ON THE HIGHLY OSCILLATORY PLATEAU PROBLEM

ABSTRACT.

1. Introduction

In this paper, we study the homogenization of a highly oscillatory Plateau problem. We expect to get a nontrivial limit, for instance in the energy minimizer case the weak limit converges to the average of the boundary (In this case the energy is bounded, we need to show this!), so we would like to show that the limit is different than the average of the boundary condition. The problem of Plateau is the following:

Problem. Let $g : S^1 \to [-M, M]$ be a function, for some $M > 0$, and the graph of $u$ is a minimal surface on $B_1(0) \subseteq \mathbb{R}^2$ such that $u|_{S^1} = g$.

What is the least regularity of $g$ for the existence of $u$? What can we prove about the properties of $u$ under the least possible regularity on $g$, such as the behavior near the boundary $S^1$?

Example 1.1. When we consider the solution even for a noncontinuous boundary condition we can still talk about the existence of a smooth minimal surface: Let us assume that

$$g(x) = \begin{cases} 2 & \text{on } S^1 \cap \{x|\langle x, e_1 \rangle < 0\}, \\ -2 & \text{on } S^1 \cap \{x|\langle x, e_1 \rangle > 0\}. \end{cases}$$

Then the minimal surface (Is it unique???? Need to check some comparison principles!) can be found by analyzing the problem by rotation, since minimal surface is a minimal surface from any direction and this case we also have a symmetry on the data so we get a figure below, Figure 1: In order to show that $u$ is a graph in $B_1(0)$, we can show that outer normal directional derivative is negative in $B_1(0)$. This implies that $u$ is decreasing in this direction. Thus, it forms a graph. In order to check the sign of this directional derivative we can use a maximum principle since, this derivative has a sign on the boundary. However, the question that we need to answer is what kind of a PDE equation this derivative satisfies. Since $n = 2$, we can write the minimal surface equation as

$$(1 + |u_x|^2)u_{yy} - 2u_xu_yu_{xy} + (1 + |u_y|^2)u_{xx} = 0$$
First, we analyze an asymptotic behavior of solutions $u_\varepsilon$ as $\varepsilon \to 0$.

\[
\begin{cases}
\text{div} \left( \frac{\nabla u_\varepsilon}{\sqrt{1 + |\nabla u_\varepsilon|^2}} \right) = 0 \quad \text{in } B_1(0), \\
u_\varepsilon = g\left(\frac{x}{\varepsilon}\right) \quad \text{on } S^1
\end{cases}
\]  

for some continuous function $g$.

**Example 1.2.** In this example, we consider the solution for a highly oscillatory boundary condition. Let us assume that $g(e^{i\theta}) = 2 + 2\cos(\theta)$ for $e^{i\theta} \in S^1$. Here, $g_\varepsilon$ do not converge pointwise, they do converge in $L_1$ norm, which only guarantees the pointwise convergence of a subsequence. If we were dealing with a pointwise convergence of the boundary condition, we could have directly obtained a convergence of the minimal surfaces to a minimal surface up to the boundary, but now, we need to be careful about the trace (if it exists???) of the limit. In the interior we still have a minimal surface but we don’t know its behavior on the boundary. That is the problem that we’d like to analyze. In this example, in the limit we obtain two cups with densities 1/2 and they are attached on their bottoms at the height of 2, (one cup is upside down), it can be considered as a multi valued function of the form

\[
g(x) = \begin{cases} 
4 & \text{on } S^1 \text{ with density } 1/2, \\
2 & \text{on } S^1 \text{ with density } 1, \\
0 & \text{on } S^1 \text{ with density } 1/2.
\end{cases}
\]  

At a level of 2, we obtain the minimal surface which is exactly a disc as the limit (That is because of dyadic rotational invariance of the limit, as the dyadic rotation period converges to 0, we obtain a minimal surface that is radial and this can be only a disc).
Question: Let us consider a sequence of graphs $\{\gamma_i\}$ in $\mathbb{R}^2$ that encloses the sequence of sets $\{A_i\}$ in $\mathbb{R}^2$ so that $|A_i| < M$. Uniform mass bound gives us the convergence of $\chi_{A_i}$ in $L^1$ to a positive measure $\mu$. And this implies that the convergence of $\partial A_i = \gamma_i$ to the derivative of $\mu$ which is a distribution.

In terms of currents, what does the limit look like? We know that $\text{Mass}(\text{graph}(u_\epsilon)) \leq \text{Area}(S^1) + \text{SurfaceArea}(\text{CylinderHeight}A)$ which gives

$$T_\epsilon(\varphi) \to T(\varphi), \forall \varphi.$$ 

Since $\partial(T_\epsilon(\varphi)) = T_\epsilon(\partial \varphi)$, we get

$$\partial T_\epsilon(\varphi) \to \partial T(\varphi), \forall \varphi.$$ 

In this example, we get a current whose boundary can be considered as a multi valued function $g(x)$ given above.

we can even increase the number of the stages that the multi valued function can attain by putting more stages on periodic data $g_\epsilon$.

**Lemma 1.3.** $h(x,y) = \cos xe^y$ is a super-minimal surface for $|x| < \frac{\pi}{2}$.

**Proof.**

$$\text{MinSur}(h) = (1 + \cos^2 xe^{2y})(-\cos xe^y) + (1 + \sin^2 xe^{2y})(\cos xe^y) - 2(\sin^2 xe^{2y})$$

$$= e^{3y}(-\cos^3 x + \sin^2 x \cos x - 2 \sin^2 x \cos x)$$

$$= e^{3y}(-\cos^3 x - \sin^2 x) \cos x$$

$$= -e^{3y} \cos x$$

We next rescale to a narrow finger:

$$h_\epsilon(x,y) = \cos \frac{x}{\epsilon} e^{y/\epsilon}$$ for $y < 0$ and $|x| < \frac{\pi \epsilon}{2}$. For $y = 0$, $h_\epsilon = \cos \frac{x}{\epsilon}$ but it has exponential decay. For $y = -1$, $h_\epsilon \leq e^{-1/\epsilon}$. 

$\square$

**APPENDIX A**

In this section we will state the known facts about the minimal surfaces:

**Theorem A.1** (From [2], Thm 1.2). *The solutions of the Plateau problem are regular up to the boundary if the boundary is smooth.*

Check whether convexity is necessary or not!!!

**Theorem A.2** (From [3], Thm 16.8). *Let $\Omega$ be a bounded $C^2$ domain in $\mathbb{R}^n$. Then the Dirichlet problem $Mu = 0$ in $\Omega$, $u = \varphi$ on $\partial \Omega$, is solvable for arbitrary $\varphi \in C(\Omega)$ if and only if the mean curvature of the boundary $\partial \Omega$ is everywhere non-negative.*

**Remark A.3.** Since, mean curvature equation is a quasilinear, locally uniformly elliptic equation we can apply the following comparison principle. As along as we have continuous solutions up to the boundary we have the comparison, so we get uniqueness for continuous solutions.

**Theorem A.4** (From [3], Thm 10.1). *Let $u, v \in C(\bar{\Omega}) \cap C^2(\Omega)$ with $Mu \geq Mv$ in $\Omega$ and $u \leq v$ on $\partial \Omega$ then $u \leq v$ in $\Omega$.*

**Remark A.5.** Next theorem gives us the interior regularity of the minimal surface. As long as it is in $C^2(\Omega)$, it becomes smooth with uniformly bounded derivatives.

**Theorem A.6** (From [3], Corollary 16.7). *Let $\Omega$ be a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$ whose graph has mean curvature $H \in C^k(\Omega)$, $k \geq 1$. Then $u \in C^{k+1}(\Omega)$ and for any point $y \in \Omega$ and multi-index $\beta$, $|\beta| = k + 1$, $|D^\beta u(y)| \leq C$, where $C = C(n, k, |H|_k, \text{dist}(y, \partial \Omega), \sup |u|)$.*

**Remark A.7.** Next theorems give us the existence and characterization results for $L^1$ boundary data. In [4], M. Miranda works on a more generalized (non homogenous) variational problem and we adapt his results to our case.

**Theorem A.8** (From [4] on trace operator). *Let $\Omega$ be a domain in $\mathbb{R}^n$ with locally Lipschitz boundary, then for any $u \in BV(\Omega)$ there exists a unique function in $L^1(\partial \Omega)$ which we call trace of $u(x)$*.

**Theorem A.9** (From [4]). *Let $\Omega$ be a domain in $\mathbb{R}^n$ with $C^2$ boundary, $g(x)$ is a continuous function at $x \in \partial \Omega$, and the principal curvatures of $\partial \Omega$ at every point $x$ is strictly positive, then there exists a continuous minimal surface $u(x)$ a.e. $x \in \Omega$ and satisfies $u(x) = g(x)$, $x \in \partial \Omega$.*

**Remark A.10.** When we only assume $L^1$ boundary data, i.e. $g(x) \in L^1(\partial \Omega)$, we can consider the variational problem and analyze a/the minimizer of the functional:

$$J(u) = \int_\Omega (1 + |\nabla u(x)|^2)^{1/2} dx + \int_{\partial \Omega} |u(x) - g(x)| dH_{n-1}.$$  

We have the following results by M. Miranda.
**Theorem A.11** (From [4]). Let $\Omega$ be a domain in $\mathbb{R}^n$ with locally Lipschitz boundary, $g(x) \in L^1(\partial \Omega)$, then the functional $J(u)$, (1.4), attains its minimum in $BV(\Omega)$.

**Theorem A.12** (From [4]). Let $u \in BV(\Omega)$ minimizes the functional $J(u)$ at a finite value, then $u(x)$ is continuous a.e. in $\Omega$. 
References


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