MATH 499 Exercises

February 20, 2008

1. Let Y and Z be curves. If $P \in Y$ and $Q \in Z$ are analytically isomorphic plane curve singularities, show that the multiplicities $\mu_P(Y)$ and $\mu_Q(Z)$ are the same.

2. The plane curves $y^2 = x^2(x+1)$ and $xy = 0$ both have a singularity at the origin. Show that these singularities are analytically isomorphic by exhibiting an explicit formal power series change of variables.

3. (i) Suppose that f and g are homogeneous polynomials of degree s and t, respectively, that do not share any linear factors. Show that $\langle f, g\rangle_d = k[x,y]_d$ for $d \geq s+t-1$ (that is, f and g generate all the dth degree polynomials).

   (ii) Let $f = f_r + f_{r+1} + f_{r+2} + \cdots \in k[[x,y]]$, where f_k is the kth degree homogeneous part of f. Suppose that f_r factors as $f_r = g_s h_t$ for homogeneous polynomials g_s and h_t of degrees s and t, respectively, and that g_s and h_t share no common linear factor. Show that there are formal power series $f, g \in k[[x,y]]$ such that $f = gh$.

   [Hint: Generalize the previous problem, and use part (i).]

4. Recall that the Milnor number of a plane curve singularity is defined as

$$\mu_{f,0} = \dim \frac{\mathbb{C}[[x,y]]}{\langle \partial f/\partial x, \partial f/\partial y\rangle}$$

   and the Tjurina number is defined to be

$$\tau_{f,0} = \dim \frac{\mathbb{C}[[x,y]]}{\langle f, \partial f/\partial x, \partial f/\partial y\rangle}$$

   Calculate the Milnor and Tjurina numbers for $f(x,y) = x^3 + y^3 - x^2y$ at the origin.

5. Calculate the Milnor and Tjurina numbers at the origin for $f(x,y) = y^3 - x^7 + x^5y$.

6. Calculate the Milnor and Tjurina numbers at the origin for $f(x, y) = x^p + y^q$, for integers p and q.

7. The Tjurina and Milnor numbers of a singularity are invariants of the analytic equivalence class of the singularity (that is, if two singularities are analytically equivalent, they have the same Tjurina and Milnor numbers). Sketch a proof of this fact.