More series! (But didn't we just get tested on that?!) 

Sorry...

What was good; i.e. easy, about a power series?

\[ f(x) = \sum_{n=0}^{\infty} c_n x^n \]

It's easy to find derivatives/integrals!

For this reason, we want to study them a little more.

**EXERCISE**

Given a general power series \( f(x) = \sum_{n=0}^{\infty} c_n x^n \)

Find:

1. \( f(0) \)
2. \( f'(0) \)
3. \( f''(0) \)
4. \( f'''(0) \)
5. \( f^{(4)}(0) \)

Is there a pattern emerging?

If using the abstract \( f(x) = \sum c_n x^n \) is too difficult, do the same with \( g(x) = \sum_{n=0}^{\infty} 1 x^n \)
\[ f^{(n)}(0) = n(n-1) \ldots 3 \cdot 2 \cdot 1 c_n = n! \]

OR, equivalently, we determine the coefficients \( c_n \) by
\[ c_n = \frac{f^{(n)}(0)}{n!} \]

**Theorem** In general, for the power series representation of \( f(x) \) centered at \( a \),
\[ f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad |x-a| < R \]
the coefficients are given by
\[ c_n = \frac{f^{(n)}(a)}{n!} \]

i.e.
\[ f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \]

*This is called the **Taylor Series** for \( f \) at \( a \).*

When the Taylor series is centered at \( 0 \), it's called the Maclaurin series.

**IMPORTANT:** The theorem says that IF \( f(x) \) has a power series representation, it has this form. It does not guarantee the existence of some power series which always converges to \( f(x) \) for any \( x \)!
So finding a Taylor series (or Maclaurin series) shouldn't be too bad.

E.g. Find the Taylor series for $e^x = f(x)$ at $1$

$f(1) = e^1 = e$

$f'(x) = e^x \Rightarrow f'(1) = e$

$f''(x) = e^x \Rightarrow f''(1) = e$

$f'''(x) = e^x \Rightarrow f'''(1) = e$

$\ldots$

$\Rightarrow f^{(n)}(1) = e$ for all $n$

So if $f(x) = e^x$ holds at all Taylor series at $1$, it is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n = \sum_{n=0}^{\infty} \frac{e}{n!} (x-1)^n$$

We must also find the radius of convergence for this power series...

By the ratio test

$$\lim_{n \to \infty} \left| \frac{\frac{e}{n!} (x-1)^{n+1}}{\frac{e}{n!} (x-1)^n} \right| = \lim_{n \to \infty} \left| \frac{n!}{(n+1)!} (x-1) \right|$$

$$= |x-1| \lim_{n \to \infty} \frac{1}{n+1} = 0$$

$\Rightarrow$ The power series converges for every $x$!
But is \( f(x) \) EQUAL to this Taylor series?

We need a way to ensure that
\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]
actually converges to \( f(x) \).

Look AT THE PARTIAL SUMS!

For any \( n \), set
\[
T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k
= f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)}{n!} (x-a)^n
\]
This partial sum is called the \( n \)-th Taylor polynomial
for \( f(x) \) at \( a \).

*The Taylor series converges to \( f(x) \) when \( f(x) \) is the limit
\[
f(x) = \lim_{n \to \infty} T_n(x)
\]

For any \( n \), define the remainder \( R_n(x) \) by
\[
R_n(x) = f(x) - T_n(x)
\]
so it must be \( f(x) = T_n(x) + R_n(x) \)
We can equivalently show that the remainder \( \to 0 \)

Theorem: If \( f(x) = T_n(x) + R_n(x) \) and \( \lim_{n \to \infty} R_n(x) = 0 \)
for \( |x-a| < R \) \( \Rightarrow \) \( f \) is equal to its Taylor series for \( |x-a| < R \).
Q: Is $f(x) = e^x$ equal to its Taylor series at $1$?

Taylor @ $1$: $\sum_{n=0}^{\infty} \frac{e}{n!(x-1)^n}$ for any $x$

By the Taylor inequality,

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-1|^{n+1}$$

What can we use for $M$?

Since $f^{(n)}(1) = e$ for all $n$, set $M = e$

$$\Rightarrow |R_n(x)| \leq \frac{M}{(n+1)!} |x-1|^{n+1}$$

By the ratio test,

$$\lim_{n \to \infty} \left| \frac{\frac{e}{(n+2)!} |x-1|^{n+2}}{\frac{e}{(n+1)!} |x-1|^{n+1}} \right|$$

$$= \lim_{n \to \infty} \frac{|(n+1)!| |x-1|}{(n+2)!} |x-1|$$

$$= |x-1| \lim_{n \to \infty} \frac{1}{n+2} = 0$$

Since $R_n(x) \to 0$ for any $x$,

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{e}{n!} (x-1)^n$$
Find the Maclaurin series for \( f(x) = \sin(x) \) and prove that it is equal to \( f(x) \) for any \( x \).

### Table of Maclaurin Series

1. \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... \)

2. \( e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \)

3. \( \sin(x) = \)

4. \( \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \)

5. \( \tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \)

6. \( (1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + ... \)
Special Note: $(1+x)^k$ is called the binomial series (or, at least the series is). Those coefficients are called the binomial coefficients and are notated by:

\[
\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \quad \text{or} \quad \frac{k!}{(k-n)!n!}
\]

You'll usually see $(\binom{k}{n})$ or "$k$ choose $n$" comes up frequently (especially in probability).

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**Taylor Series** let us integrate!

* This function is "impossible" to integrate using normal techniques. $\psi(x) = e^{-x^2}$

But we can integrate it using Taylor series:

Since $e^y = \sum_{n=0}^{\infty} \frac{1}{n!} e^y$

\[
e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}
\]

\[
\int e^{-x^2} \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \, dx
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n} \, dx
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{2n+1} x^{2n+1} + C
\]
Taylor Series help us find limits!

Try to evaluate:

$$\lim_{x \to 0} \frac{x - \tan^{-1}(x)}{x^3}$$

Try to evaluate the Taylor series representation:

$$\frac{x - \tan^{-1}(x)}{x^3} = \frac{x - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}{x^3}$$

$$= \frac{x}{x^3} - \frac{x^3}{x^3} + \frac{x^5}{5x^3} - \frac{x^7}{7x^3} + \frac{x^9}{9x^3} - \ldots$$

$$= \frac{1}{3} - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \ldots$$

$$\Rightarrow \lim_{x \to 0} \left( \frac{1}{3} - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \ldots \right) = \frac{1}{3}$$

Then all go to 0!

E.g. Evaluate the Indefinite integral and limit:

$$\int \frac{\cos x - 1}{x} \, dx \quad \lim_{x \to 1} \frac{\cos x - 1}{x}$$
Just like polynomials, we can add/multiply/divide Taylor series.

e.g. Multiply: $e^x \cos x$ via Taylor Series

$$e^x \cos x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) \left(1 - \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right)$$

$$= 1 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + \frac{x}{1!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)$$

$$+ \frac{x^2}{2!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + \frac{x^3}{3!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{x}{1!} - \frac{x^3}{1!2!} + \frac{x^5}{1!4!} - \frac{x^7}{1!6!} + \ldots$$

$$+ \frac{x^2}{2!} - \frac{x^4}{2!2!} + \frac{x^6}{2!4!} - \frac{x^8}{2!6!} + \ldots + \frac{x^3}{3!} - \frac{x^5}{3!2!} + \frac{x^7}{3!4!} - \frac{x^9}{3!6!} + \ldots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^4}{2!2!} + \frac{x^3}{3!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^4}{2!2!} + \ldots$$

Combining like terms:

$$= 1 + x + \left(\frac{1}{3!} - \frac{1}{2!}\right)x^3 + \left(\frac{1}{4!} - \frac{1}{2!2!}\right)x^4 + \ldots$$

Lucky: we usually just need these first few terms.

This process is tedious, but not hard, so don't become overwhelmed!
Division is done via "long division" which if you remember how to do long division, isn't too bad...

Taylor Series and Taylor Polynomials help us approximate Tons of functions, which would be very difficult to evaluate honestly.

E.g. Approximate \( f(x) = \ln(1+2x) \) by a Taylor polynomial of degree 3 at \( x = 1 \).

\[
f'(1) = \frac{1}{1+2} = \frac{1}{3}
\]

\[
f''(1) = \frac{-2}{(1+2)^2} = \frac{-2}{9}
\]

\[
f'''(1) = \frac{6}{(1+2)^3} = \frac{6}{27} = \frac{2}{9}
\]

\[
T_3(x) = \ln(3) + \frac{1}{3} x - \frac{2}{9} \cdot x^2 + \frac{2}{9} \cdot 3^2 x^3
\]

(2) Use Taylor's inequality to estimate the accuracy of \( T_3 \) for \( 0.5 \leq x \leq 1.5 \), i.e. \( |x-1| \leq 0.5 \).

\[
R_3(x) \leq \frac{M}{4!} |x-1|^4 \quad \text{where} \quad |f^{(4)}(1)| \leq M
\]

Since \( f^{(4)}(x) = \frac{6(-3)(2)}{(1+2x)^4} = \frac{-36}{(1+2x)^4} \)

\[
\Rightarrow f^{(4)}(1) = \frac{-36}{81} = \frac{-4}{9}
\]

Set \( M = \frac{4}{9} \)

\[
R_3(x) \leq \frac{(4/9)}{4!} |x-1|^4 \leq \frac{1}{6 \cdot 9} \cdot (0.5)^4 = \frac{1}{54} \cdot \frac{1}{16} \approx 0.001157...
\]
e.g. (a) Find a Taylor polynomial of degree 4 for \( f(x) = x \sin x \) about \( a = 0 \).

(b) Use Taylor inequality to estimate the accuracy of \( T_4(x) \) when \( -1 \leq x \leq 1 \).