Math 101, Spring 2008, Exam 2
Solutions

The exam consists of 8 questions. You must show all your work to receive full credit. Please indicate your final answer clearly. Write the honor pledge on your exam when you are finished. Good luck!

In case you need them, here are some formulas:

$$\begin{aligned}\Sigma_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\ \Sigma_{i=1}^{n} i^3 &= \frac{n^2(n+1)^2}{4} = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\end{aligned}$$

1) Find the limit:

a) $\lim_{t\to\infty} \frac{e^t+t^2}{e^t-t}$

There are a few ways to do this one. My favorite is just to apply L'Hôpital's rule a bunch. Since it is in the indeterminate form $\frac{\infty}{\infty}$, we can apply L'Hôpital's rule.

$$\begin{aligned}&\lim_{t\to\infty} \frac{e^t+t^2}{e^t-t} \\ =& \lim_{t\to\infty} \frac{e^t+2t}{e^t-1} \\ =& \lim_{t\to\infty} \frac{e^t+2}{e^t} \\ =& \lim_{t\to\infty} \frac{e^t}{e^t} \\ =& 1\end{aligned}$$

b) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$

This has the indeterminate form 1^∞, so we need to take the natural logarithm to get it into a different form first. First, I'm going to let $L = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$.

$$\begin{aligned}L &= \lim_{x\to 0}(1+x)^{\frac{1}{x}} \\ \ln L &= \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}} \\ &= \lim_{x\to 0}\frac{1}{x}\ln(1+x)\end{aligned}$$

That has the indeterminate form $\infty \cdot 0$, but that is easily fixed by rewriting it as $\lim_{x\to 0}\frac{\ln(1+x)}{x}$, which has the indeterminate form $\frac{0}{0}$. Now we can use L'Hôpital's rule.

$$\begin{aligned}\ln L &= \lim_{x\to 0}\frac{\ln(1+x)}{x} \\ &= \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} \\ &= 1\end{aligned}$$

So now we know $\ln L = 1$. To find L itself, notice that $L = e^{\ln L} = e^1 = e$. So the limit is e.

2) Compute the definite integral:

a) $\displaystyle\int_0^{\frac{\pi}{4}} (\sin 2x - 2\cos 5x)dx$

$$\begin{aligned}
&= \left[\frac{-\cos 2x}{2} - \frac{2\sin 5x}{5}\right]_0^{\frac{\pi}{4}} \\
&= \left[\frac{-\cos\frac{\pi}{2}}{2} - \frac{2\sin\frac{5\pi}{4}}{5}\right] - \left[\frac{-\cos 0}{2} - \frac{2\sin 0}{5}\right] \\
&= \left[0 - \frac{2}{5}\cdot\frac{-\sqrt{2}}{2}\right] - \left[\frac{-1}{2} + 0\right] \\
&= \frac{\sqrt{2}}{5} + \frac{1}{2}
\end{aligned}$$

b) $\displaystyle\int_1^2 \frac{x^2}{2} - \frac{2}{x^2}dx$

$$\begin{aligned}
&= \left[\frac{x^3}{6} + \frac{2}{x}\right]_1^2 \\
&= \left[\frac{8}{6} + 1\right] - \left[\frac{1}{6} + 2\right] \\
&= \frac{14}{6} - \frac{13}{6} \\
&= \frac{1}{6}
\end{aligned}$$

3) Find the indefinite integral:

a) $\displaystyle\int \sec\frac{t}{3}\tan\frac{t}{3}dt$

$3\sec\frac{t}{3} + C$

b) $\displaystyle\int \frac{3dx}{x\sqrt{x}}$

$\frac{3}{x\sqrt{x}} = 3x^{\frac{-3}{2}}$, so the integral is $-6x^{\frac{-1}{2}} + C$

4) Consider the function $f(x) = 3x + 6$ on the interval $I = [0, 4]$.

a) Using a partition of I into 4 regular subintervals, compute the right-endpoint Riemann sum for $f(x)$ associated with that partition.

The length of each subinterval is 1, and the right endpoints are 1, 2, 3, and 4. The Riemann sum is $1\cdot 9 + 1\cdot 12 + 1\cdot 15 + 1\cdot 18 = 54$

b) Using the Riemann sum definition of the integral, find $\displaystyle\int_0^4 f(x)dx$.

Using formulas (15) and (16) in section 5.4, for any regular partition of this interval, $\Delta x = \frac{4}{n}$ and $x_i = 0 + \frac{4i}{n}$, so $f(x_i) = 3(\frac{4i}{n}) + 6 = \frac{12i}{n} + 6$. The Riemann sum definition says

$$\begin{aligned}
&\int_0^4 f(x)dx \\
=\ & \lim_{n\to\infty} \Sigma_{i=1}^n \left(\frac{12i}{n}+6\right)\cdot\frac{4}{n} \\
=\ & \lim_{n\to\infty} \Sigma_{i=1}^n \left(\frac{48i}{n^2}+\frac{24}{n}\right) \\
=\ & \lim_{n\to\infty} \frac{48(n)(n+1)}{2n^2}+24 \\
=\ & 24\lim_{n\to\infty} \frac{n^2+n}{n^2}+24 \\
=\ & 24+24 \\
=\ & 48
\end{aligned}$$

5) Find the derivative of $F(x)$:

a) $F(x)=\int_0^x \sin^2 t dt$

By the fundamental theorem of calculus, $F'(x)=\sin^2 x$.

b) $F(x)=\int_{\frac{\pi}{3}}^{e^{2x}} \sin^2 t dt$

Let $u(x)=e^{2x}$, so $\frac{du}{dx}=2e^{2x}$. Let $y(u)=\int_{\frac{\pi}{3}}^{u} \sin^2 t dt$. The fundamental theorem of calculus says $\frac{dy}{du}=\sin^2 u$. The chain rule says $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\sin^2 u\cdot 2e^{2x}$. Substituting back in for u, this is $2e^{2x}\sin^2 e^{2x}$.

6)

$$\begin{aligned}
f(x) &= 3x^5-5x^3 \\
f'(x) &= 15x^2(x+1)(x-1) \\
f''(x) &= 30x(2x^2-1)
\end{aligned}$$

a) Find and classify all critical points of $f(x)$ and determine the intervals on which f is increasing or decreasing.

Since the first derivative is always defined, the only critical points will be when $f'(x)=0$. This occurs when $x=0$, -1, or 1. To classify them, we need to figure out whether they are maxima, minima, or neither. $f''(-1)<0$, so $f(-1)$ is a local maximum. $f''(1)>0$, so $f(1)$ is a local minimum. $f''(0)=0$, so the second derivative test is inconclusive, so we need to apply other methods to figure out whether $f(0)$ is a local extremum. Since the derivative does not change sign at 0, $f(0)$ is not a local extremum.

The derivative is positive when $x<-1$, so the function is increasing on the interval $(-\infty,-1)$. It is negative between -1 and 1, so the function is decreasing on the interval $(-1,1)$, and the derivative is positive when $x>1$, so the function is increasing on the interval $(1,\infty)$.

b) Find any inflection points of $f(x)$ and determine the intervals on which f is concave up or concave down.

The possible inflection points occur when $f''(x) = 0$, which in this case are $x = 0, \pm\sqrt{\frac{1}{2}}$. To determine whether they are inflection points, we need to check and make sure the second derivative changes sign at all of them. It does, so they are all inflection points.

The second derivative is negative when $x, -\sqrt{\frac{1}{2}}$, so the function is concave down there. The second derivative is positive when $-\sqrt{\frac{1}{2}} < x < 0$, so the function is concave up on that interval. The second derivative is negative when $0 < x < \sqrt{\frac{1}{2}}$, so the function is concave down there, and the second derivative is positive when $x > \sqrt{\frac{1}{2}}$, so the function is concave up there.

7)

$$f(x) = \frac{x^2+1}{x-2}$$

Find any vertical, horizontal, or slant asymptotes of $f(x)$.

The function is not defined at $x = 2$, and the discontinuity is not removable, so the graph has a vertical asymptote there.

$\lim_{x\to\pm\infty}$ does not exist, so the function has no horizontal asymptotes.

The function can be rewritten as $f(x) = x + 2 + \frac{5}{x-2}$, so the line $y = x + 2$ is a slant asymptote.

8) Find $\frac{dy}{dx}$, assuming y is defined implicitly as a function of x by the equation

$$\begin{aligned}
\cos x + \cos y &= xy \\
-\sin x - \sin y\frac{dy}{dx} &= y + x\frac{dy}{dx} \\
-\sin x - y &= (x + \sin y)\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{-(\sin x + y)}{x + \sin y}
\end{aligned}$$

Bonus: Tell me something you have learned in another class this semester.