Computing the Log Canonical Threshold on a Plane Curve

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Now we'll describe how to calculate the log canonical threshold for the blowup of a plane curve. The general idea will be to use information about the blowup of the curve (in particular, the multiplicities of the exceptional curves on the blowup) and compare this to the information given purely by the blowup of the plane (the same exceptional curves, except now we want the multiplicities given by the blowup of the plane).

We already have basically all of the tools necessary for calculating the log canonical threshold. We just need to describe some notation to make them more convenient to work with.

Earlier in the seminar, we described how to find the multiplicity of a curve. In particular, we talked about how to calculate the multiplicity of an exceptional curve on a blowup. Now, if we have a blowup $\pi : \tilde{Y} \to Y$, we'll write the *total transform* of the curve C as

$$\pi^*(C) = \tilde{C} + \sum a_i E_i$$

where $\tilde{C}$ is the proper transform of C, and the E_i are exceptional curves of multiplicity a_i. Here, π^* is the *pullback morphism.* We can think of this as the inverse of the blowup map π.

Here, we are essentially describing the proper transform as a *divisor* on the blowup, and expressing it in the notation of divisors. We can think of a *divisor* D on a variety Y of dimension n as basically a dimension $n-1$ subvariety on Y. In our case, the divisors we are using are curves (dimension one subvarieties) on either the plane or a blow-up of the plane. For our purposes, we will only make use of some facts about divisors, as a full description is a bit beyond what we need.

We can similarly calculate the multiplicities of the exceptional curves on the blowup of the plane. This information will be captured by the *canonical divisor* K_π of the blowup. This, however, is not quite as straight-forward to calculate as for the curve, so for now we'll rely on a fact about divisors for this calculation.

Lemma 1 *Let $\pi : \tilde{Y} \to Y$ be the blowup of a point $P \in Y$, and let E be the exceptional curve. Then*

$$K_{\tilde{Y}} = \pi^*(K_Y) + E$$

where π^* is the pullback morphism from the blowup.

This formula will allow us to calculate the canonical divisor in somewhat the same way as the total transform.

Now we have enough information to describe the log canonical threshold. In it's full generality, this is fairly difficult to calculate. In the case of the normal resolution at a point of a plane curve, we have a fairly simple formula. Given the canonical divisor of the blowup expressed as $K_{\pi} = \pi^*(K_Y) + \sum a_i E_i$ and the total transform of the curve C expressed as $\pi^*(C) = \tilde{C} + \sum b_i E_i$, the *log canonical threshold* of C at the point of blowup is given by

$$c_0(C) = \min \left\{ \frac{a_i + 1}{b_i} \right\}.$$

Therefore, we can calculate the log canonical threshold of a point on a plane curve by blowing up the point until we have a normal resolution, and then calculating the total transform and canonical divisor of the blowup. An example will probably clear up what we are trying to do here.

Example 1 *Let C be the curve given by $y^2 - x^3$. Calculate the log canonical threshold of C at the origin.*

First, let's calculate the total transform for the blowup of the curve. We have previously described the normal resolution of this curve using three blowups, but let's do it again, since we will want to keep closer track of what happens to the exceptional curves.

First, let's blowup by the relation $u_1 x = t_1 y$. On the first patch, we have

$$u_1 = 1, \quad x = t_1 y$$
$$y^2 - x^3 = y^2 - t_1^3 y^3 = y^2(1 - t_1^3 y)$$

Here, the exceptional curve does not intersect the proper transform, so we can safely ignore this patch. On the other patch, we have

$$t_1 = 1, \quad y = u_1 x$$
$$y^2 - x^3 = u_1^2 x^2 - x^3 = x^2(u_1^2 - x)$$

This will give us proper transform and exceptional curve

$$\begin{aligned} &C_1 : u_1^2 - x \\ &E_1 : x^2 \end{aligned}$$

We have successfully resolved the singularity on C in just one blowup. However, the exceptional curve E_1 is tangent to C_1, so we will need to blow-up again.

On the second blowup, let's use the relation $t_2 u_1 = u_2 x$. On the first patch, we have

$$\begin{aligned} &t_2 = 1, \quad u_1 = u_2 x \\ &x^2(u_1^2 - x) = x^2(u_2^2 x^2 - x) = x^3(u_2^2 x - 1) \end{aligned}$$

Again, the exceptional divisor will not intersect the proper transform, so we can ignore this patch. On the other patch, we have

$$\begin{aligned} &u_2 = 1, \quad x = t_2 u_1 \\ &x^2(u_1^2 - x) = t_2^2 u_1^2(u_1^2 - t_2 u_1) = t_2^2 u_1^3(u_1 - t_2) \end{aligned}$$

and so we have the proper transform and exceptional curves

$$\begin{aligned} &C_2 : u_1 - t_2 \\ &E_1 : t_2^2 \\ &E_2 : u_1^3 \end{aligned}$$

Notice that we don't have any tangency. However, both E_1 and E_2 intersect C_2 at the same point, so this is not yet a normal resolution. We will have to blow-up again.

Now let's blow-up by the relation $t_3 u_1 = u_3 t_2$. On the first patch, this will give us

$$\begin{aligned} &t_3 = 1, \quad u_1 = u_3 t_2 \\ &t_2^2 u_1^3(u_1 - t_2) = t_2^2 u_3^3 t_2^3(u_3 t_2 - t_2) = u_3^3 t_2^6(u_3 - 1) \end{aligned}$$

Now we have a normal resolution of the singularity. Notice, however, that on this patch, the exceptional divisor E_1 disappears (it does not lie on this patch at all). The multiplicity we wanted was completely determined by the first blowup, but if we still want to see this exceptional divisor, we can calculate the blow-up on the other patch.

$$\begin{aligned} &u_3 = 1, \quad t_2 = t_3 u \\ &t_2^2 u_1^3(u_1 - t_2) = t_3^2 u_1^2 u_1^3(u_1 - t_3 u_1) = t_3^2 u_1^6(1 - t_3) \end{aligned}$$

Now (up to some changes in the variables) we have proper transform and exceptional divisors

$$\begin{aligned}
&C_3 : u_3 - 1 \\
&E_1 : t_3^2 \\
&E_2 : u_3^3 \\
&E_3 : t_2^6
\end{aligned}$$

Therefore, we can write the total transform of C as

$$\pi^\star(C) = C_3 + 2E_1 + 3E_2 + 6E_3.$$

Now let's calculate the canonical divisor. Using the lemma above, this is simply a matter of considering each blow-up separately, since we gain an exceptional curve from each. To be clear, the resolution computed above consists of a sequence of three blowups:

$$Y_3 \xrightarrow{\pi_3} Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y$$

From the first blowup, we simply apply the lemma, to get

$$K_{Y_1} = \pi_1^*(K_X) + E_1$$

On the second blowup, we again apply the lemma. This time, we need to interpret the pullback of E_1. Since π_2 maps both E_1 and E_2 to E_1, we have

$$\pi_2^*(E_1) = E_1 + E_2.$$

Therefore,

$$\begin{aligned}
K_{Y_2} &= \pi_2^*(K_{Y_1}) + E_2 \\
&= \pi_2^*(\pi_1^*(K_X) + E_1) + E_2 \\
&= \pi_2^*(\pi_1^*(K_X)) + (E_1 + E_2) + E_2 \\
&= \pi_2^*(\pi_1^*(K_X)) + E_1 + 2E_2
\end{aligned}$$

Finally, on the third blowup, we apply the lemma again. Here, we need to pullback both E_1 and E_2 by π_3^* (you may want to draw the resolution diagram to see what happens).

$$\begin{aligned}
K_{Y_3} &= \pi_3^*(K_{Y_2}) + E_3 \\
&= \pi_3^*(\pi_2^*(\pi_1^*(K_X)) + E_1 + 2E_2) + E_3 \\
&= \pi_3^*(\pi_2^*(\pi_1^*(K_X))) + (E_1 + E_3) + 2(E_2 + E_3) + E_3 \\
&= \pi_3^*(\pi_2^*(\pi_1^*(K_X))) + E_1 + 2E_2 + 4E_3
\end{aligned}$$

So we have canonical divisor $K_\pi = \pi^*(K_X) + E_1 + 2E_2 + 4E_3$.
Plugging these into our formula for the log canonical threshold, we get that

$$\begin{aligned} c_0(C) &= \min\left\{\frac{a_i+1}{b_i}\right\} \\ &= \min\left\{\frac{2}{2}, \frac{3}{3}, \frac{5}{6}\right\} \\ &= \frac{5}{6} \end{aligned}$$

Exercises

1. Calculate the total transform and canonical divisor for the normal resolution of the curve $y^6 + x^6 - xy$ at the origin, and write them in divisor notation. Then calculate the log canonical threshold of the curve at the origin. It may be helpful to draw the resolution diagram at each step of the blowup.

2. Repeat the previous problem for the curve $y^2 - x^3 + x^6 + y^6$.