

The Effect of Infecting Curves on Knot Concordance

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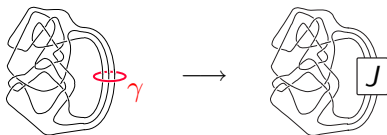
GOAL: Show that by varying the infecting curve in an iterated infection produces infinitely many nonconcordant knots in $\mathfrak{F}_2/\mathfrak{F}_{2.5}$.

PLAN:

- Brief reprise of infection, the n -solvable filtration, and previous results.
- Build up and state the main theorem.
- Discuss Blanchfield form restrictions.
- Give an example: 9_{46} .
- Sketch the proof.

Generating Knots: Infection

If J, R are knots in S^3 and $\gamma \subset S^3 \setminus R$ is unknotted in S^3 , we form $K = R(\gamma, J)$ by tying the strands of R which “go through” γ into the knot J .



Equivalently,

- 1 Remove a tubular neighborhood of $\gamma \subset M(R)$
- 2 Remove a tubular neighborhood of $J \subset S^3$
- 3 Glue $M(R) \setminus \nu(\eta)$ to $S^3 \setminus \nu(J)$ along their toric boundaries, ensuring orientation is preserved.

Infection and n -Solvability

Via infection, we create a new knot K denoted

$$K \equiv R(\gamma, J)$$

Question: How does this process affect the n -solvable filtration?

Proposition (Cochran-Orr-Teichner, 2003)

If R is a ribbon knot, J is n -solvable, and γ is a curve such that $[\gamma] \in \pi_1(R)^{(\ell)}$, then $K \equiv R(\gamma, J)$ is $(n + \ell)$ -solvable.

Previous Results

Due to work by Levine, Milnor, Casson-Gordon, Cochran-Orr-Teichner, Cochran-Harvey-Leidy,

$$\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \subset \mathfrak{F}_n / \mathfrak{F}_{n.5} \quad \forall n.$$

The technique by Cochran-Harvey-Leidy used iterated infections to produce knots in \mathfrak{F}_n .

$$K_n = R_n(K_{n-1})$$

$$\vdots$$

$$K_1 = R_1(K_0)$$

where each R_i was a robust doubling operator, and $K_i \in \mathfrak{F}_i$

Previous Results

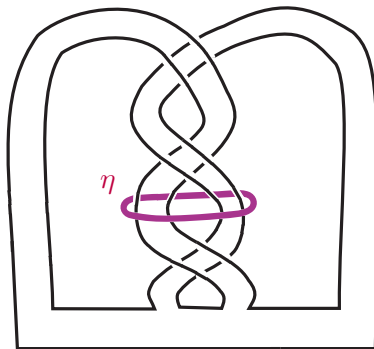
In each, $\mathfrak{F}_n/\mathfrak{F}_{n.5}$,

- the first linearly independent set was produced by varying classical signatures of the base knot K_0 .
- Next, by taking $\{R_i\}$ to be *robust doubling operators* with varying strongly coprime Alexander polynomials, they produced n additional parameters.

Question: How does varying the infection curve, γ , but fixing the base knot K_0 and ribbon knots R_i affect concordance?

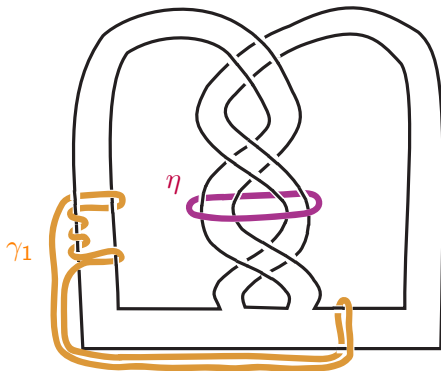
A Common Robust Doubling Operator

In many previous results, $\mathfrak{R} \equiv \mathfrak{9}_{46}$ is used as a *robust doubling operator* with infection curve η below.



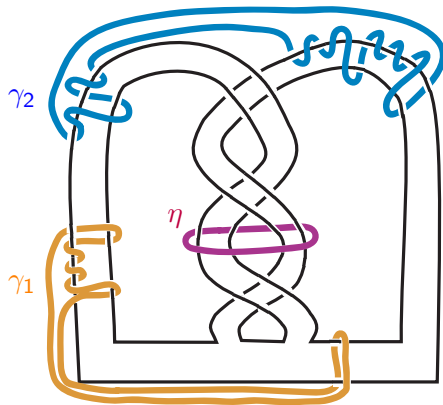
A Common Robust Doubling Operator

But what if instead of infection along $\eta \subset S^3 \setminus \mathfrak{R}$, we instead infect along γ_1 ?



A Common Robust Doubling Operator

Or would infecting along γ_2 lead to a completely separate concordance class?

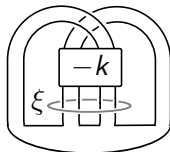


Our Knots: Infection in Two Steps

- Take j be an Arf-invariant 0 knot.
 j must be “special” in a way we’ll discuss later.
- Take R to be the ribbon knot of the form below.
The Alexander polynomials of R is

$$p(t) = (kt - (k + 1))((k + 1)t - k)$$

- Take $\xi \subset S^3 \setminus R$ to generate the rational Alexander module of R .
- Define $J = R(\xi, j)$. (J is 1-solvable!)



Our Knots: Infection Step 2

- Let \mathfrak{K} be *any* knot.
- Take $\gamma_1, \gamma_2 \subset S^3 \setminus \mathfrak{K}$ to be unknotted and have linking number 0 with \mathfrak{K} .
- Make sure $\mathcal{B}l_{\mathfrak{K}}(\gamma_1, \gamma_1) \neq \mathcal{B}l_{\mathfrak{K}}(\gamma_2, \gamma_2)$ and $\mathcal{B}l_{\mathfrak{K}}(\gamma_1, \gamma_1) \neq 0$.
- Define $K_i \equiv \mathfrak{K}(\gamma_i, J)$.

Note: If \mathfrak{K} is ribbon, then both K_i are 2-solvable.

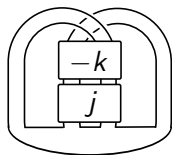
How must j be “special”? We need j to have $|\rho_0(j)|$ greater than the sum of the Cheeger-Gromov constant of $\mathfrak{K}\# - \mathfrak{K}$ and twice that of R .

The Cheeger-Gromov constant of a knot is the positive constant, $C_K > 0$, such that $|\rho(M(K), \Phi)| < C_K \quad \forall \Phi$.

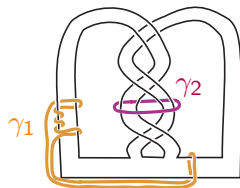
Main Theorem

Theorem (F., 2010)

- 1 If j is "special",
 - 2 $J \equiv R(\xi, j)$,
 - 3 and $K_i \equiv \mathfrak{R}(\gamma_i, J)$
 - 4 such that $\mathcal{B}l_{\mathfrak{R}}(\gamma_1, \gamma_1) \neq \mathcal{B}l_{\mathfrak{R}}(\gamma_2, \gamma_2)$ and $\mathcal{B}l_{\mathfrak{R}}(\gamma_1, \gamma_1) \neq 0$,
- then K_1 and K_2 are not concordant.



$R(\xi, j)$



\mathfrak{R}

Remarks

- R does not necessarily need to be of the form described!
Need only $p(t^n), p(t^m)$ to have no common roots unless $n = \pm m$.
- Could have $\mathcal{Bl}_{\mathfrak{R}}(\gamma_2, \gamma_2) = 0$.
- $K_1\# - K_2$ is 2-solvable. We show $K_1\# - K_2$ is not 2.5-solvable.

Before proving this theorem, we discuss some properties of the Blanchfield form and give an example to illustrate the implications of this theorem.

Properties of the Blanchfield Form

The Blanchfield Form of \mathfrak{K} , $\mathcal{B}l_{\mathfrak{K}}$, is a sesquilinear form on $\mathcal{A}^{\mathbb{Z}}(\mathfrak{K})$.

$$\mathcal{A} \times \mathcal{A} \xrightarrow{\mathcal{B}l} \frac{\mathbb{Q}(t)}{\mathbb{Z}[t, t^{-1}]}$$

Because it is sesquilinear, it satisfies the following properties:

- $\mathcal{B}l_{\mathfrak{K}}(\alpha_1 + \alpha_2, \beta) = \mathcal{B}l_{\mathfrak{K}}(\alpha_1, \beta) + \mathcal{B}l_{\mathfrak{K}}(\alpha_2, \beta)$
 $\mathcal{B}l_{\mathfrak{K}}(\alpha, \beta_1 + \beta_2) = \mathcal{B}l_{\mathfrak{K}}(\alpha, \beta_1) + \mathcal{B}l_{\mathfrak{K}}(\alpha, \beta_2)$
- $\mathcal{B}l_{\mathfrak{K}}(x(t)\alpha, \beta) = x(t)\mathcal{B}l_{\mathfrak{K}}(\alpha, \beta)$
 $\mathcal{B}l_{\mathfrak{K}}(\alpha, y(t)\beta) = \overline{y}(t)\mathcal{B}l_{\mathfrak{K}}(\alpha, \beta)$

Here, $\overline{y}(t)$ denotes the image of $y(t)$ under the group ring involution

$$\overline{\sum n_i g_i} = \sum n_i g_i^{-1}$$

The Blanchfield form

We will be using the Blanchfield form with coefficients in $\mathcal{Q} \equiv \mathbb{Z}[\frac{1}{2}] \subset \mathbb{Q}$, and $\mathcal{A}^{\mathcal{Q}} = \mathcal{A}^{\mathbb{Z}}(\mathfrak{X}) \otimes_{\mathbb{Z}} \mathcal{Q} = H_1(M(\mathfrak{X}), \mathcal{Q}[t, t^{-1}])$

$$\mathcal{Bl}_{\mathfrak{X}}^{\mathcal{Q}}: \mathcal{A}^{\mathcal{Q}} \times \mathcal{A}^{\mathcal{Q}} \rightarrow \frac{\mathcal{Q}(t)}{\mathcal{Q}[t, t^{-1}]}$$

An example:

$$\begin{aligned} \mathcal{Bl}(x\alpha + y\beta, x\alpha + y\beta) &= \mathcal{Bl}(x\alpha, x\alpha) + \mathcal{Bl}(x\alpha, y\beta) + \mathcal{Bl}(y\beta, x\alpha) + \mathcal{Bl}(y\beta, y\beta) \\ &= x^2\mathcal{Bl}(\alpha, \alpha) + xy\mathcal{Bl}(\alpha, \beta) + xy\mathcal{Bl}(\beta, \alpha) + y^2\mathcal{Bl}(\beta, \beta) \end{aligned}$$

where $x, y \in \mathcal{Q}$, $\alpha, \beta \in \mathcal{A}^{\mathcal{Q}}(\mathfrak{X})$.

The Blanchfield form

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An example:

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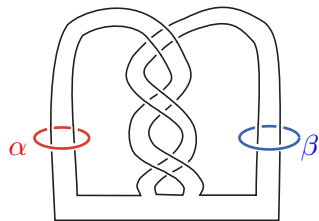
where $x, y \in \mathcal{Q}$, $\alpha, \beta \in \mathcal{A}^{\mathcal{Q}}(\mathfrak{X})$.

Suppose in this case, $\mathcal{Bl}(\alpha, \alpha) = \mathcal{Bl}(\beta, \beta) = 0$, then this would yield

$$\mathcal{Bl}(x\alpha + y\beta, x\alpha + y\beta) = xy(\mathcal{Bl}(\alpha, \beta) + \mathcal{Bl}(\beta, \alpha))$$

Example: $\mathfrak{R} = 9_{46}$

- $\Delta_{\mathfrak{R}}(t) = (-2t + 1)(t - 2)$
- α and β generate $\mathcal{A}^{\mathbb{Z}}(\mathfrak{R})$.
- $\eta = \alpha + \beta$ generates $\mathcal{A}^{\mathbb{Q}}(\mathfrak{R})$.



In $\mathcal{A}^{\mathbb{Z}}$, we have the equations

$$(2t - 1)\alpha \equiv 0 \implies 2t\alpha \equiv \alpha$$

$$(t - 2)\beta \equiv 0 \implies t\beta \equiv 2\beta$$

Taking $\mathcal{A}^{\mathbb{Q}}$, we see that if $\gamma_i \in \mathcal{A}^{\mathbb{Q}}$, we may write

$$\gamma_i = x_i\alpha + y_i\beta, \quad x_i, y_i \in \mathbb{Q}.$$

$\mathcal{B}l_{\mathfrak{X}}(\gamma_i, \gamma_i)$

The (Q) Blanchfield form yields...

$$\begin{aligned}\mathcal{B}l_{\mathfrak{X}}(\gamma_i, \gamma_i) &\equiv \mathcal{B}l_{\mathfrak{X}}(x_i\alpha + y_i\beta, x_i\alpha + y_i\beta) \\ &\equiv x_i^2 \mathcal{B}l_{\mathfrak{X}}(\alpha, \alpha) + x_i y_i \mathcal{B}l_{\mathfrak{X}}(\alpha, \beta) \\ &\quad + x_i y_i \mathcal{B}l_{\mathfrak{X}}(\alpha, \beta) + y_i^2 \mathcal{B}l_{\mathfrak{X}}(\beta, \beta) \\ &\equiv x_i y_i (\mathcal{B}l(\alpha, \beta) + \mathcal{B}l(\beta, \alpha)) \\ &\equiv x_i y_i \mathcal{B}l(\eta, \eta)\end{aligned}$$

$\mathcal{B}l_{\mathfrak{R}}(\gamma_i, \gamma_i)$

The (\mathcal{Q}) Blanchfield form yields...

$$\begin{aligned}\mathcal{B}l_{\mathfrak{R}}(\gamma_i, \gamma_i) &\equiv \mathcal{B}l_{\mathfrak{R}}(x_i\alpha + y_i\beta, x_i\alpha + y_i\beta) \\ &\equiv x_i^2 \mathcal{B}l_{\mathfrak{R}}(\alpha, \alpha) + x_i y_i \mathcal{B}l_{\mathfrak{R}}(\alpha, \beta) \\ &\quad + x_i y_i \mathcal{B}l_{\mathfrak{R}}(\alpha, \beta) + y_i^2 \mathcal{B}l_{\mathfrak{R}}(\beta, \beta) \\ &\equiv x_i y_i (\mathcal{B}l(\alpha, \beta) + \mathcal{B}l(\beta, \alpha)) \\ &\equiv x_i y_i \mathcal{B}l(\eta, \eta)\end{aligned}$$

Since η generates $\mathcal{A}^{\mathcal{Q}}(\mathfrak{R})$, it has

$$\mathcal{B}l_{\mathfrak{R}}(\eta, \eta) = \frac{f(t)}{\Delta_{\mathfrak{R}}(t)}$$

for some $f(t)$ coprime to $\Delta_{\mathfrak{R}}(t)$.

$\mathcal{B}l_{\mathfrak{X}}(\gamma_i, \gamma_i)$

We have

$$\mathcal{B}l_{\mathfrak{X}}(\gamma_i, \gamma_i) \equiv x_i y_i \mathcal{B}l(\eta, \eta) = x_i y_i \frac{f(t)}{\Delta_{\mathfrak{X}}(t)}$$

Therefore, if $\mathcal{B}l_{\mathfrak{X}}(\gamma_1, \gamma_1) = \mathcal{B}l_{\mathfrak{X}}(\gamma_2, \gamma_2)$, this implies

$$\begin{aligned}(x_1 y_1 - x_2 y_2) \frac{f(t)}{\Delta_{\mathfrak{X}}(t)} &\in \mathcal{Q}[t, t^{-1}] \\ \implies x_1 y_1 - x_2 y_2 &= h(t) \Delta_{\mathfrak{X}}(t)\end{aligned}$$

Since $x_i, y_i \in \mathcal{Q}$, this is only possible when $x_1 y_1 - x_2 y_2 = 0$.

Knots not concordant to $\mathfrak{R}(\eta, J)$

Fix $\gamma_1 = \eta = 1\alpha + 1\beta$ and set $K_1 \equiv \mathfrak{R}(\eta, J)$

Question: On what curves γ may I infect \mathfrak{R} by J to obtain $K_2 \equiv \mathfrak{R}(\gamma, J)$ not concordant to K_1 ?

$\gamma = x\alpha + y\beta$ in $\mathcal{A}^{\mathcal{Q}}$. We must show that $1 - xy \neq 0$ If it does hold, then

$$xy = 1 \in \mathcal{Q} \implies \begin{cases} x &= \pm 2^{-r}, \\ y &= \pm 2^r \end{cases}$$

Knots not concordant to $\mathfrak{R}(\eta, J)$

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So $\gamma = \pm(2^{-r}\alpha + 2^r\beta) \equiv \pm(t^r\alpha + t^r\beta) = \pm t^r\eta$, which represents a curve equivalent to $\pm\eta$ in $S^3 \setminus \mathfrak{R}$.

Knots not concordant to $\mathfrak{K}(\eta, J)$

Fix $\gamma_1 = \eta = 1\alpha + 1\beta$ and set $K_1 \equiv \mathfrak{K}(\eta, J)$

Question: On what curves γ may I infect \mathfrak{K} by J to obtain $K_2 \equiv \mathfrak{K}(\gamma, J)$ not concordant to K_1 ?

$\gamma = x\alpha + y\beta$ in $\mathcal{A}^{\mathcal{Q}}$. We must show that $1 - xy \neq 0$. If it does hold, then

$$xy = 1 \in \mathcal{Q} \implies \begin{cases} x = \pm 2^{-r}, \\ y = \pm 2^r \end{cases}$$

So $\gamma = \pm(2^{-r}\alpha + 2^r\beta) \equiv \pm(t^r\alpha + t^r\beta) = \pm t^r\eta$, which represents a curve equivalent to $\pm\eta$ in $S^3 \setminus \mathfrak{K}$.

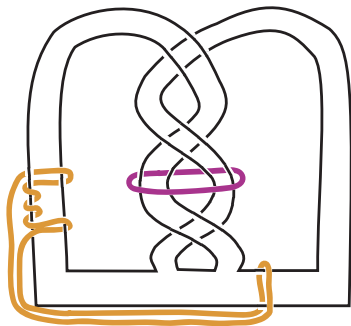
Therefore, the only curves on which I can infect \mathfrak{K} by J and obtain a concordant knot to $K_1 \equiv \mathfrak{K}(\eta, J)$ is $\pm\eta$

In the following diagram,

$$\eta = \alpha + \beta$$

$$\gamma_1 = (t + t^{-1})\alpha + \beta$$

$$= \frac{5}{2}\alpha + \beta$$



Since $c(\eta) = 1$, $c(\gamma_1) = \frac{5}{2}$, our results show that $K_1 = \mathfrak{R}(\eta, J)$, $K_2 = \mathfrak{R}(\gamma_1, J)$ are not concordant.

Concordance Classes of $\mathfrak{R}(\gamma, J)$

Fixing $\mathfrak{R} = 9_{46}$ and $J = R(J')$, there are infinitely many concordance classes of the form $\mathfrak{R}(\gamma, J)$.

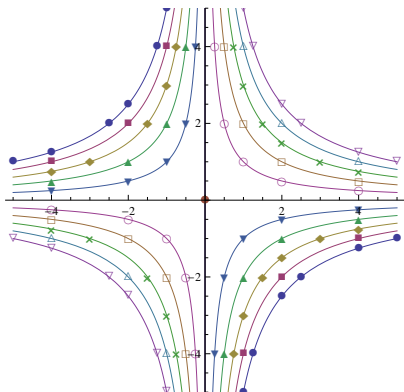
For every $\gamma = x\alpha + y\beta$ in $\mathcal{A}^{\mathcal{Q}}$, each $c(\gamma) = xy$ in $\mathbb{Z} \left[\frac{1}{2} \right]$ identifies a unique family, $\mathcal{F}_c \subset \mathcal{C}$.

$|\mathcal{F}_c|$ is finite.

Concordance Classes of $\mathfrak{R}(\gamma, J)$

Families \mathcal{F}_c where $c = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.

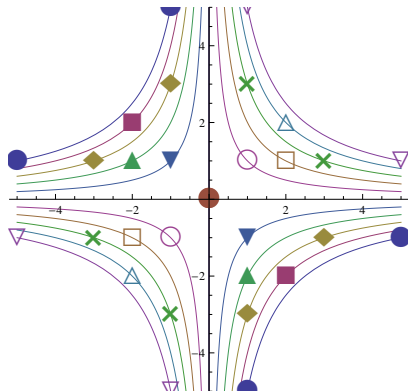
$$(x, y) = (x, c/x) \in \mathbb{Z} \left[\frac{1}{2} \right]$$



Concordance Classes of $\mathfrak{R}(\gamma, J)$

Mod out by t action \rightsquigarrow distinct concordance classes in \mathcal{F}_c .

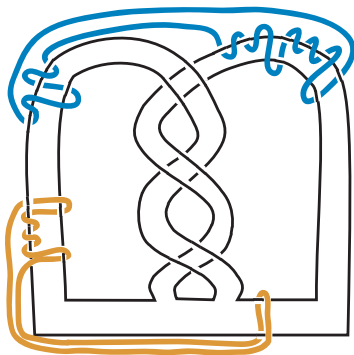
Distinct concordance classes, $c = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.



In the following diagram,

$$\gamma_1 = (t + t^{-1})\alpha + \beta = \frac{5}{2}\alpha + \beta$$

$$\begin{aligned}\gamma_2 &= t\alpha + (t^2 + 1)\beta \\ &= \frac{1}{2}\alpha + 5\beta\end{aligned}$$

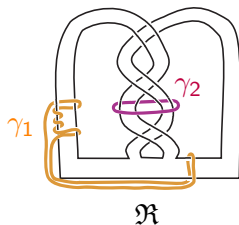
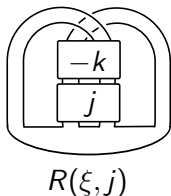


Since $c(\gamma_1) = c(\gamma_2) = \frac{5}{2}$, our methods are unable to distinguish between $K_1 = \mathfrak{R}(\gamma_1, J)$, $K_2 = \mathfrak{R}(\gamma_2, J)$.

Main Theorem

Theorem (F., 2010)

- 1 If j is "special",
 - 2 $J \equiv R(\xi, j)$,
 - 3 and $K_i \equiv \mathfrak{R}(\gamma_i, J)$
 - 4 such that $\mathcal{B}l_{\mathfrak{R}}(\gamma_1, \gamma_1) \neq \mathcal{B}l_{\mathfrak{R}}(\gamma_2, \gamma_2)$ and $\mathcal{B}l_{\mathfrak{R}}(\gamma_1, \gamma_1) \neq 0$,
- then K_1 and K_2 are not concordant.



Proof (ish)

We suppose $K_1\# - K_2$ is a 2.5-solvable knot and seek a contradiction.

Distributivity of satellite operations yields:

$$\begin{aligned}K_1\# - K_2 &\equiv \mathfrak{R}(\gamma_1, J)\# - \mathfrak{R}(\gamma'_2, J) \\ &\equiv (\mathfrak{R}\# - \mathfrak{R})(\gamma_1, \gamma'_2; J, -J)\end{aligned}$$

So we think of $K_1\# - K_2$ to be the result of a *double infection* on the slice knot $\mathfrak{R}\# - \mathfrak{R}$. Therefore, $K_1\# - K_2$ is always 2-solvable regardless of whether \mathfrak{R} is actually ribbon.

Note: We denote γ'_2 the copy of the curve and Alexander module element γ_2 in $\mathcal{A}(-\mathfrak{R}) \subset \mathcal{A}(\mathfrak{R}\# - \mathfrak{R})$.

Coefficient Systems, ρ -invariants, and Signature Defects

Define a coefficient system on W by the projection:

$$\Phi : \pi_1(W) \rightarrow \pi_1(W)/\pi_1(W)_S^{(3)}$$

“Facts”:

- 1 The restriction of Φ to $\pi_1(M(j))$ factors nontrivially through \mathbb{Z} .
- 2 The restriction of Φ to $\pi_1(\overline{M(j)})$ is trivial.

We consider the L^2 -signature defect of W induced by this coefficient system,

$$\sigma^{(2)}(W, \Phi) - \sigma(W) = \rho(\partial W, \Phi|_{\partial})$$

Signature defects, ρ -invariants \rightsquigarrow Contradiction

Due to additivity of the signature defect, we have

$$\begin{aligned}\sigma^{(2)}(W, \Phi) - \sigma(W) &= \sigma^{(2)}(V, \Phi) - \sigma(V) + \sigma^{(2)}(E, \Phi) - \sigma(E) \\ &\quad + \sigma^{(2)}(F, \Phi) - \sigma(F) + \sigma^{(2)}(\bar{F}, \Phi) - \sigma(\bar{F}) \\ &= 0\end{aligned}$$

Signature defects, ρ -invariants \rightsquigarrow Contradiction

Due to additivity of the signature defect, we have

$$\begin{aligned}\sigma^{(2)}(W, \Phi) - \sigma(W) &= \sigma^{(2)}(V, \Phi) - \sigma(V) + \sigma^{(2)}(E, \Phi) - \sigma(E) \\ &\quad + \sigma^{(2)}(F, \Phi) - \sigma(F) + \sigma^{(2)}(\overline{F}, \Phi) - \sigma(\overline{F}) \\ &= 0\end{aligned}$$

However,

$$\begin{aligned}\sigma^{(2)}(W, \Phi) - \sigma(W) &\equiv \rho(M(j), \Phi) + \rho(\overline{M(j)}, \Phi) + \rho(M(R), \Phi) \\ &\quad + \rho(\overline{M(R)}, \Phi) + \rho(M(\mathfrak{R}\# - \mathfrak{R}), \Phi)\end{aligned}$$

Signature defects, ρ -invariants \rightsquigarrow Contradiction

Due to additivity of the signature defect, we have

$$\begin{aligned}\sigma^{(2)}(W, \Phi) - \sigma(W) &= \sigma^{(2)}(V, \Phi) - \sigma(V) + \sigma^{(2)}(E, \Phi) - \sigma(E) \\ &\quad + \sigma^{(2)}(F, \Phi) - \sigma(F) + \sigma^{(2)}(\overline{F}, \Phi) - \sigma(\overline{F}) \\ &= 0\end{aligned}$$

However,

$$\begin{aligned}\sigma^{(2)}(W, \Phi) - \sigma(W) &\equiv \rho(M(j), \Phi) + \rho(\overline{M(j)}, \Phi) + \rho(M(R), \Phi) \\ &\quad + \rho(\overline{M(R)}, \Phi) + \rho(M(\mathfrak{R}\# - \mathfrak{R}), \Phi)\end{aligned}$$

Because of the “Facts” before, $\rho(M(j), \Phi) = \rho_0(j)$, $\rho(\overline{M(j)}, \Phi) = 0$.

$$\rho_0(j) = -\rho(M(R), \Phi) - \rho(\overline{M(R)}, \Phi) - \rho(M(\mathfrak{R}\# - \mathfrak{R}), \Phi)$$

$\Rightarrow \Leftarrow$

Commutator Series

Problem: We need a refinement of the filtration to really get a contradiction. So we use a well chosen commutator series.

Definition

Let

$$G_S^{(0)} \equiv G$$
$$G_S^{(1)} = G_r^{(1)} \equiv \ker \left(G \rightarrow \frac{G}{[G, G]} \otimes_{\mathbb{Z}} \mathbb{Q} \right)$$

and for $i = 1, 2$

$$G_S^{(i+1)} \equiv \ker \left(G_S^{(i)} \rightarrow \frac{G_S^{(i)}}{[G_S^{(i)}, G_S^{(i)}]} \otimes_{\mathbb{Z}[G/G_S^{(i)}]} \mathbb{Q}[G/G_S^{(i)}] S_i^{-1} \right).$$

Commutator Series

Before, the S_i are right divisor sets defined by

$$S_1 \equiv S_1(G) = \{q_1(\mu) \cdots q_r(\mu) \mid (\Delta_{\mathfrak{A}}(t), q_j(t)) = 1; G/G_r^{(1)} \cong \langle \mu \rangle\} \\ \subset \mathbb{Q}[G/G_r^{(1)}]$$

$S_2 = S_2(G) \subset \mathbb{Q}[G_S^{(1)}/G_S^{(2)}] \subset \mathbb{Q}[G/G_S^{(2)}]$ is the multiplicative set generated by

$$\{q_1(a_1) \cdots q_r(a_r) \mid \widetilde{(p, q_j)} = 1; q_j(1) \neq 1; a_j \in G_S^{(1)}/G_S^{(2)}\} \\ \cup \{p(\mu^i \gamma_2 \mu^{-i}) \mid i \in \mathbb{Z}\}.$$

A Game Plan

It is straightforward to check that the submodules generated by γ_1 and $\gamma_1 + \gamma_2'$ are nontrivial in $\frac{\pi_1(W)^{(1)}}{\pi_1(W)_S^{(2)}}$.

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It is straightforward to check that the submodules generated by γ_1 and $\gamma_1 + \gamma_2'$ are nontrivial in $\frac{\pi_1(W)^{(1)}}{\pi_1(W)_S^{(2)}}$.

Note: $\pi_1(M(j))$ is normally generated by μ_j which is isotopic in W to ξ .
Furthermore, we have that $\pi_1(M(j)) \subset \pi_1(M(J))^{(1)} \subset \pi_1(W)^{(2)}$.
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A Game Plan

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Therefore, if we wish to show

- 1 The restriction of Φ to $\pi_1(M(j))$ factors nontrivially through \mathbb{Z} .
- 2 The restriction of Φ to $\pi_1(\overline{M(j)})$ is trivial.

where $\Phi : \pi_1(W) \rightarrow \frac{\pi_1(W)}{\pi_1(W)_S^{(3)}}$ it will suffice to show that ξ is nontrivial whereas $\bar{\xi}$ is trivial in $\frac{\pi_1(W)}{\pi_1(W)_S^{(3)}}$.

Consider the diagram, where $\pi = \pi_1(W)$, $\mathcal{K} = \mathbb{Q} \left[\pi / \pi_S^{(2)} \right] \mathcal{S}_2^{-1}$.

$$\begin{array}{ccc}
 \pi_1(M(J))^{(1)} \oplus \pi_1(\overline{M(J)})^{(1)} & \rightarrow & \pi^{(2)} & \longrightarrow & \frac{\pi_S^{(2)}}{\pi_S^{(3)}} \\
 \downarrow & & & & \downarrow \phi \\
 (\mathcal{A}(J) \oplus \mathcal{A}(\overline{J})) \otimes \mathcal{K} & \longrightarrow & \frac{\pi_S^{(2)}}{[\pi_S^{(2)}, \pi_S^{(2)}]} \otimes \mathcal{R} & &
 \end{array}$$

ϕ is injective since this kernel is defined to be $\pi_1(W)_S^{(3)}$

$\Phi|_{\pi_1(\overline{M(j)})}$ is trivial

To determine if $\xi, \bar{\xi}$ are trivial in $\pi_S^{(2)}/\pi_S^{(3)}$, look at their images under the map

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$\bar{\xi}$ generates $\mathcal{A}^{\mathbb{Q}}(\bar{J})$. So we have

$$\begin{aligned} \langle \bar{\xi} \rangle &\cong \mathcal{A}^{\mathbb{Q}}(\bar{J}) \cong \frac{\mathbb{Q}[t, t^{-1}]}{\rho(t)\mathbb{Q}[t, t^{-1}]} \\ \langle \bar{\xi} \otimes 1 \rangle &\cong \mathcal{A}(\bar{J}) \otimes \mathcal{K} \cong \left(\frac{\mathbb{Q}[t, t^{-1}]}{\rho(\gamma'_2)\mathbb{Q}[t, t^{-1}]} \right) S_2^{-1} \equiv 0 \end{aligned}$$

since $\rho(\gamma'_2) \in S_2$ by definition.

$\Phi|_{\pi_1(M(j))}$ factors nontrivially through \mathbb{Z}

The kernel of $\mathcal{A} \otimes \mathcal{K} \rightarrow \frac{\pi_S^{(2)}}{[\pi_S^{(2)}, \pi_S^{(2)}]} \otimes \mathcal{K}$ is isotropic with respect to $\mathcal{B}l^{\mathcal{K}}$, but since $\xi \otimes 1$ generates $\mathcal{A}(J) \otimes \mathcal{K}$ and the Blanchfield form is nonsingular, we must show

$$\langle \xi \otimes 1 \rangle \cong \mathcal{A}(J) \otimes \mathcal{K} \cong \left(\frac{\mathbb{Q}[t, t^{-1}]}{\rho(\gamma_1)\mathbb{Q}[t, t^{-1}]} \right) S_2^{-1} \neq 0.$$

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Under the map

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the kernel is the S_2 torsion submodule. So we ask if $\rho(\gamma_1) \in S_2$.

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Recall: S_2 generated by

$$\{q_1(a_1) \dots q_r(a_r) \mid \widetilde{(p, q_j)} = 1; q_j(1) \neq 1; a_j \in G_S^{(1)}/G_S^{(2)}\} \\ \cup \{\rho(\mu^i \gamma_2 \mu^{-i}) \mid i \in \mathbb{Z}\}.$$

If $p(\gamma_1) \in S_2 \Rightarrow$

$$p(\gamma_1) = \prod_i p(\mu^i \gamma'_2 \mu^{-i}) = \prod_i p(\tau_*^i(\gamma'_2)) \rightsquigarrow \begin{cases} \gamma_1 = x^m, \\ \tau_*^i(\gamma'_2) = x^n \end{cases}$$

But $p(x^n), p(x^m)$ share no common roots unless $n = \pm m$. So,

$$\tau_*(\gamma'_2) = \pm \gamma_1$$

This equation must hold in $\mathcal{A}(W)$, but since γ_1, γ'_2 are really curves in $M(\mathfrak{R}\# - \mathfrak{R})$ and $\mathcal{B}l_{\mathfrak{R}}(\gamma_1 \pm \tau_*^i(\gamma'_2), \gamma_1 \pm \tau_*^i(\gamma'_2)) \neq 0$, it may also be interpreted in $\mathcal{A}(\mathfrak{R}\# - \mathfrak{R}) \cong \mathcal{A}(\mathfrak{R}) \oplus \mathcal{A}(-\mathfrak{R})$.

This is impossible since

$$\begin{aligned} \tau_*(0 \oplus \mathcal{A}(-\mathfrak{R})) &\subset 0 \oplus \mathcal{A}(-\mathfrak{R}) \\ \tau_*(\mathcal{A}(\mathfrak{R}) \oplus 0) &\subset \mathcal{A}(\mathfrak{R}) \oplus 0 \end{aligned}$$

QED!

This finishes the “Facts” we needed for our proof.

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And my talk. Thank you!