MATH 499 Exercises

January 16, 2008

1. Consider the sets $U_i \subseteq \mathbb{P}^n$ given by $U_i = \{p \in \mathbb{P}^n : a_i \neq 0\}$, and the maps

$$\begin{aligned}
&\phi_i : \mathbb{A}^n \to \mathbb{P}^n \\
&\phi_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_n), \\
&\psi_i : U_i \to \mathbb{A}^n \\
&\psi_i(a_0, \ldots, a_n) = \left(\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}\right)
\end{aligned}$$

   Verify that there is a bijection from each set U_i to $\mathbb{A}^n$ by showing that the maps ϕ_i and ψ_i are inverses.

2. Consider the maps

$$\begin{aligned}
&\mu_i : k[x_0, \ldots, x_n] \to k[y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n]_h \\
&\mu_i(f) = f(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) \\
&\nu_i : k[y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n]_h \to k[x_0, \ldots, x_n] \\
&\nu_i(f) = x_i^d f(x_0/x_1, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n)
\end{aligned}$$

   (i) Show that $\mu_i(\nu_i(f)) = f$ for f an arbitrary polynomial.

   (ii) Show that μ_i and ν_i are not inverses.

3. Consider the map

$$\phi(x_0, x_1, x_2, x_3) = (x_0x_1, x_0x_2, x_0x_3, x_1x_2, x_1x_3, x_2x_3)$$

   (i) Does this give a well-defined map from $\mathbb{P}^3(\mathbb{Q}) \to \mathbb{P}^5(\mathbb{Q})$?

   (ii) Consider the projective variety

$$X = \left\{[x_0, x_1, x_2, x_3] : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\right\} \subseteq \mathbb{P}^3(\mathbb{C})$$

   Does the map ϕ give a well-defined map from $X \to \mathbb{P}^5(\mathbb{C})$.