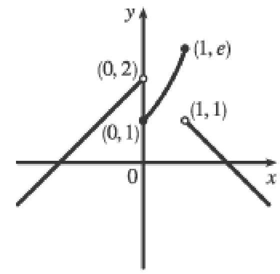
49. (a) $2^{x-5} = 3 \quad \Leftrightarrow \quad \log_2 3 = x - 5 \quad \Leftrightarrow \quad x = 5 + \log_2 3$.

Or: $2^{x-5} = 3 \quad \Leftrightarrow \quad \ln(2^{x-5}) = \ln 3 \quad \Leftrightarrow \quad (x-5)\ln 2 = \ln 3 \quad \Leftrightarrow \quad x - 5 = \dfrac{\ln 3}{\ln 2} \quad \Leftrightarrow \quad x = 5 + \dfrac{\ln 3}{\ln 2}$

(b) $\ln x + \ln(x-1) = \ln(x(x-1)) = 1 \quad \Leftrightarrow \quad x(x-1) = e^1 \quad \Leftrightarrow \quad x^2 - x - e = 0$. The quadratic formula (with $a = 1$, $b = -1$, and $c = -e$) gives $x = \frac{1}{2}\left(1 \pm \sqrt{1+4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}\left(1 + \sqrt{1+4e}\right)$.

39. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x > 1 \end{cases}$



f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0, 1)$ since it is an exponential.

Now $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(x+2) = 2$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} e^x = 1$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0. Also $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} e^x = e$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(2-x) = 1$, so f is discontinuous at 1. Since $f(1) = e$, f is continuous from the left at 1.

40. $y = \sqrt[4]{\dfrac{x^2+1}{x^2-1}} \quad \Rightarrow \quad \ln y = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1) \quad \Rightarrow \quad \dfrac{1}{y}y' = \dfrac{1}{4}\cdot\dfrac{1}{x^2+1}\cdot 2x - \dfrac{1}{4}\cdot\dfrac{1}{x^2-1}\cdot 2x \quad \Rightarrow$

$y' = \sqrt[4]{\dfrac{x^2+1}{x^2-1}}\cdot\dfrac{1}{2}\left(\dfrac{x}{x^2+1} - \dfrac{x}{x^2-1}\right) = \dfrac{1}{2}\sqrt[4]{\dfrac{x^2+1}{x^2-1}}\left(\dfrac{-2x}{x^4-1}\right) = \dfrac{x}{1-x^4}\sqrt[4]{\dfrac{x^2+1}{x^2-1}}$

28. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad dy = \dfrac{1}{2\sqrt{x}}\,dx$. When $x = 100$ and $dx = -0.2$, $dy = \dfrac{1}{2\sqrt{100}}(-0.2) = -0.01$, so $\sqrt{99.8} = f(99.8) \approx f(100) + dy = 10 - 0.01 = 9.99$.

40. This limit has the form $\infty \cdot 0$. $\lim\limits_{x\to-\infty} x^2 e^x = \lim\limits_{x\to-\infty}\dfrac{x^2}{e^{-x}} \overset{\text{H}}{=} \lim\limits_{x\to-\infty}\dfrac{2x}{-e^{-x}} \overset{\text{H}}{=} \lim\limits_{x\to-\infty}\dfrac{2}{e^{-x}} = \lim\limits_{x\to-\infty} 2e^x = 0$

55. $y = (1-2x)^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln(1-2x)$, so $\lim\limits_{x\to 0}\ln y = \lim\limits_{x\to 0}\dfrac{\ln(1-2x)}{x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{-2/(1-2x)}{1} = -2 \quad \Rightarrow$

$\lim\limits_{x\to 0}(1-2x)^{1/x} = \lim\limits_{x\to 0} e^{\ln y} = e^{-2}$.

50. Call the two integers x and y. Then $x + 4y = 1000$, so $x = 1000 - 4y$. Their product is $P = xy = (1000-4y)y$, so our problem is to maximize the function $P(y) = 1000y - 4y^2$, where $0 < y < 250$ and y is an integer. $P'(y) = 1000 - 8y$, so $P'(y) = 0 \quad \Leftrightarrow \quad y = 125$. $P''(y) = -8 < 0$, so $P(125) = 62{,}500$ is an absolute maximum. Since the optimal y turned out to be an integer, we have found the desired pair of numbers, namely $x = 1000 - 4(125) = 500$ and $y = 125$.

57. $F(x) = \int_1^x f(t)\,dt \quad \Rightarrow \quad F'(x) = f(x) = \int_1^{x^2} \frac{\sqrt{1+u^4}}{u}\,du \quad \left[\text{since } f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u}\,du\right] \quad \Rightarrow$

$$F''(x) = f'(x) = \frac{\sqrt{1+(x^2)^4}}{x^2} \cdot \frac{d}{dx}(x^2) = \frac{\sqrt{1+x^8}}{x^2} \cdot 2x = \frac{2\sqrt{1+x^8}}{x}. \text{ So } F''(2) = \sqrt{1+2^8} = \sqrt{257}.$$

36. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\sin x\,dx = -du$, so

$$\int \frac{\sin x}{1+\cos^2 x}\,dx = \int \frac{-du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

46. Let $u = x^2 + 1$ [so $x^2 = u - 1$]. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,x\,dx = \int (u-1)\sqrt{u}\,\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int (u^{3/2} - u^{1/2})\,du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right) + C = \tfrac{1}{5}(x^2+1)^{5/2} - \tfrac{1}{3}(x^2+1)^{3/2} + C.$$

Or: Let $u = \sqrt{x^2+1}$. Then $u^2 = x^2 + 1 \quad \Rightarrow \quad 2u\,du = 2x\,dx \quad \Rightarrow \quad u\,du = x\,dx$, so

$$\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,x\,dx = \int (u^2-1)\,u \cdot u\,du = \int (u^4 - u^2)\,du$$

$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}(x^2+1)^{5/2} - \tfrac{1}{3}(x^2+1)^{3/2} + C.$$

Note: This answer can be written as $\frac{1}{15}\sqrt{x^2+1}(3x^4 + x^2 - 2) + C$.

67. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2-1) = 2.$$

28. The curves $y = 3x^2$ and $y = -4x + 4$ intersect when $3x^2 = -4x + 4 \quad [\text{for } x \geq 0] \quad \Leftrightarrow$

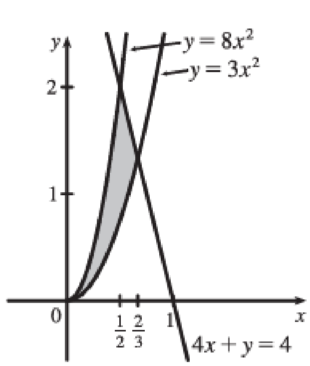
$3x^2 + 4x - 4 = 0 \quad \Leftrightarrow \quad (3x-2)(x+2) = 0 \quad \Rightarrow$

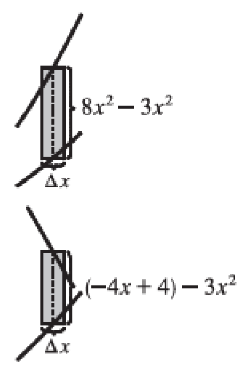
$x = \frac{2}{3}$. The curves $y = 8x^2$ and $y = -4x + 4$

intersect when $8x^2 = -4x + 4 \quad [\text{for } x \geq 0] \quad \Leftrightarrow$

$8x^2 + 4x - 4 = 0 \quad \Leftrightarrow \quad 2x^2 + x - 1 = 0 \quad \Leftrightarrow$

$(2x-1)(x+1) = 0 \quad \Rightarrow \quad x = \frac{1}{2}$.





$$A = \int_0^{1/2} (8x^2 - 3x^2)\,dx + \int_{1/2}^{2/3} \left[(-4x+4) - 3x^2\right]dx$$

$$= \int_0^{1/2} 5x^2\,dx + \int_{1/2}^{2/3} (-3x^2 - 4x + 4)\,dx = \left[\tfrac{5}{3}x^3\right]_0^{1/2} + \left[-x^3 - 2x^2 + 4x\right]_{1/2}^{2/3}$$

$$= \tfrac{5}{3}\left(\tfrac{1}{2}\right)^3 - 0 + \left[-\left(\tfrac{2}{3}\right)^3 - 2\left(\tfrac{2}{3}\right)^2 + 4\left(\tfrac{2}{3}\right)\right] - \left[-\left(\tfrac{1}{2}\right)^3 - 2\left(\tfrac{1}{2}\right)^2 + 4\left(\tfrac{1}{2}\right)\right] = \tfrac{5}{24} - \tfrac{8}{27} - \tfrac{8}{9} + \tfrac{8}{3} + \tfrac{1}{8} + \tfrac{1}{2} - 2$$

$$= \tfrac{45}{216} - \tfrac{64}{216} - \tfrac{192}{216} + \tfrac{576}{216} + \tfrac{27}{216} + \tfrac{108}{216} - \tfrac{432}{216} = \tfrac{68}{216} = \tfrac{17}{54} \quad [\approx 0.315]$$

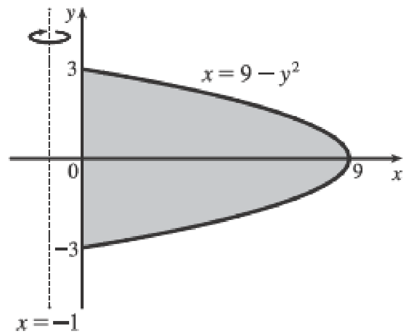
19. $\rho_{\text{ave}} = \dfrac{1}{8}\displaystyle\int_0^8 \frac{12}{\sqrt{x+1}}\,dx = \frac{3}{2}\int_0^8 (x+1)^{-1/2}\,dx = \left[3\sqrt{x+1}\right]_0^8 = 9 - 3 = 6 \text{ kg/m}$

9. $V = \pi \int_{-3}^{3} \left\{ \left[(9-y^2)-(-1)\right]^2 - \left[0-(-1)\right]^2 \right\} dy$

$= 2\pi \int_0^3 \left[(10-y^2)^2 - 1\right] dy = 2\pi \int_0^3 (100 - 20y^2 + y^4 - 1)\,dy$

$= 2\pi \int_0^3 (99 - 20y^2 + y^4)\,dy = 2\pi \left[99y - \frac{20}{3}y^3 + \frac{1}{5}y^5\right]_0^3$

$= 2\pi\left(297 - 180 + \frac{243}{5}\right) = \frac{1656}{5}\pi$



25. Equilateral triangles with sides measuring $\frac{1}{4}x$ meters have height $\frac{1}{4}x\sin 60^\circ = \frac{\sqrt{3}}{8}x$. Therefore,

$A(x) = \frac{1}{2}\cdot\frac{1}{4}x\cdot\frac{\sqrt{3}}{8}x = \frac{\sqrt{3}}{64}x^2$. $\quad V = \int_0^{20} A(x)\,dx = \frac{\sqrt{3}}{64}\int_0^{20} x^2\,dx = \frac{\sqrt{3}}{64}\left[\frac{1}{3}x^3\right]_0^{20} = \frac{8000\sqrt{3}}{64\cdot 3} = \frac{125\sqrt{3}}{3}\text{ m}^3$.