Math 101 Fall 2004 Exam 2 **Solutions**

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour and fifteen minutes**. Do all 7 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work will receive no credit.
Please print you name clearly here.

Print name: ____

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

____

Grader's use only:

1. ________ /10

2. ________ /10

3. ________ /25

4. ________ /10

5. ________ /10

6. ________ /15

7. ________ /20

1. [10 points] Compute the first three derivatives of the following function.

$$f(t) = (t^2 + 3t)\ln\left(t^2 + 3t\right)$$

$$f'(t) = (2t+3)\ln\left(t^2 + 3t\right) + (t^2+3t)\frac{2t+3}{t^2+3t} = (2t+3)\ln\left(t^2 + 3t\right) + 2t + 3.$$

$$f''(t) = 2\ln(t^2 + 3t) + (2t+3)\frac{2t+3}{t^2+3t} + 2 = 2\ln(t^2+3t) + \frac{(2t+3)^2}{t^2+3t} + 2.$$

$$\begin{aligned} f'''(t) &= 2\frac{2t+3}{t^2+3t} + \frac{4(2t+3)(t^2+3t) - (2t+3)^2 \cdot (2t+3)}{(t^2+3t)^2} \\ &= \frac{2(2t+3)}{t^2+3t} - \frac{9(2t+3)}{(t^2+3t)^2}. \end{aligned}$$

2. [10 points] Evaluate the following limits, if they exist.

(a) $\lim_{t\to 0} \frac{1-\cos 3t}{t \sin t}$

This limit is indeterminate of type 0/0 so applying L'Hôpital (twice) gives

$$\lim_{t\to 0} \frac{1-\cos 3t}{t\sin t} = \lim_{t\to 0} \frac{3\sin 3t}{\sin t + t\cos t} = \lim_{t\to 0} \frac{9\cos 3t}{2\cos t - t\sin t} = \frac{9}{2}.$$

(b) $\lim_{x\to 0} (1-3x)^{1/(2x)}$

This limit is indeterminate of type 1^∞ so rearranging and using L'Hôpital gives

$$\begin{aligned}\lim_{x\to 0}(1-3x)^{1/(2x)} &= \lim_{x\to 0} \exp\left(\frac{\ln(1-3x)}{2x}\right) = \exp\left(\lim_{x\to 0}\frac{\ln(1-3x)}{2x}\right) \\ &= \exp\left(\lim_{x\to 0}\frac{\frac{-3}{1-3x}}{2}\right) = \exp(-3/2) = e^{-3/2}.\end{aligned}$$

3. [25 points] Evaluate the following integrals:

(a) $\int (e^t + 1)^2 \, dt$

$$\int (e^t + 1)^2 dt = \int (e^{2t} + 2e^t + 1) dt = \frac{1}{2} e^{2t} + 2e^t + t + C.$$

(b) $\int x^2 \sec^2\left(x^3 + 1\right) \, dx$
Subsituting $u = x^3 + 1$ so $du = 3x^2 \, dx$ gives

$$\int x^2 \sec^2\left(x^3 + 1\right) \, dx = \frac{1}{3} \int \sec^2 u \, du = \frac{\tan u}{3} + C = \frac{\tan\left(x^3 + 1\right)}{3} + C.$$

(c) $\int \sin^5 3z \cos 3z \, dz$
Substituting $u = \sin(3z)$, $du = 3\cos(3z) \, dz$ gives

$$\int \sin^5 3z \cos 3z \, dz = \frac{1}{3} \int u^5 \, du = \frac{1}{18} u^6 + C = \frac{1}{18} \sin^6(3z) + C.$$

(d) $\int_0^1 x(2 - x^2)^3 \, dx$
Substituting $u = 2 - x^2$, so $du = -2x \, dx$ and $x = 0$ means $u = 2$, $x = 1$ means $u = 1$ gives

$$\int_0^1 x(1 - x^2)^3 \, dx = -\frac{1}{2} \int_2^1 u^3 \, du = \frac{1}{2} \int_1^2 u^3 du = \frac{1}{8} u^4 \Big|_1^2 = 2 - \frac{1}{8} = \frac{15}{8}.$$

(e) $\int_0^{\pi/2} e^{\sin x} \cos x \, dx$
Substituting $u = \sin x$, so $du = \cos x \, dx$ and $x = 0$ means $u = 0$, $x = \pi/2$ means $u = 1$ gives

$$\int_0^{\pi/2} e^{\sin x} \cos x \, dx = \int_0^1 e^u \, du = e^u \big|_0^1 = e - 1.$$

4. [10 points] Evaluate the definite integral below directly from the definition. That is, compute $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ for a regular partition of the given interval of integration.

$$\int_0^3 (3x^2 + 1)\, dx$$

The following formulas may be helpful

$$\sum_{i=1}^{n} 1 = n, \quad \sum_{i=1}^{n} i = \frac{1}{2}n^2 + \frac{1}{2}n, \quad \sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

Since $a = 0$, $b = 3$ and $f(x) = 3x^2 + 1$ we have $\Delta x = (b - a)/n = 3/n$ and $x_i = a + i\Delta x = 3i/n$. Hence

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} \left(3\left(\frac{3i}{n}\right)^2 + 1 \right) \frac{3}{n} = \sum_{i=1}^{n} \left(\frac{81i^2}{n^3} + \frac{3}{n} \right) \\
&= \frac{81}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1 = \frac{81}{n^3} \left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \right) + \frac{3}{n}n \\
&= 27 + \frac{81}{2n} + \frac{27}{2n^2} + 3 = 30 + \frac{81}{2n} + \frac{27}{2n^2}.
\end{aligned}$$

Hence

$$\int_0^3 (3x^2 + 1)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \left(30 + \frac{81}{2n} + \frac{27}{2n^2} \right) = 30.$$

5. [10 points] Find the area of the region in the plane bounded by

$$y = x^3 - 3x^2 + 2x \quad \text{and} \quad y = 2x.$$

The two curves intersect when $y = x^3 - 3x^2 + 2x = 2x$, hence $x^3 = 3x^2$, so $x = 0$ or $x = 3$. Plugging in $x = 1$, we get $y = 0$ for the cubic and $y = 2$ for the line. Hence the line is higher and the region is below $y = 2x = f(x)$, above $y = x^3 - 3x^2 + 2x = g(x)$ between $x = 0 = a$ and $x = 3 = b$. So

$$\begin{aligned} A &= \int_a^b (f(x) - g(x))\, dx = \int_0^3 \big(2x - (x^3 - 3x^2 + 2x)\big)\, dx \\ &= \int_0^3 (3x^2 - x^3)\, dx = \left(x^3 - \frac{1}{4}x^4\right)\Big|_0^3 \\ &= \left(27 - \frac{81}{4}\right) - 0 = \frac{27}{4}. \end{aligned}$$

6. [15 points] We define the plane region R to be bounded by

$$y = x^2 \quad \text{and} \quad x = y^2.$$

Consider the volume V generated by rotating the region R around the x-axis.

(a) Using the method of cross-sections, compute the volume V described above.

To do cross-sections, we need the region described in terms of x. The two parabolas intersect at $x = 0 = a$ and $x = 1 = b$. Between $x = 0$ and $x = 1$, the higher parabola is $y = f(x) = \sqrt{x}$ and the lower curve is $y = g(x) = x^2$. Hence the volume is

$$\begin{aligned} V &= \int_a^b \pi[f(x)^2 - g(x)^2]\, dx = \int_0^1 \pi\left[(\sqrt{x})^2 - (x^2)^2\right]\, dx \\ &= \pi \int_0^1 \left(x - x^4\right)\, dx = \pi\left(\frac{1}{2}x^2 - \frac{1}{5}x^5\right)\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}. \end{aligned}$$

(b) Using the method of cylindrical shells, compute the volume V described above. Note: You should get the same result as in part 6a.

To do shells, we need the region described in terms of y. The two parabolas intersect at $y = 0 = c$ and $y = 1 = d$. Between $y = 0$ and $y = 1$ the rightmost curve is $y = x^2$ or $x = \sqrt{y} = h(y)$ and the leftmost curve is $x = y^2 = k(y)$. Hence the volume is

$$\begin{aligned} V &= \int_c^d 2\pi y[h(y) - k(y)]\, dy = \int_0^1 2\pi y(\sqrt{y} - y^2)\, dy \\ &= 2\pi \int_0^1 (y^{3/2} - y^3)\, dy = 2\pi\left(\frac{2}{5}y^{5/2} - \frac{1}{4}y^4\right)\Big|_0^1 = 2\pi\left(\frac{2}{5} - \frac{1}{4}\right) \\ &= \frac{3\pi}{10}. \end{aligned}$$

7. [20 points] For the function $f(x) = \frac{\sqrt{x^2+1}}{x+5}$, the first two derivatives are $f'(x) = \frac{5x-1}{(x+5)^2\sqrt{x^2+1}}$ and $f''(x) = \frac{(3-2x)(5x^2+6x+9)}{(x+5)^3(x^2+1)^{3/2}}$. YOU NEED NOT VERIFY THESE FORMULAS.

(a) Find (and justify) all horizontal and vertical asymptotes of the graph $y = f(x)$. At any vertical asymptotes compute both the left and right hand limits of $f(x)$.

Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+5} = \lim_{x\to\infty} \frac{\sqrt{1+x^{-2}}}{1+5/x} = \frac{\sqrt{1+0}}{1+0} = 1,$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+5} = \lim_{x\to-\infty} -\frac{\sqrt{1+x^{-2}}}{1+5/x} = -\frac{\sqrt{1+0}}{1+0} = -1,$$

we see f has two horizontal asymptotes $y = 1$ approached at ∞ and $y = -1$ approached at $-\infty$. The only possible vertical asymptote is at the discontinuity $x = -5$. Since the numerator is positive $f(x) > 0$ for $x > 5$ and $f(x) < 0$ for $x < 5$. Hence

$$\lim_{x\to 5^+} f(x) = +\infty, \text{ and } \lim_{x\to 5^-} f(x) = -\infty.$$

and $x = -5$ is a vertical asymptote.

(b) Find the intervals on which $f(x)$ is increasing and those on which it is decreasing.

Note that $f'(x)$ is undefined at the vertical asymptote $x = -5$. The denominator of $f'(x)$ is always non-negative so $f'(x) > 0$ for $x > 1/5$ and $f'(x) < 0$ for $-5 < x < 1/5$ and for $x < -5$. Thus f is decreasing on $(-\infty, -5)$ and on $(-5, 1/5]$ and f is increasing on $[1/5, \infty)$.

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(c) Find the critical points of $f(x)$ and classify them as local maxima, local minima or neither.

$f'(x)$ is undefined at $x = -5$, but this is the asymptote and $f(-5)$ is also undefined, hence it is not a critical point. Solving $f'(x) = 0$ gives $x = 1/5$ as the only critical point. Since by (b), f' switches from negative to positive at $x = 1/5$, by the First Derivative Test, $x = 1/5$ is a local minimum.

(d) Find the intervals on which $f(x)$ is concave upward and those on which it is concave downward. (It may be helpful to notice that $5x^2 + 6x + 9 = 4x^2 + (x + 3)^2$ is positive for all x.)

The factors $5x^2+6x+9$ and $(x^2+1)^{3/2}$ are both positive. $3-2x$ is positive for $x < 3/2$ and negative for $x > 3/2$. $(x + 5)^3$ is positive for $x > -5$ and negative for $x < -5$. Hence $f''(x)$ is negative on $(-\infty, -5)$ and on $(3/2, \infty)$ and $f''(x)$ is positive on $(-5, 3/2)$. Hence f is concave down on $(-\infty, -5)$ and on $(3/2, \infty)$ and f is concave up on $(-5, 3/2)$.

(e) On the next page sketch the graph of $y = \frac{\sqrt{x^2+1}}{x+5}$ showing the results of (a)-(d). (The following values may be helpful $f(1/5) = 1/\sqrt{26} \approx 0.196$, $f(3/2) = 1/\sqrt{13} \approx 0.277$.)