Obstructions to Knot Concordance

Christopher Davis and Bridget Franklin

Abstract

Knots are embeddings of circles in 3-dimensional space. An active field in low-dimensional topology is that of knot concordance – classes of knots distinguished by a 4-dimensional equivalence relation. This endows a (quite complicated) group structure to knots called the knot concordance group which we then study via filtration. Our current research strives to present a fractal nature to this group using satellite operations on knots and non-commutative algebra with vast consequences for 3-manifold topology.

Classical Knotting

Definition. *Classically, a **Knot** is an oriented embedding of the circle in S^3, the 3-dimensional sphere.*

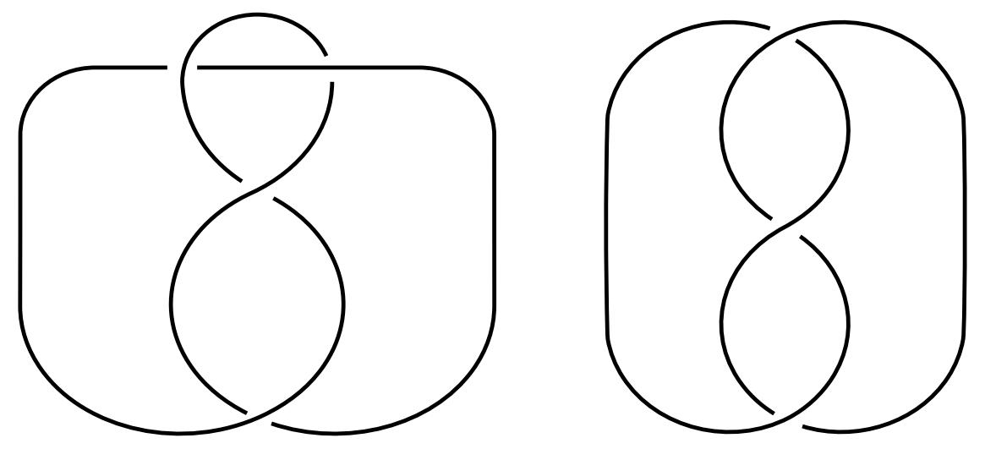
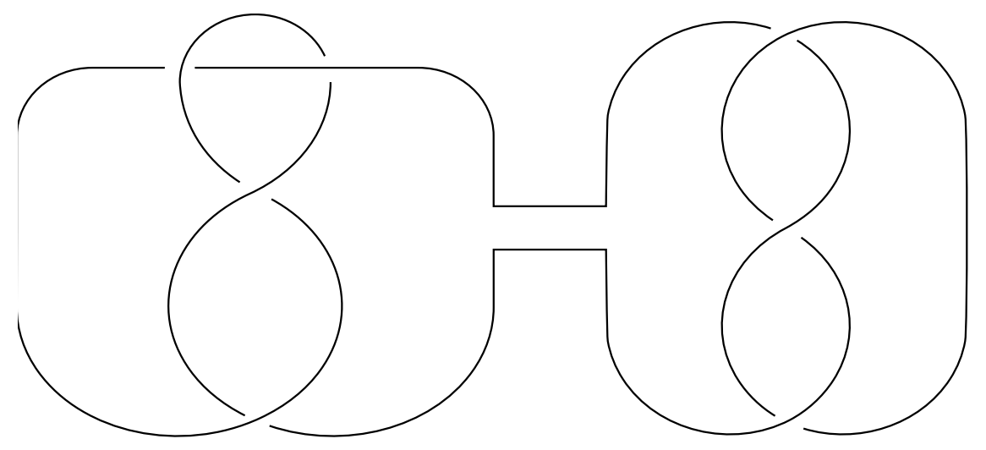


Figure 8 Trefoil

Given two knots, K and L we may form a $K\#L$ called the **connected sum**.



Under connected sum, the set of all knots forms a monoid. (i.e. This set has an identity element given by the trivial knot but knots do not have inverses under connected sum.)

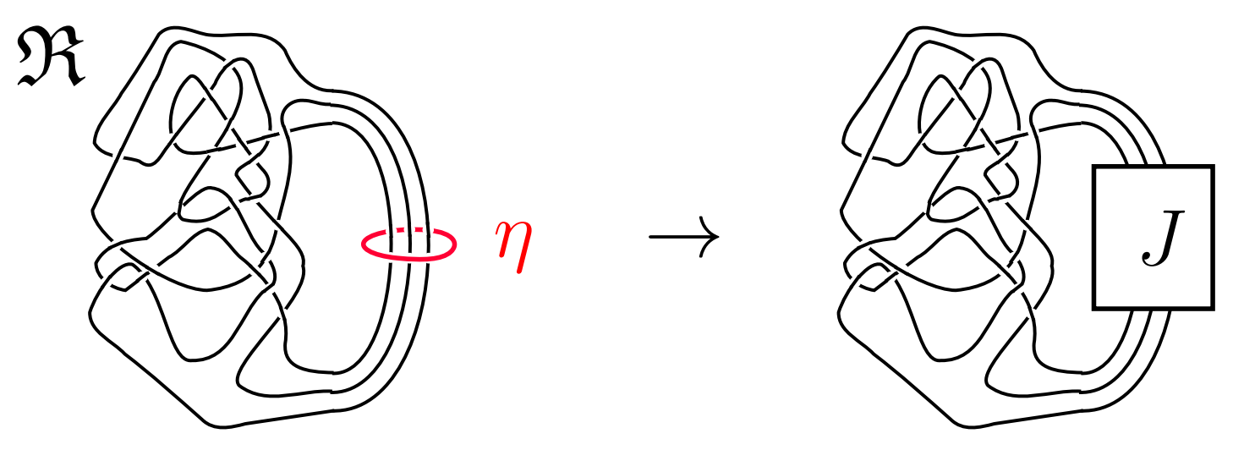
To a knot K one can associate a 3-manifold M_K in a natural way via a process called **zero surgery** on K in S^3.

Infection

If $\mathfrak{R}$ and J are two knots, we can perform another operation called **infection** given by

$$K \equiv \mathfrak{R}(\eta, J)$$

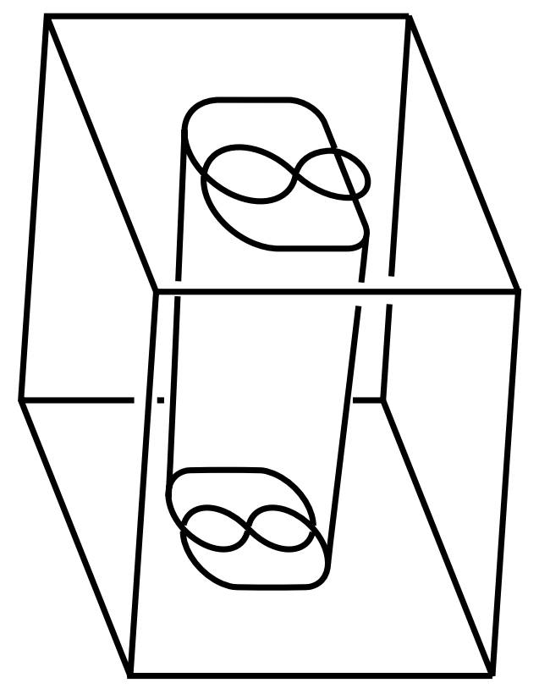
where we cut the strings of $\mathfrak{R}$ which go through the curve η and tie these strings into the knot J.



Iterating this procedure yields knots with progressively subtler details.

Knot Concordance

Two knots in S^3 are **concordant** if their union cobounds an annulus in $S^3 \times [0,1]$.



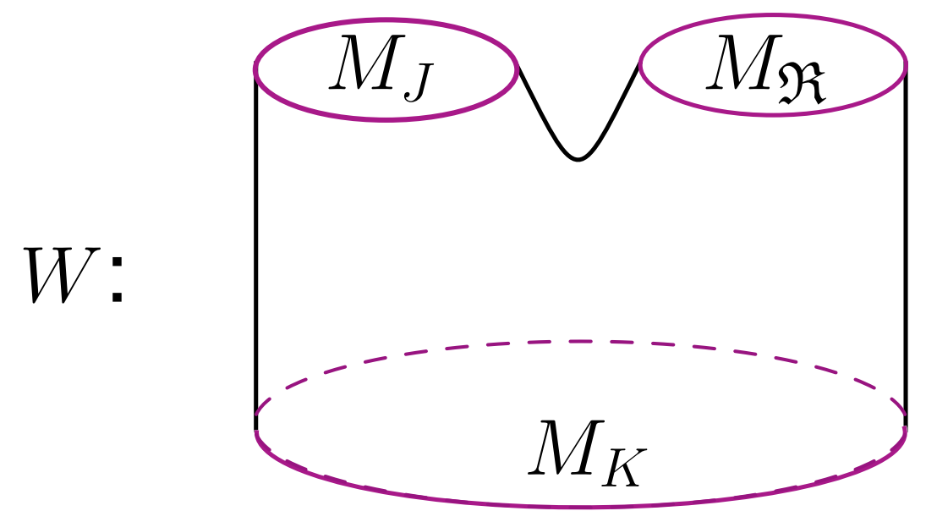
An (inaccurate) concordance between knots

The set of concordance classes of knots under the connected sum operation forms a complicated group called **The Knot Concordance Group**, denoted $\mathcal{C}$.

If a knot if concordant to the trivial knot, it bounds a disk in a 4-dimensional ball, and is called **slice**. The disk bounded by the knot is its **slice disk**.

Infections and Cobordisms

A **cobordism**, W, between 3-dimensional manifolds M_1, M_2 is a 4-dimensional manifold whose boundary is the disjoint union of M_1 and M_2. Under any infection operation, there is a simple cobordism between 3-manifolds associated to the base knot, the infecting knot, and the resultant knot:



ρ-Invariants & Blanchfield Form

To a 4-manifold, W, (such as the cobordism above) one can associate two real numbers, $\sigma(W)$, the signature, and $\sigma^{(2)}(W)$, an L^2-signature. The difference $(\sigma^{(2)} - \sigma)$ depends only on the boundary of W, and is called the **von Neumann ρ-invariant** of the boundary.

Theorem. *The signature and L^2 signatures of the above cobordism, W, are zero. Therefore $\rho(M_K) - \rho(M_J) - \rho(M_{\mathfrak{R}}) = 0$*

The **Blanchfield form** is a form on the Alexander module of a knot $\mathcal{A}(K)$. And for any curves $\alpha, \beta \in \mathcal{A}$, we have $\mathcal{B}\ell(\alpha, \beta)$ is an element of $\mathbb{Q}(t) \mod \mathbb{Z}[t, t^{-1}]$.

Theorem. *If a knot K is slice via some slice disk complement W, then there is a half-dimension submodule of $\mathcal{A}(K)$ such that for any element η in this submodule $\mathcal{B}\ell(\eta, \eta) = 0$.*

Classical results:

Theorem (Cochran-Orr-Teichner). *If a knot is slice then the ρ-invariants which extend over the complement of the slice disk are all zero.*

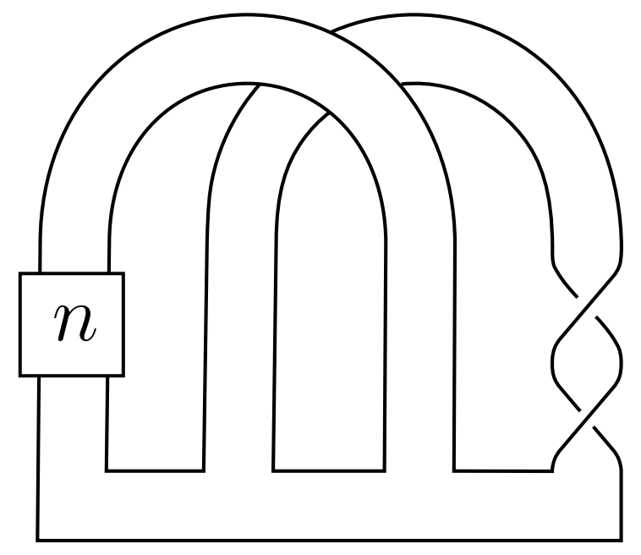
Theorem. *Abelianization extends over slice disk complements, so that the corresponding ρ-invariant is zero for slice knots*

Theorem (Levine). *The knot concordance group surjects to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$. The knot concordance group is infinitely generated.*

Results

Theorem. *If knots have distinct prime Alexander polynomials and nonzero ρ^1-invariants, then the knots are linearly independent in the concordance group.*

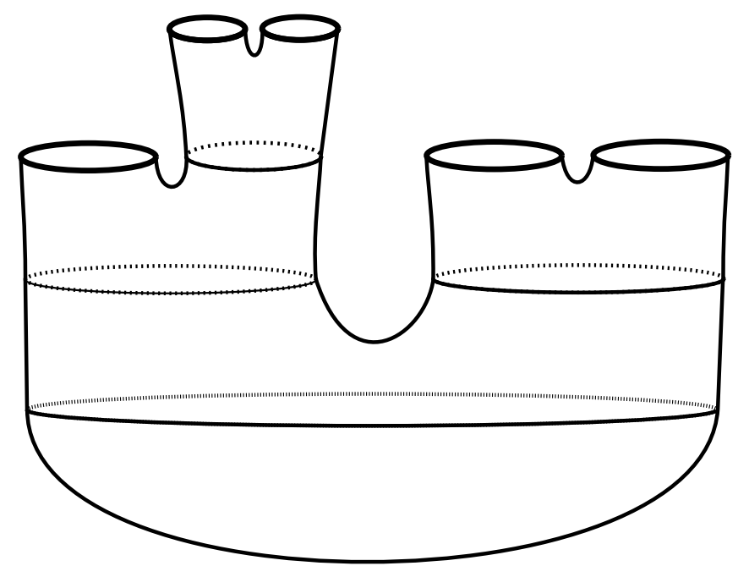
Corollary. *There is an infinite linearly independent set of twist knots.*



Twist knots

Theorem. *If $K_1 \equiv \mathfrak{R}(\eta_1, J)$ and $K_2 \equiv \mathfrak{R}(\eta_2, L)$ where $\mathcal{B}\ell(\eta_1, \eta_1) \neq \mathcal{B}\ell(\eta_2, \eta_2)$ (with some properties on J and L), then K_1 and K_2 are distinct concordance classes in $\mathcal{C}$.*

Corollary. *By varying the infecting curve in $\mathfrak{R} = 9_{46}$ (a particular slice knot), we can produce infinitely many concordance classes of $K_i \equiv \mathfrak{R}(\eta_i; J)$*



Future Work

- **Approximation:** Von Neumann ρ-invariants are inherently difficuly to compute. It is hard to find a 4-manifold suited to computing the desired ρ-invariant. Furthermore, even if one is found there is no general strategy to compute $\sigma^{(2)}(W)$. In order to get concrete applications one must find a means to compute the ρ-invariants of interest. The strategy we pursue is to reduce them to an easier setting, while keeping track of what information is lost.
- **Linear Independence** (with regards to infecting curves): Previous results have only shown that infections upon curves with different “self Blanchfield forms” produce distinct concordance classes in $\mathcal{C}$. There are problems when considering linear independence, however. In particular, if proven, this gives further evidence for the “fractal” nature of the knot concordance group.