I have adhered to the Rice honor code in completing this test.

Signature: 
Name: 
Date: 
Time:

Please read the following information:

• This exam is closed book, closed notes. Calculators are not allowed.
• There is no time limit, but the exam must be completed in one sitting.
• The exam is worth 100 points total.
• Show all your work for full credit.
• Clearly indicate your final answers (box or circle them, or write “Answer:”).
• Justify answers using complete sentences.
• No credit will be given for correct but unsupported answers.
• If you have a question, I will be available at (860) 748-3771.
• Have fun!
1. (Worth 15 points) Compute. Answers should be in polar form:

(a) \((x + iy)^2\).

(b) \(2e^{i\pi} + 3e^{i\pi/2}\).

Answer to part 1a:
Let

\[
\begin{align*}
    r^2 &= x^2 + y^2 \\
    \theta &= \arctan \frac{y}{x}.
\end{align*}
\]

Then \(x + iy = re^{i\theta}\), so \((x + iy)^2 = (re^{i\theta})^2 = r^2 e^{2i\theta}\).

Answer to part 1b:

\[
\begin{align*}
    2e^{i\pi} &= 2(\cos \pi + i \sin \pi) \\
             &= -2. \\
    3e^{i\pi/2} &= 2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) \\
               &= 3i.
\end{align*}
\]

So \(2e^{i\pi} + 3e^{i\pi/2} = -2 + 3i = \sqrt{13}e^{i\arctan(-3/2)}\)
2. *(Worth 15 points)* Deduce a solution to the family of equations

\[ y' + p(x)y^n = 0, \]

where \( n \) is an integer strictly greater than 1, and you may assume \( y \geq 0 \).

The easy way everyone found is that this is a separable equation:

\[
\begin{align*}
\frac{dy}{dx} + p(x)y^n &= 0 \\
\Rightarrow \frac{dy}{y^n} &= -p(x)dx \\
\Rightarrow -\frac{1}{(n-1)y^{n-1}} &= -\int p(x)dx + C \\
\Rightarrow y(x) &= \left( (n-1)\int p(x)dx + C \right)^{1/(1-n)}. 
\end{align*}
\]
3. *(Worth 15 points)* Find the equation defining a family of curves orthogonal to the ellipses

\[ y^2 + nx^2 = 1. \]

Differentiating both sides with respect to \(x\) gives

\[ 2yy' + 2nx = 0. \]

Substituting in

\[ n = \frac{1 - y^2}{x^2} \]

gives

\[ y' = -\frac{1 - y^2}{xy}, \]

so we solve the (separable) ODE

\[ y' = \frac{xy}{1 - y^2} \]

to find

\[ \log |y| - \frac{y^2}{2} = \frac{x^2}{2} + C, \]

or

\[ y^2 = Ae^{x^2+y^2}. \]
4. (Worth 10 points each) Solve the following

(a) \( y' \sin y = x^2 \).
(b) \( 2y'' + 2y' + 3y = 0 \).
(c) \( (x^2 + 2y')y'' + 2xy' = 0 \), subject to the initial conditions \( y'(0) = 0 \), and \( y(0) = 1 \).
(d) \( y'x = 1 \).

Part (a): This is separable, so
\[
\begin{align*}
\frac{dy}{dx} \sin y &= x^2 \\
\Rightarrow \sin y \, dy &= x^2 \, dx \\
\Rightarrow -\cos y &= \frac{x^3}{3} + C \\
\Rightarrow y(x) &= \arccos(-\frac{x^3}{3} + C).
\end{align*}
\]
The solution is
\[
y(x) = \arccos(-\frac{x^3}{3} + C).
\]
Part (b): We find the roots of the associated polynomial, \( 2r^2 + 2r + 3 = 0 \) to be
\[r = -\frac{1}{2} \pm \frac{i\sqrt{5}}{2}.
\]
Thus the solution is
\[
y(x) = e^{-x/2} \left( A \cos \left( \frac{\sqrt{5}}{2} x \right) + B \sin \left( \frac{\sqrt{5}}{2} x \right) \right).
\]
Part (c):
There is no \( y \) in this equation, so we may reduce order by substituting \( y' = p \), and \( y'' = p' \). Then we need to solve
\[
\begin{align*}
(x^2 + 2p)p' + 2xp &= 0 \\
\Rightarrow (x^2 + 2p) \frac{dp}{dx} + 2xp &= 0,
\end{align*}
\]
which is an exact equation, since \( M_x - N_p = 2x - 2x = 0 \). Now
\[
\int x^2 + 2p \, dp = px^2 + p^2 + C(x).
\]
Also,
\[
\frac{\partial}{\partial x} (px^2 + p^2 + C(x)) = 2xp + C'(x),
\]
combined with \( N(x, p) = 2xp \), we conclude that \( C'(x) = 0 \). Hence our solution is

5
\[ x^2 y' + (y')^2 = C. \]

From the initial condition \( y'(0) = 0 \), we find that \( C = 0 \). Then dividing out by \( y' \), we have the easy equation

\[ x^2 + y' = 0 \Rightarrow y = -\frac{1}{3}x^3 + C. \]

The last initial condition gives us \( C = 1 \), so \( y(x) = -\frac{1}{3}x^3 + 1 \).

Part (d):
This is a separable equation, giving \( dy = dx/x \), so \( y = \log|x| + C \).
5. (Worth 15 points) A tank contains 40 gallons of pure water. Saltwater with 3 lbs of salt per gallon flows into the tank at the rate of 2 gal/min. Simultaneously, water is being let out of the tank at the rate of 3 gal/min. Assume everything mixes perfectly. How much salt is in the tank after time $t$? (where $0 \leq t \leq 40$)

We solve by calculating that

\[ \text{(rate in)} = \frac{3 \text{lbs}}{\text{gal}} \cdot \frac{2 \text{gal}}{\text{min}} = \frac{6 \text{ lbs}}{\text{min}} \]

\[ \text{(rate out)} = \frac{3 \text{gal}}{\text{min}} \cdot \frac{x(t) \text{lbs}}{(40 - t) \text{gal}} = \frac{3x}{40 - t} \text{ lbs} \min. \]

Hence we need to solve

\[ x' = 6 - \frac{3x}{40 - t}. \]

This is a linear equation, with $a(x) = 3/(40 - t)$, so

\[ \int a(x) \, dx = -3 \log |40 - t|, \]

and $e^{\int a(x)} = (40 - t)^{-3}$, where we drop the absolute value signs since $0 \leq t \leq 40$. Now we compute

\[ \frac{x}{(40 - t)^3} = \int 6(40 - t)^{-3} \, dt \]

\[ = \frac{3}{(40 - t)^2} + C, \]

\[ \Rightarrow x(t) = 3(40 - t) + C(40 - t)^3. \]

Since $x(0) = 0$, we have that

\[ C = -\frac{120}{40^3} = -\frac{3 \cdot 40}{40^3} = -\frac{3}{40^2}. \]

We conclude that

\[ x(t) = 3(40 - t) - \frac{3}{40^2} (40 - t)^3 \]