Rice University
Department of Mathematics
Solutions

Math 101 - Fall 2009
Midterm Exam II

Problem 1. Evaluate the following limits:

(a) (5 points)
$$\lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$
(L'Hospital's Rule was applied once for the $\frac{0}{0}$ case.)

(b) (5 points)
$$\lim_{x\to 0^+}\sin x\cdot\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sin x}}=\lim_{x\to 0^+}\frac{1/x}{-\frac{\cos x}{\sin^2 x}}=\lim_{x\to 0^+}\frac{\sin^2 x}{-x\cos x}=\lim_{x\to 0^+}\frac{2\sin x\cos x}{-\cos x+x\sin x}=\frac{0}{1}=0$$
(L'Hospital's Rule was applied first for the $\frac{\infty}{\infty}$ case and then for the $\frac{0}{0}$ case.)

(c) (5 points)
$$\lim_{x\to\infty}(e^x+x)^{1/x}=\lim_{x\to\infty}e^{\ln(e^x+x)^{1/x}}=\lim_{x\to\infty}e^{\frac{\ln(e^x+x)}{x}}=e^{\lim_{x\to\infty}\frac{\ln(e^x+x)}{x}}$$
Compute the exponent separately.
$$\lim_{x\to\infty}\frac{\ln(e^x+x)}{x}=\lim_{x\to\infty}\frac{\frac{e^x+1}{e^x+x}}{1}=\lim_{x\to\infty}\frac{e^x+1}{e^x+x}=\lim_{x\to\infty}\frac{e^x}{e^x+1}=\lim_{x\to\infty}\frac{e^x}{e^x}=1$$
(L'Hospital's Rule was applied three times first for the $\frac{\infty}{\infty}$ case).

The original limit is therefore equal to $e^1=e$.

Problem 2. Consider the following curve:
$$f(x)=x+\frac{4}{x^2}$$

(a) (2 points) What is the domain of this curve?
$D=(-\infty,0)\cup(0,+\infty)$

(b) (2 points) What are the critical points of the curve in (a)?
$f'(x)=1-\frac{8}{x^3}=\frac{x^3-8}{x^3}=0$ implies $x^3=8$, that is $x=2$ is the critical point.
The derivative is not defined for $x=0$ but we don't count it as a critical number because $0\notin D$.

(c) (4 points) On what intervals is the curve in (a) increasing and decreasing? Also, identify the critical points that are either a local max or a local min.

(d) (3 points) On what intervals is the curve in (a) concave up or concave down? Also, identify any inflection points.
$$f''(x)=\frac{24}{x^4}$$

x	x^3-8	x^3	$f'(x)$	x^4	$f''(x)$	$f(x)$
$(-\infty,0)$	$-$	$-$	$+$	$+$	$+$	increasing, concave up
$(0,2)$	$-$	$+$	$-$	$+$	$+$	decreasing, concave up
$(2,+\infty)$	$+$	$+$	$+$	$+$	$+$	increasing, concave up

$f(2)=3$ is a local minimum. There are no inflection points.

(e) (5 points) Identify any asymptotes for the curve in (a)
Hint: The curve $y = x + \frac{4}{x^2}$ can be rewritten as $y = \frac{x^3+4}{x^2}$

$x = 0$ is a vertical asymptote.

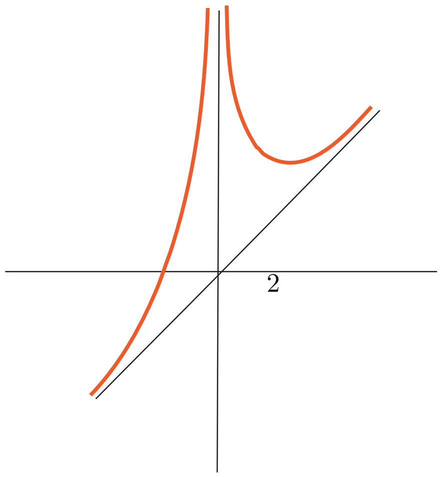
$$\lim_{x \to 0^+} f(x) = +\infty$$
$$\lim_{x \to 0^-} f(x) = +\infty$$

As

$$\lim_{x \to \pm\infty} (f(x) - x) = \lim_{x \to \pm\infty} \frac{4}{x^2} = 0,$$

the line $y = x$ is a slant asymptote at $\pm\infty$

(f) (4 points) Use all of the information that you found to sketch the curve.



Problem 3. (10 points) Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius 3. (Hint: the image below is what it looks like to inscribe a rectangle in a quarter-circle of radius 3.)

The function representing the area of the rectangle is $A(x) = 4x\sqrt{9-x^2}$. We want to maximize $A(x)$ on the interval $[0,3]$

$A'(x) = 4\sqrt{9-x^2} + 4x \cdot \frac{-2x}{2\sqrt{9-x^2}} = 4\sqrt{9-x^2} - \frac{-4x^2}{\sqrt{9-x^2}} = \frac{4(9-x^2)-4x^2}{\sqrt{9-x^2}} = \frac{36-8x^2}{\sqrt{9-x^2}}$.

$A'(x) = 0$ yields $36 - 8x^2 =$ or $x = \pm\frac{3}{\sqrt{2}}$. We pick $x = \frac{3}{\sqrt{2}} \in [0,3]$.

Note that for $x \in [0,3], x < \frac{3}{\sqrt{2}}$, $A'(x) > 0$

while for $x \in [0,3], x > \frac{3}{\sqrt{2}}$, $A'(x) < 0$.

This means $A(\frac{3}{\sqrt{2}})$ is indeed a global maximum.

Dimensions of rectangle are : length $= 2 \cdot \frac{3}{\sqrt{2}} = 3\sqrt{2}$, height $= 2 \cdot \sqrt{9 - (3/\sqrt{2})^2} = 3\sqrt{2}$.

It's a square !!!

Remark : One can also maximize $S(x) = A^2(x) = 16x^2(9-x^2)$ and avoid dealing with the square roots.

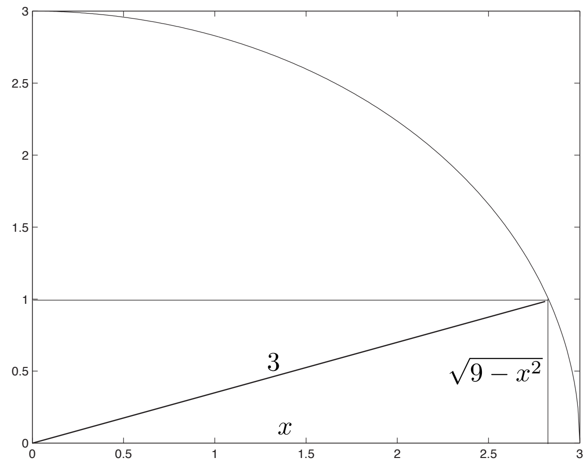


Figure 1: Rectangle Inscribed in a Quarter-Circle

Problem 4.

(a) (4 points) Show that the equation $3x + 2\cos x + 5 = 0$ has ***at least*** one real root.
Let $f(x) = 3x + 2\cos x + 5$. Use the Intermediate Value Property, say on $[-\pi, \pi]$
$f(-\pi) = -3\pi - 2 + 5 = -3\pi + 3 < 0$
$f(\pi) = 3\pi - 2 + 5 = 3\pi + 3 > 0$
By IVP, for $-3\pi + 3 < 0 < 3\pi + 3$, there exists $c \in (-\pi, \pi)$ such that $f(c) = 0$. Thus, there is at least one root.

(b) (6 points) Show that the equation $3x + 2\cos x + 5 = 0$ has ***at most*** one real root.
$f'(x) = 3 - 2\sin x > 0$, for all $x \in \mathbb{R}$. This implies that $f(x)$ is a strictly increasing function and so its graph can only cross the $x-$axis once. This means that c from before is a unique solution to $f(x) = 0$.

Problem 5. (10 points) Find the function $f(x)$ given the following:

$f''(x) = 2 - 12x$, $f(0) = 0$, $f(1) = 1$

Computing the antiderivatives yields $f'(x) = 2x - 6x^2 + C$, and $f(x) = x^2 - 6\frac{x^3}{3} + Cx + D$.
$f(0) = D = 0$
$f(1) = 1 - 2 + C = 1$ implies $C = 2$.
$f(x) = x^2 - 2x^3 + 2x$

Problem 6. (10 points) The side length of a square is increasing at a rate of 8 cm/s. At what rate is the area of the square increasing when the side length of the square is 16 cm?

Let l represent the length of the side of the square. Then the area is $A(l) = l^2$.

$\frac{dA}{dt} = 2l \cdot \frac{dl}{dt} = 2 \cdot 16 \cdot 8 = 256$ cm^2/s.