An example:

Consider \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \).

Since \( \lim_{n \to \infty} \frac{1}{n} = 0 \) and

\( \frac{1}{n+1} < \frac{1}{n} \) for all \( n \) \( \Rightarrow \) the series converges by the alternating series test.

What about \( \sum_{n=1}^{\infty} \frac{1}{n} \)?

This is the harmonic series which we know diverges!

\( \Rightarrow \) So although

\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \text{ CONVERGES, } \sum_{n=1}^{\infty} |(-1)^{n-1}\frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ DIVERGES.} \]

It's actually common for a series \( \sum a_n \) to converge while \( \sum |a_n| \) is divergent. When they both converge, we give the series a special name...

**defn** A series \( \sum a_n \) is **ABSOLUTELY CONVERGENT** if the series \( \sum |a_n| \) is convergent.

The alternating series \( \sum (-1)^{n-1} \frac{1}{n} \) is convergent but **NOT** absolutely convergent.

**defn** A series which is convergent but **NOT** absolutely convergent is **CONDITIONALLY CONVERGENT**.
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{3^{n-1}} \]

Is the series convergent? Absolutely convergent?

**Theorem:** If a sequence is absolutely convergent, it must also be convergent.

\[ \Rightarrow \quad \text{ABSOLUTE CONVERGENCE IS STRONGER THAN CONVERGENCE} \]

Why? If \( \Sigma |a_n| \) converges, then \( \Sigma |a_n| = S \).

\[ \Rightarrow \quad |\Sigma a_n| \leq \Sigma |a_n| = S \quad (*) \]

So \( \Sigma a_n \) converges by the “comparison test.”

*This is the triangle inequality: \(|x+y| \leq |x| + |y|\).*

\[ \square \]

We can test for absolute convergence too...

**The Ratio Test:** Given \( \sum_{n=1}^{\infty} a_n \).

1. \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \Rightarrow \sum a_n \text{ is absolutely convergent} \)
   (and so it's convergent)

2. \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \text{ or } \infty \Rightarrow \sum a_n \text{ diverges} \)

3. \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \Rightarrow \sum a_n \text{ is ?} \) (Just can't say!)
* The ratio test is AWESOME and you will learn to love it.

e.g. \( \sum_{n=1}^{\infty} \frac{(1.1)^n}{n^4} \).

Use the ratio test:

\[
\lim_{n \to \infty} \left| \frac{(1.1)^{n+1}/(n+1)^4}{1.1^n/n^4} \right| = \lim_{n \to \infty} \left| \frac{(1.1)^{n+1}n^4}{(1.1)^n(n+1)^4} \right|
\]

\[
= \lim_{n \to \infty} 1.1 \frac{n^4}{(n+1)^4}
\]

\[
= 1.1 \lim_{n \to \infty} \frac{n^4}{n^4 + 4n^3 + 6n^2 + 4n + 1}
\]

\[
= 1.1 \lim_{n \to \infty} \frac{1}{1 + \frac{4}{n} + \frac{6}{n^2} + \frac{4}{n^3} + \frac{1}{n^4}}
\]

\[
= 1.1
\]

\( \implies \) The series diverges.

e.g. \( \sum_{n=0}^{\infty} \frac{(-10)^n}{n!} \).
One last test...
* This test is helpful whenever we have n's in an exponent!

The **Root Test**

Given $\Sigma a_n$,

1. $\lim_{n \to \infty} \sqrt[n]{|a_n|} < 1 \Rightarrow \Sigma a_n$ is ABSOLUTELY CONVERGENT.
2. $\lim_{n \to \infty} \sqrt[n]{|a_n|} > 1 \Rightarrow \Sigma a_n$ DIVERGES!
3. $\lim_{n \to \infty} \sqrt[n]{|a_n|} = 1 \Rightarrow$ NO RESULT.

[* If the ratio test fails, so does the root test. Unfortunately, and vice versa.*]

**e.g.** \[\sum_{n=2}^{\infty} \left[ \frac{-2n}{n+1} \right]^{5n} \]

Use the root test...

\[
\lim_{n \to \infty} \sqrt[n]{\left[ \frac{-2n}{n+1} \right]^{5n}} = \lim_{n \to \infty} \left| \frac{-2n}{n+1} \right|^{5n} = \left[ \lim_{n \to \infty} \frac{-2n}{n+1} \right]^{5} = \left[ 2 \lim_{n \to \infty} \frac{n}{n+1} \right]^{5} = 2^{5}
\]

m> The series diverges.

**e.g.** \[\sum_{n=2}^{\infty} \frac{n}{(ln n)^n} \]

\[(Hint: \lim_{n \to \infty} n^{1/n} = \lim_{n \to \infty} e^{ln(n^{1/n})} = e^{\lim_{n \to \infty} \frac{ln(n)}{n}})\]
e.g. *Recursively defined series.*

If $a_1 = 1$ and $a_{n+1} = \frac{2 + \cos(n)}{\sqrt{n}} a_n$

Does $\sum a_n$ converge? diverge? ...

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**Q1** Is the series a special type?

(a) Geometric: $\sum_{n=1}^{\infty} ar^{n-1}$

Converges when ____________________________

Diverges when ____________________________

(Sum is ____________________________)

(b) $p$-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Converges when ____________________________

Diverges when ____________________________

No? Then we gotta use a test...
Q2 Does $\lim_{n \to \infty} a_n = 0$? 
If not...
What test is this? ______________

Q3 Is $\sum a_n$ similar to one of a special type? 
@ YES! Are all the terms positive? 
(Yes! Use a comparison test! (limit comparison or regular))
(No... Use comparison test with $|a_n|$.

B No? Shucks...

Q4 Is it alternating??
If so Alternating series test.
$\sum_{n=1}^{\infty} (-1)^n a_n$ converges when 
1) __________________
2) __________________

Q5 Is there an "easily integrable $f(x)$ so that $f(n)$ is $a_n$? INTEGRAL TEST!
Need $f(x)$ to be 1) ________ 2) ________ 3) ________
$\sum a_n$ converges when ______________
$\sum a_n$ diverges when __________________
Q6. Is an a power like $(b_n)^n$? 

→ Try the Root Test.

\[ \sum a_n \text{ converges when } \ldots \]
\[ \sum b_n \text{ diverges when } \ldots \]

Q7. Ratio test, maybe?

\[ \sum a_n \text{ converges when } \ldots \]
\[ \sum a_n \text{ diverges when } \ldots \]

Remember:
1. The comparison tests only work for series with POSITIVE terms.
2. But you could use them to test for absolute convergence (and thus convergence)
3. The $f(x)$ in the integral test must be positive, decreasing, and continuous!

Which tests do you like the best? Why? Discuss pros & cons of them all...
\[ \sum_{n=1}^{\infty} \frac{1}{n + 3^n} \]

\[ \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln(n)} \]

\[ \sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} \]

\[ \sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k} \]