrac{1}{3} + \tfrac{16}{3} - 4 + \tfrac{1}{3} = 6 - 4 = 2\end{aligned}$$

In Exercises 39 and 40, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

50. (a) We want to choose a so that

$$\int_1^a \frac{1}{x^2}\,dx = \int_a^4 \frac{1}{x^2}\,dx \quad\Rightarrow\quad \left[\frac{-1}{x}\right]_1^a = \left[\frac{-1}{x}\right]_a^4 \quad\Rightarrow\quad -\frac{1}{a} + 1 = -\frac{1}{4} + \frac{1}{a} \quad\Rightarrow\quad \frac{5}{4} = \frac{2}{a} \quad\Rightarrow\quad a = \frac{8}{5}.$$

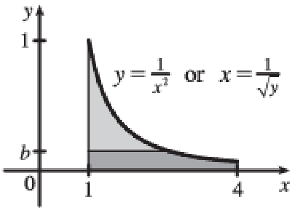
(b) The area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$ is $\frac{3}{4}$ [take $a = 4$ in the first integral in part (a)]. Now the line $y = b$ must intersect the curve $x = 1/\sqrt{y}$ and not the line $x = 4$, since the area under the line $y = 1/4^2$ from $x = 1$ to $x = 4$ is only $\frac{3}{16}$, which is less than half of $\frac{3}{4}$. We want to choose b so that the upper area in the diagram is half of the total area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$. This implies that

$\int_b^1 \left(1/\sqrt{y} - 1\right)dy = \frac{1}{2}\cdot\frac{3}{4} \quad\Rightarrow\quad \left[2\sqrt{y} - y\right]_b^1 = \frac{3}{8} \quad\Rightarrow\quad 1 - 2\sqrt{b} + b = \frac{3}{8} \quad\Rightarrow$

$b - 2\sqrt{b} + \frac{5}{8} = 0$. Letting $c = \sqrt{b}$, we get $c^2 - 2c + \frac{5}{8} = 0 \quad\Rightarrow$

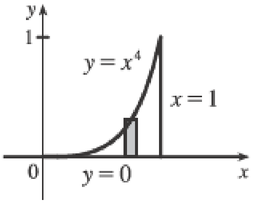
$8c^2 - 16c + 5 = 0$. Thus, $c = \frac{16 \pm \sqrt{256 - 160}}{16} = 1 \pm \frac{\sqrt{6}}{4}$. But $c = \sqrt{b} < 1 \quad\Rightarrow$

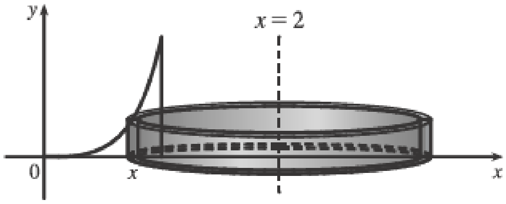
$c = 1 - \frac{\sqrt{6}}{4} \quad\Rightarrow\quad b = c^2 = 1 + \frac{3}{8} - \frac{\sqrt{6}}{2} = \frac{1}{8}\left(11 - 4\sqrt{6}\right) \approx 0.1503.$



15. The shell has radius $2 - x$, circumference $2\pi(2 - x)$, and height x^4.

$$\begin{aligned}V &= \int_0^1 2\pi(2-x)x^4\,dx \\ &= 2\pi\int_0^1 (2x^4 - x^5)\,dx \\ &= 2\pi\left[\tfrac{2}{5}x^5 - \tfrac{1}{6}x^6\right]_0^1 \\ &= 2\pi\left[\left(\tfrac{2}{5} - \tfrac{1}{6}\right) - 0\right] = 2\pi\left(\tfrac{7}{30}\right) = \tfrac{7}{15}\pi\end{aligned}$$





43. Use shells:

$$V = 2\int_0^r 2\pi x\sqrt{r^2 - x^2}\,dx$$

$$= -2\pi\int_0^r (r^2 - x^2)^{1/2}(-2x)\,dx$$

$$= \left[-2\pi \cdot \tfrac{2}{3}(r^2 - x^2)^{3/2}\right]_0^r$$

$$= -\tfrac{4}{3}\pi(0 - r^3) = \tfrac{4}{3}\pi r^3$$