Yesterday, we talked a lot about sequences and some about series.

Turns out sequences aren't really what we care about: We care what happens when we add up all the terms.

⇒ For calculus we really care about SERIES.

Today, we discuss a few special types of series and try to determine when a series converges

(Since this means some INTEGRAL will converge ... )
**Important Series**

The geometric series: \( a \) and \( r \) are real numbers

\[
\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots
\]

- If \( r = 1 \) \( \Rightarrow \) \( S_n = a + a + \ldots + a = na \) and \( S_n \to \infty \)
  and \( \sum ar^{n-1} \) diverges.

- If \( r \neq 1 \) \( \Rightarrow \)

  \[
  S_n = a + ar + \ldots + ar^{n-1} \\
  rS_n = ar + \ldots + ar^n
  \]

  \[
  \Rightarrow S_n - rS_n = a - ar^n \\
  \Rightarrow S_n (1-r) = a - ar^n = a \left( 1 - r^n \right) \\
  \Rightarrow S_n = \frac{a (1 - r^n)}{1 - r}
  \]

Q: Does \( \lim_{n \to \infty} S_n \) exist?

Using limit laws

\[
\lim_{n \to \infty} a (1-r^n) = a \lim_{n \to \infty} 1 - r^n = a - a \lim_{n \to \infty} r^n
\]

\[
\lim_{n \to \infty} 1-r = 1-r
\]

So \( \lim_{n \to \infty} S_n = \frac{a - a \lim_{n \to \infty} r^n}{1-r} \)

\[
\lim_{n \to \infty} r^n = \begin{cases} 
0 & -1 < r < 1 \\
\infty & \text{otherwise}
\end{cases} 
\Rightarrow \lim_{n \to \infty} S_n = \begin{cases} 
\frac{a}{1-r} & -1 < r < 1 \\
\pm \infty & \text{otherwise}
\end{cases}
\]
This means when $|r| < 1$, the series converges
and $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

Otherwise it diverges.

e.g. Use the geometric series.
Show the series is convergent and find the sum.

(a) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

\[ \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} \cdot \frac{(-3)^{n-1}}{4^{n-1}} = \frac{1}{4} \left( -\frac{3}{4} \right)^{n-1}. \quad r = -\frac{3}{4} \]

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{4} \left( -\frac{3}{4} \right)^{n-1} = \frac{\frac{1}{4}}{1 - \left( -\frac{3}{4} \right)} = \frac{1}{7}$

(b) $\sum_{n=1}^{\infty} \frac{n^n}{3^{n+1}}$

If $\sum_{n=1}^{\infty} a_n$ converges, we clearly need $\lim_{n \to \infty} a_n = 0$. However, just because $\lim_{n \to \infty} a_n = 0$ does not mean $\sum_{n=1}^{\infty} a_n$ converges.
e.g. The harmonic series
\[ \sum_{n=1}^{\infty} \frac{1}{n} \] diverges!

* Related to why \( \int_{1}^{\infty} \frac{1}{x} \, dx \) diverges.
(For a current proof, see book)

Test for Divergence

If \( \lim_{n \to \infty} a_n \neq 0 \) or does not exist, then \( \sum a_n \)
diverges.

Properties of Sums

If \( \sum a_n, \sum b_n \) converge and \( c \) is a #,

- \[ \sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n \]
- \[ \sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \]
- \[ \sum_{n=1}^{\infty} a_n - b_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n \]

e.g. Does the series converge? If so, find the sum

\[ \sum_{n=1}^{\infty} \frac{3}{5^n} + \frac{2}{n} \]
\[ \sum_{n=1}^{\infty} \ln \left( \frac{n^2 + 1}{2n^2 + 1} \right) \]
Consider the series:
\[ \sum_{n=1}^{\infty} \frac{0.1}{n^2+1}. \]

What are some partial sums?

\[ S_1 = \]
\[ S_2 = \]
\[ S_3 = \]

Note that the sequence \( \{ \frac{1}{n^2+1} \} \) is related to the function \( f(x) = \frac{1}{x^2+1} \).

The series \( \sum \frac{1}{n^2+1} \) is related to Riemann sums for this function.

So it's clear that \( \sum \frac{1}{n^2+1} \) should somehow be related to the integral \( \int_{1}^{\infty} \frac{1}{x^2+1} \).

Q: Is \( \sum \frac{1}{n^2+1} \) exactly \( \int_{1}^{\infty} \frac{1}{x^2+1} \)?

Why or why not?

*We can use this integral to test for convergence of the series!*

**The Integral Test**

If \( f \) is continuous, positive, decreasing on the interval \([1, \infty)\), and \( a_n = f(n) \) for every integer \( n \), then...
the series \[ \sum_{n=1}^{\infty} \frac{1}{n^p} \] converges exactly whenever the integral \[ \int_{1}^{\infty} x^{p-1} dx \] converges.

*e.g.* Does our series \[ \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \] converge or diverge?

Important Integral Test example

\textbf{p-series}: \[ \sum_{n=1}^{\infty} \frac{1}{n^p} \]

Using the integral test, determine if the p-series converges:

\[ p \neq 1: \quad \int_{1}^{\infty} \frac{1}{x^p} \, dx = \lim_{n \to \infty} \int_{1}^{n} \frac{1}{x^p} \, dx \]

\[ = \lim_{n \to \infty} \left[ \frac{1}{1-p} \cdot \frac{1}{x^{p-1}} \right]_{1}^{n} \]

\[ = \frac{1}{1-p} \lim_{n \to \infty} \left( \frac{1}{n^{p-1}} - 1 \right) \]

*So \( \sum \frac{1}{n^p} \) converges exactly when this limit converges? For what \( p \) does this limit converge?*

\textbf{HINT}: Make sure to consider all possible \( p \).

\textbf{e.g.}: \( p=3, p=0, p=-3, p=\ldots \)
$$\int_1^\infty \frac{1}{x} \, dx = \lim_{n \to \infty} \int_1^n \frac{1}{x} \, dx$$

$$= \lim_{n \to \infty} \ln |x| \bigg|_1^n$$

$$= \infty$$

What does this mean for the series $\sum_{n=1}^\infty \frac{1}{n}$?

**Theorem**

$\sum_{n=1}^\infty \frac{1}{n^p}$ converges when $p$ and diverges when

**WARNING**

Most of the time, the sum of the series $\sum_{n=1}^\infty a_n$ and $\int_1^\infty f(x) \, dx$ are **NOT** the same!

E.g., $\sum_{n=1}^\infty \frac{1}{n^2 + 1}$ is the area:

$\int_1^\infty \frac{1}{x^2 + 1} \, dx$ is the area:

*however the integral does give us an estimate on the sum.*

Notice that by taking $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty f(n)$, these are "upper estimates" for $\int_1^\infty f(x) \, dx$.

(since $\sum_{n=1}^\infty a_n$ is more area than $\int_1^\infty f(x) \, dx$).
If instead of starting at \( n = 1 \), we started at \( n = 2 \), the series \( \sum_{n=2}^{\infty} a_n \) gives a lower bound for \( \int_1^{\infty} f(x) \, dx \).

\[
\Rightarrow \quad \sum_{n=2}^{\infty} a_n \leq \int_{1}^{\infty} f(x) \, dx \leq \sum_{n=1}^{\infty} a_n
\]

In fact, for any \( k \), we can see

\[
\sum_{n=k+1}^{\infty} a_n \leq \int_{k}^{\infty} f(x) \, dx \leq \sum_{n=k}^{\infty} a_n
\]

Or equivalently

\[
\int_{k+1}^{\infty} f(x) \, dx \leq \sum_{n=k+1}^{\infty} a_n \leq \int_{k}^{\infty} f(x) \, dx.
\]
So... That's cool what does it mean?
It means if we found the partial sum $S_n$, we have a decent bound on the actual sum $S$.

Since

$$S = S_n + \sum_{n=k+1}^{\infty} a_n$$

Let $R_n$ be the remainder $= S - S_n$.

**Remainder Estimate for Integral Test**

If $f(n) = a_n$ and $f$ is continuous
- positive
- (eventually) decreasing

and $\sum a_n$ converges, then

$$\int_{n+1}^{\infty} f(x) \, dx \leq S - S_n \leq \int_{n}^{\infty} f(x) \, dx$$

* Use this theorem to estimate

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

to 3 decimal places.