MATH 101: SAMPLE PROBLEMS FOR MIDTERM II

Most of the exercises below come from Stewart. This list is not meant to be totally comprehensive. In particular, there are no exercises that cover Section 5.2 of Stewart in the following list of problems.

(1) The volume of a cube is increasing at a rate of $10\,\mathrm{cm}^3/\mathrm{min}$. How fast is the surface area increasing when when the length of an edge is $30\,\mathrm{cm}^3$?

(2) Use a linear approximation to estimate the value of $(8.06)^{2/3}$.

(3) The circumference of a sphere was measured to be 84 cm with a possible error of 0.5cm. Use differentials to estimate the maximum error if we use our measurement to calculate the volume of the sphere. What is the relative error.

(4) Calculate dy/dx for the following:
   (a) $y = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$ (Hint: logarithms.)
   (b) $y = \cos(3^{\sqrt{\tan 3x}})$.
   (c) $y = \ln\left|\dfrac{x^2-4}{2x+5}\right|$.
   (d) $xe^y = y - 1$.
   (e) $xy^4 + x^2y = x + 3y$.
   (f) $y = (\sin x)^{\ln x}$.

(5) The length of a rectangle is increasing at a rate of $8cm/s$ and its width is increasing at a rate of $3cm/s$. When the length is $20cm$ and the width is $10cm$, how fast is the area of the rectangle increasing?

(6) Evaluate the following limits:
   (a) $\lim_{x\to(\pi/2)^-} (\tan x)^{\cos x}$.
   (b) $\lim_{x\to\infty} \dfrac{e^{4x}-1-4x}{x^2}$.
   (c) $\lim_{x\to 1^+} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$.
   (d) $\lim_{x\to\infty} xe^{1/x} - x$.
   (e) $\lim_{x\to\infty} \left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)^x$.
   (f) $\lim_{x\to\infty} \sqrt{x^2+3x} - x$.

(7) Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

(8) Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root.

(9) Show that the equation $x \ln x = 3$ has exactly one solution in the interval $[2, 4]$.

(10) If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

(11) Find the absolute maximum and absolute minimum values of f on the given interval.
   (a) $f(x) = x^4 - 2x^2 + 3$ on $[-2, 3]$.
   (b) $f(t) = \sqrt[3]{t}(8 - t)$ on $[0, 8]$.
   (c) $f(x) = x - \ln x$ on $[1/2, 2]$.
   (d) $x - 2\cos x$ on $[-2, 0]$.

(12) Let $f(x) = xe^{-x^2}$.
   (a) What is the domain of f?
   (b) What are the intercepts of the graph of f?
   (c) Does the function have any symmetry? What kind?
   (d) Find the horizontal asymptote(s) of the function.
   (e) Compute $f'(x)$. Use it to find the intervals of increase and decrease of the function.
   (f) You are given that $f''(x) = -2x(3 + x^2)e^{-1/x^2}$. Use this information to find the intervals of concavity of f.
   (g) Does f have local maxima/minima or inflection points? If so, what are they?
   (h) Use all the information you've gathered to sketch a graph of $f(x)$.

(13) Let $f(x) = x\sqrt{5 - x}$.
   (a) What is the domain of f?
   (b) What are the intercepts of the graph of f?
   (c) Compute $f'(x)$. Use it to find the intervals of increase and decrease of the function.
   (d) You are given that $f''(x) = \dfrac{3x - 20}{4(5 - x)^{3/2}}$. Use this information to find the intervals of concavity of f.
   (e) Does f have local maxima/minima or inflection points? If so, what are they?
   (f) Use all the information you've gathered to sketch a graph of $f(x)$.

(14) Let $f(x) = \sqrt[3]{x^3 + 1}$.
   (a) What is the domain of f?
   (b) What are the intercepts of the graph of f?
   (c) Compute $f'(x)$. Use it to find the intervals of increase and decrease of the function.
   (d) You are given that $f''(x) = \dfrac{2x}{\sqrt[3]{(x^3 + 1)^5}}$. Use this information to find the intervals of concavity of f.
   (e) Does f have local maxima/minima or inflection points? If so, what are they?
   (f) Use all the information you've gathered to sketch a graph of $f(x)$.

(15) Let $f(x) = \dfrac{2x^3 + x^2 + 1}{x^2 + 1}$.

(a) What is the domain of f?
(b) What are the intercepts of the graph of f?
(c) Find the slant asymptote(s) of the function.
(d) Compute $f'(x)$. Use it to find the intervals of increase and decrease of the function.
(e) You are given that $f''(x) = \dfrac{4x(3-x^2)}{(x^2+1)^3}$. Use this information to find the intervals of concavity of f.
(f) Does f have local maxima/minima or inflection points? If so, what are they?
(g) Use all the information you've gathered to sketch a graph of $f(x)$.

(16) Let $f(x) = \dfrac{\sin x}{2+\cos x}$.
(a) What is the domain of f?
(b) What are the intercepts of the graph of f?
(c) Does the function have any symmetry? What kind(s)?
(d) Compute $f'(x)$. Use it to find the intervals of increase and decrease of the function.
(e) You are given that $f''(x) = \dfrac{-2\sin x(1-\cos x)}{(2+\cos x)^3}$. Use this information to find the intervals of concavity of f.
(f) Does f have local maxima/minima or inflection points? If so, what are they?
(g) Use all the information you've gathered to sketch a graph of $f(x)$.

(17) Let $f(x) = \frac{x^2}{x+8}$.
(a) What is the domain of f?
(b) What are the intercepts of the graph of f?
(c) Find the asymptote(s) of the function.
(d) Compute $f'(x)$. Use it to find the intervals of increase and decrease of the function.
(e) You are given that $f''(x) = \dfrac{128}{(x+8)^3}$. Use this information to find the intervals of concavity of f.
(f) Does f have local maxima/minima or inflection points? If so, what are they?
(g) Use all the information you've gathered to sketch a graph of $f(x)$.

(18) If $1200cm^2$ of material is available to make a box with square base and open top, find the largest possible volume of the box.

(19) Find the point on the line $y = 4x + 7$ that is closest to the origin.

(20) Find the points on the ellipse $4x^2y^2 = 4$ that are farthest from the point $(1, 0)$

(21) The top and bottom margins of a poster are each 6 cm and the side margins are 4 cm. If the area of printed material on the poster is fixed at $384cm^2$, find the dimensions of the poster with the smallest area.

(22) Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

(23) At which points on the curve $1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

(24) An oil field containing 20 wells has been producing 4000 barrels of oil daily (200 barrels per well). For each new well drilled, the production of each well decreases by 5 barrels. How many new wells should be drilled to maximize the total daily production of the oil field?

(25) A weather balloon is rising vertically at the rate of 5 meters per second. An observer is standing on the ground 300 meters from the point where the balloon was released. At what rate is the distance between the observer and the balloon changing when the balloon is 400 meters high?

(26) Find f

(a) $f'(x) = \cos x - (1 - x^2)^{-1/2}$.

(b) $f'(x) = \dfrac{x^2 + \sqrt{x}}{x}, \quad f(1) = 3$.

(c) $f'(x) = 2e^x + \sec x \tan x$.

(d) $f''(x) = 2 - 12x, \quad f(0) = 9, f(2) = 15$.

(e) $f'(x) = \dfrac{2 + x^2}{1 + x^2}$.

(27) Find an expression for the area under the graph of $f(x) = x \cos x$ between 0 and $\pi/2$ as a limit. Do not evaluate the limit.

(28) Find an expression for the area under the graph of $f(x) = x^3$ between 0 and 1 as a limit. Use the fact that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

to evaluate the limit.

(29) Determine a region whose area is equal to $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(3 + \frac{7i}{5}\right)^{19}$.