Math 101 Fall 2004 Exam 1 **Solutions**

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour and fifteen minutes**. Do all 8 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work will receive no credit.
Please print you name clearly here.

Print name: ____________________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

____

Grader's use only:

1. ________ /10

2. ________ /15

3. ________ /10

4. ________ /20

5. ________ /10

6. ________ /10

7. ________ /15

8. ________ /10

1. [10 points] Evaluate the following limits, if they exist.

   (a) $\lim_{x\to 1} \frac{x^2+x-2}{x^2-4x+3}$

   Plugging in $x = 1$ gives $0/0$, so we factor out an $(x-1)$ from the numerator and denominator to give

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-4x+3} = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)(x-3)} = \lim_{x\to 1} \frac{x+2}{x-3} = \frac{1+2}{1-3} = -\frac{3}{2}.$$

   (b) $\lim_{\theta\to 0} \frac{1}{\theta} \sin \frac{\theta}{3}$

   As θ tends to zero so does $\theta/3$, so

$$\lim_{\theta\to 0} \frac{1}{\theta} \sin \frac{\theta}{3} = \frac{1}{3} \lim_{\theta\to 0} \frac{\sin(\theta/3)}{\theta/3} = \frac{1}{3}.$$

2. [15 points] Suppose c is a constant and the function f is given by:

$$f(x) = \begin{cases} c^2 x, & x < 1 \\ 3cx - 2, & x \geq 1 \end{cases}$$

(a) Calculate $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

(b) Find all values of the constant c so that the function f is continuous everywhere.

(a) For $x < 1$, we have $f(x) = c^2 x$ so

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} c^2 x = c^2.$$

For $x > 1$, we have $f(x) = 3cx - 2$ so

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3cx - 2 = 3c - 2.$$

(b) For $x < 1$, $f(x) = c^2 x$ is polynomial, hence continuous and for $x > 1$, $f(x) = 3cx - 2$ is polynomial, hence continous. The only place where f can fail to be continuous is at $x = 1$ where the two regions on which f is defined meet. For f to be continuous at $x = 1$ we need

$$\lim_{x\to 1} f(x) = f(1),$$

or equivalently

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1).$$

From part (a), this means $c^2 = 3c - 2$ or $c^2 - 3c + 2 = (c-1)(c-2) = 0$. Hence $c = 1$ and $c = 2$ are the only two values of c which make f continuous

3. [10 points] (a) Give the formal, mathematical definition of the derivative of a function $f(x)$ at the point $x = a$.

(b) Find the derivative of $f(x) = \frac{x}{1-2x}$ **using the definition of the derivative**. (No credit will be given for finding the derivative by other means.)

(a)

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

(b)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{x+h}{1-2(x+h)} - \frac{x}{1-2x}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)(1-2x) - x(1-2x-2h)}{h(1-2x)(1-2x-2h)} \\
&= \lim_{h\to 0} \frac{x + h - 2x^2 - 2xh - x + 2x^2 + 2hx}{h(1-2x)(1-2x-2h)} \\
&= \lim_{h\to 0} \frac{h}{h(1-2x)(1-2x-2h)} = \lim_{h\to 0} \frac{1}{(1-2x)(1-2x-2h)} \\
&= \frac{1}{(1-2x)^2}.
\end{aligned}$$

4. [20 points] Calculate the derivative for each of the following functions:

(a) $g(x) = (8x^2 - 5x)(13\sqrt{x} + 4)$

$$\begin{aligned} g'(x) &= \frac{d(8x^2 - 5x)}{dx}(13\sqrt{x} + 4) + (8x^2 - 5x)\frac{d(13\sqrt{x} + 4)}{dx} \\ &= (16x - 5)(13\sqrt{x} + 4) + (8x^2 - 5x)\frac{13}{2\sqrt{x}}. \end{aligned}$$

(b) $f(x) = \frac{\sec x}{2x^2 - 4x + 8}$

$$\begin{aligned} f'(x) &= \frac{\frac{d\sec x}{dx}(2x^2 - 4x + 8) - \sec x\frac{d(2x^2-4x+8)}{dx}}{(2x^2 - 4x + 8)^2} \\ &= \frac{(2x^2 - 4x + 8)\sec x\tan x - (4x - 4)\sec x}{(2x^2 - 4x + 8)^2}. \end{aligned}$$

(c) $k(\theta) = \cos^2\left(e^{3\theta+1}\right)$

$$\begin{aligned} k'(\theta) &= 2\cos(e^{3\theta+1})\frac{d\cos(e^{3\theta+1})}{d\theta} = -2\cos(e^{3\theta+1})\sin(e^{3\theta+1})\frac{de^{3\theta+1}}{d\theta} \\ &= -6\cos(e^{3\theta+1})\sin(e^{3\theta+1})e^{3\theta+1}. \end{aligned}$$

(d) $f(x) = \left(7x + \ln(x^2)\right)^6$

$$\begin{aligned} f'(x) &= 6\left(7x + \ln(x^2)\right)^5 \frac{d(7x + \ln(x^2))}{dx} = 6\left(7x + \ln(x^2)\right)^5 \left(7 + \frac{1}{x^2}\frac{d(x^2)}{dx}\right) \\ &= 6\left(7x + \ln(x^2)\right)^5 \left(7 + \frac{2}{x}\right). \end{aligned}$$

5. [10 points] Find the equation of the tangent line to the graph of $y = xe^{2x}$ at $x = 1$.

Since

$$\frac{dy}{dx} = e^{2x} + 2xe^{2x}$$

the slope of the tangent line at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 3e^2.$$

Since $f(1) = e^2$ the tangent line goes through the point $(1, e^2)$ so it is

$$(y - e^2) = 3e^2(x - 1)$$

or

$$y = 3e^2 x - 2e^2.$$

6. [10 points] Find the maximum and minimum value of the function $f(x) = x\sqrt{2-x^2}$ on the interval $[-\sqrt{2}, \sqrt{2}]$. Be sure to show all the steps you need to show in order to justify that your answers really are the maximum and minimum.

Since the inside of the square root is never negative this function is continuous on $[-\sqrt{2}, \sqrt{2}]$. Therefore the max and min will occur at endpoints or critical points. We compute

$$f'(x) = \sqrt{2-x^2} + x\frac{1}{2\sqrt{2-x^2}}\frac{d(2-x^2)}{dx} = \sqrt{2-x^2} - \frac{x^2}{\sqrt{2-x^2}}.$$

The derivative fails to exist at the endpoints $x = \pm\sqrt{2}$. Setting the derivative equal to zero the other critical points occur when

$$\sqrt{2-x^2} = \frac{x^2}{\sqrt{2-x^2}},$$

which gives $2-x^2 = x^2$, or $2x^2 = 2$, so $x = \pm 1$. Since $f(-\sqrt{2}) = f(\sqrt{2}) = 0$, $f(-1) = -1$ and $f(1) = 1$, we see that the maximum value is 1 at $x = 1$ and the minimum value is -1 at $x = -1$.

7. [15 points] A grain silo is to be built in the shape of a right circular cylinder with a hemispherical top. The concrete floor of the silo costs \$10 per square foot and the sides and top cost \$3 per square foot. You have 3600π dollars to spend. What is the maximum possible volume for the silo? (Some useful geometry formulas: The volume of a right circular cylinder of height h and radius r is $V = \pi r^2 h$ and the curved part of the surface of the cylinder has area $S = 2\pi r h$. The volume and surface area of a sphere of radius r are $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, respectively.)

Let r be the radius of the silo (in feet) and h the height of the cylindrical part of the silo (also in feet). Since the volume of the silo is the volume of a cylinder or radius r and height h plus half a sphere of radius r, the total volume will be

$$V = \pi r^2 h + \frac{2}{3}\pi r^3.$$

The concrete floor will have area πr^2 hence cost $10\pi r^2$. The sides will have area $2\pi r h$ and cost $6\pi r h$ and the top will have area $2\pi r^2$ and hence cost $6\pi r^2$. Hence the total cost will be $3600\pi = 6\pi r h + 16\pi r^2$. Hence $h = 600/r - 8r/3$. Thus we get

$$V(r) = 600\pi r - 2\pi r^3.$$

Since r and h must both be nonnegative, the domain is $0 \leq r \leq 15$. Since $V(r)$ is a polynomial it is continuous and we compute

$$V'(r) = 600\pi - 6\pi r^2$$

which vanishes when $r^2 = 100$ or $r = \pm 10$ (but only $r = 10$ is in our domain). Also $V(0) = 0$, $V(10) = 4000\pi$ and $V(15) = 2250\pi$. Hence the maximum volume is 4000π cubic feet.

8. [10 points] Find dy/dx for the following by implicit differentiation.

$$(x^2 + y^2)^2 = 50xy.$$

Differentiating the left hand side with respect to x gives

$$2(x^2 + y^2)\frac{d(x^2 + y^2)}{dx} = 2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right)$$

and differentiating the right hand side with respect to x gives

$$50y + 50x\frac{dy}{dx}.$$

So

$$2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 50y + 50x\frac{dy}{dx}.$$

Moving all the dy/dx terms to the left and all other terms to the right gives

$$(4y(x^2 + y^2) - 50x)\frac{dy}{dx} = 50y - 4x(x^2 + y^2),$$

or

$$\frac{dy}{dx} = \frac{50y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - 50x}.$$