Math 101 Fall 2006 Exam 1 **Solutions**

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour and fifteen minutes**. Do all 7 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work will receive no credit.
Please print you name clearly here.

Print name: ____

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

____

Grader's use only:

1. ________ /15

2. ________ /10

3. ________ /25

4. ________ /10

5. ________ /15

6. ________ /15

7. ________ /10

1. [15 points] Evaluate the following limits, if they exist.

   (a) $\lim_{x\to -3} \frac{x^2+2x-3}{x^2+4x+3}$

   Plugging in gives 0/0, hence we factor $x+3$ out of the numerator and denominator to obtain

$$\lim_{x\to -3} \frac{x^2+2x-3}{x^2+4x+3} = \lim_{x\to -3} \frac{(x+3)(x-1)}{(x+3)(x+1)} = \lim_{x\to -3} \frac{x-1}{x+1} = \frac{-4}{-2} = 2.$$

   (b) $\lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x-4}$

   Multiplying numerator and denominator by the conjugate $\sqrt{2x+1}+3$ gives

$$\begin{aligned} \lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x-4} &= \lim_{x\to 4} \frac{(\sqrt{2x+1}-3)(\sqrt{2x+1}+3)}{(x-4)(\sqrt{2x+1}+3)} \\ &= \lim_{x\to 4} \frac{2x+1-9}{(x-4)(\sqrt{2x+1}+3)} = \lim_{x\to 4} \frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)} \\ &= \lim_{x\to 4} \frac{2}{\sqrt{2x+1}+3} = \frac{2}{6} = \frac{1}{3}. \end{aligned}$$

   (c) $\lim_{\theta\to 0} \frac{\sin(3\theta)}{\tan(4\theta)}$

   Rewriting the quantity inside the limit we see

$$\lim_{\theta\to 0} \frac{\sin(3\theta)}{\tan(4\theta)} = \lim_{\theta\to 0} \frac{\sin(3\theta)}{3\theta} \cdot \frac{4\theta}{\tan(4\theta)} \cdot \frac{3}{4} = 1 \cdot 1 \cdot \frac{3}{4} = \frac{3}{4}.$$

2. [10 points] Find the derivative of $g(t) = \frac{1}{2t+1}$ using the **definition of the derivative**. No credit will be given for derivatives obtained in any other way.

$$
\begin{aligned}
g'(t) &= \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h\to 0} \frac{\frac{1}{2(t+h)+1} - \frac{1}{2t+1}}{h} \\
&= \lim_{h\to 0} \frac{(2t+1) - (2t+2h+1)}{h(2t+1)(2t+2h+1)} = \lim_{h\to 0} \frac{-2h}{h(2t+1)(2t+2h+1)} \\
&= \lim_{h\to 0} \frac{-2}{(2t+1)(2t+2h+1)} = \frac{-2}{(2t+1)^2}
\end{aligned}
$$

3. [25 points] Calculate the derivative for each of the following functions:

(a) $g(x) = 1 + x^3 + e^{-x} + 5\sin x$

$$g'(x) = 3x^2 - e^{-x} + 5\cos x.$$

(b) $h(x) = (x+2)\tan(6x)$

$$\begin{aligned} h'(x) &= \frac{d(x+2)}{dx}\tan(6x) + (x+2)\frac{d\tan(6x)}{dx} \\ &= \tan(6x) + (x+2)\sec^2(6x)\frac{d(6x)}{dx} = \tan(6x) + 6(x+2)\sec^2(6x). \end{aligned}$$

(c) $g(\theta) = \frac{(\sec\theta)^{5/3}}{\theta}$

$$\begin{aligned} g'(\theta) &= \frac{\theta\frac{d(\sec\theta)^{5/3}}{d\theta} - (\sec\theta)^{5/3}\frac{d\theta}{d\theta}}{\theta^2} = \frac{\frac{5}{3}\theta(\sec\theta)^{2/3}\frac{d\sec\theta}{d\theta} - (\sec\theta)^{5/3}}{\theta^2} \\ &= \frac{\frac{5}{3}\theta(\sec\theta)^{2/3}\sec\theta\tan\theta - (\sec\theta)^{5/3}}{\theta^2} = \frac{\frac{5}{3}\theta(\sec\theta)^{5/3}\tan\theta - (\sec\theta)^{5/3}}{\theta^2} \end{aligned}$$

(d) $f(x) = xe^{\sqrt{x}}$

$$f'(x) = e^{\sqrt{x}} + x\frac{de^{\sqrt{x}}}{dx} = e^{\sqrt{x}} + xe^{\sqrt{x}}\frac{d\sqrt{x}}{dx} = e^{\sqrt{x}} + xe^{\sqrt{x}}\frac{1}{2\sqrt{x}} = e^{\sqrt{x}} + \frac{1}{2}\sqrt{x}e^{\sqrt{x}}.$$

(e) $y = \ln(x^4 + x^{-2} + 3)$

$$\frac{dy}{dx} = \frac{1}{x^4 + x^{-2} + 3}\frac{d(x^4 + x^{-2} + 3)}{dx} = \frac{4x^3 - 2x^{-3}}{x^4 + x^{-2} + 3}.$$

4. [10 points] Find the tangent line to the curve $y = \frac{x+1}{x-1}$ at $(2, 3)$.

The slope of the tangent line at an arbitrary point on the curve will be

$$\frac{dy}{dx} = \frac{(x-1)\frac{d(x+1)}{dx} - (x+1)\frac{d(x-1)}{dx}}{(x-1)^2} = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}.$$

Therefore at $x = 2$ the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{x=2} = \frac{-2}{(2-1)^2} = -2.$$

Hence the tangent line is $y - 3 = -2(x-2)$ or $y = -2x + 7$.

5. [15 points] Find the maximum and minimum values of $f(x) = \frac{e^x}{x^2+1}$ on the interval $[0,3]$. Be sure to show all steps in order to justify your answers really are the minimum and maximum.

The following may be helpful $e = 2.71828\ldots$, $e^2 = 7.38905\ldots$, $e^3 = 20.08553\ldots$.

Since the denominator x^2+1 is always positive, $f(x)$ is continuous. Hence the minimum and maximum exist and are either at the endpoints or at critical points. We compute

$$f'(x) = \frac{(x^2+1)\frac{de^x}{dx} - e^x\frac{d(x^2+1)}{dx}}{(x^2+1)^2} = \frac{(x^2+1)e^x - 2xe^x}{(x^2+1)^2} = \frac{(x-1)^2e^x}{(x^2+1)^2}.$$

The denominator is never zero, so the only critical points will be where $f'(x) = 0$. Since $e^x > 0$, this is just at $x = 1$. We compute

$$\begin{aligned} f(0) &= \frac{e^0}{1} = 1 \\ f(1) &= \frac{e}{2} = \frac{2.71828\ldots}{2} = 1.35914\ldots \\ f(3) &= \frac{e^3}{10} = \frac{20.08553\ldots}{10} = 2.008553\ldots \end{aligned}$$

Therefore the minimum value is 1 and the maximum value is $e^3/10$.

6. [15 points] A cylindrical steel drum is to be made from a rectangular steel sheet and two circular sheets by bending the rectangle to form the curved portion of the cylinder and welding together the resulting seam, then welding on the two circular pieces to form the top and bottom. The total length of the welds is to be 600 in. What dimensions give the maximum possible volume?

Let r be the radius of the cylinder in inches and h the height in inches. The volume is then $V = \pi r^2 h$. The weld to make the curved portion of the cyliner has length h and the top and bottom welds give two copies of the circumference of the circle $2\pi r$. Hence

$$600 = h + 4\pi r \text{ or } h = 600 - 4\pi r.$$

Eliminating we get

$$V(r) = \pi r^2(600 - 4\pi r) = 600\pi r^2 - 4\pi^2 r^3.$$

Since h and r are lengths we have $h = 600 - 4\pi r \geq 0$ or $r \leq 150/\pi$ and $r \geq 0$. Hence the range for r is $0 \leq r \leq 150/\pi$.

$V(r)$ is a polynomial, hence continuous, and

$$V'(r) = 1200\pi r - 12\pi^2 r^2 = 12\pi r(100 - \pi r)$$

exists for all r. Thus the only critical points are where $V'(r) = 0$, so $r = 0$ or $r = 100/\pi$. Since $r = 150/\pi$ gives $h = 0$ we see that $V(0) = V(150/\pi) = 0$ and the maximum is at the critical point. Setting $r = 100/\pi$ gives $h = 600 - 4\pi(100/\pi) = 200$. So the maximum volume occurs for $r = 100/\pi$ inches, $h = 200$ inches and is $V = \pi r^2 h = 2,000,000/\pi$ cubic inches.

7. [10 points] A fire has started in a dry open field and spreads in the form of a circle. The radius of the circle increases at a rate of 6 ft/min. Find the rate at which the fire area is increasing when the radius is 150 ft.

Let r be the radius of the circle in ft. Note that we are told $\frac{dr}{dt} = 6$. Let A be the area burnt. Note that we are asked for $\frac{dA}{dt}$. Since $A = \pi r^2$, we have (differentiating with respect to t)

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Hence, plugging in the particular time when $r = 150$,

$$\left.\frac{dA}{dt}\right|_{r=150} = 2\pi(150)(6) = 1800\pi.$$

Thus the fire area is increasing at 1800π ft^2/min.