Filtering smooth concordance classes of topologically slice knots

* AMS Las Vegas *

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We are interested in studying knots modulo smooth concordance.

Definition: The knot concordance group is

\[ C = \{ \text{knots in } S^3 \}/(\text{smooth concordance}) \]
In 1997, Cochran-Orr-Teichner defined the \((n)\)-solvable filtration of \(\mathcal{C} \ (n \in \mathbb{N}/2)\)

\[0 = \{\text{slice knots} \} \subseteq \ldots \subseteq \mathcal{F}_n \subseteq \ldots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}\]

- \(\mathcal{F}_0 = \text{Arf invariant zero knots}\)
- \(\mathcal{F}_{0.5} = \text{Algebraically slice knots}\)
- \(\mathcal{F}_{1.5} \subseteq \text{knots with vanishing Casson-Gordon invariants}\)
Def A knot is **(n)-solvable**

\( M_K \) (0-surgery on \( K \)) bounds a

(1) \( i_* : H_1(M_K) \overset{\cong}{\longrightarrow} H_1(W) \)

(2) \( H_2(W) \) has a basis \( \{ f_i, g_i \} \) of embedded surfaces (w/ triv. normal bundle) all disjoint except \( f_i \cdot g_i = 1 \) (geometrically)

(3) \( \pi_1(f_i), \pi_1(g_i) \subset \pi_1(W)^{(n)} \)

* If \( \pi_1(f_i) \subset \pi_1(W)^{(n+1)} \) as well then \( K \) is **(n.5)** solvable.
Def. A knot $K \subset S^3$ is **n-solvable** if there exists a smooth 4-manifold $W$ with $\partial W = S^3$ and a knot $K = K \setminus \Delta$, where $\Delta$ is a smoothly embedded $D^2$ in $W$ such that:

1. $H_1(W) = 0$
2. There exist smoothly embedded surfaces $l_i, d_i$ in $W \setminus \Delta$ with trivial normal bundles that are disjoint except $d_i \cap l_i = \text{pt}$ and

   \[ \{ l_i, d_i \}_{i=1}^{g} \] is a basis for $H_2(W)$

3. $\pi_1(l_i), \pi_1(d_i) \subset \pi_1^{(n)}(W \setminus \Delta)$ (derived series of $\pi_1(W \setminus \Delta)$)
However, this filtration **fails** to distinguish a large class of knots called the topologically slice knots.

**Def** A knot $K$ is topologically slice if $K = 2D$ where $D$ is a locally flat topological disk embedded in $B^4$. 
**Theorem (M. Freedman):** If $\Delta_K(t) = 1$ (Alexander polynomial of $K$) then $K$ is topologically slice.

**Ex:** Whitehead double of $J$

$$\text{Wh}(J) = \text{tie band into knot } J$$
\( \Delta_{\text{Wh}(J)} = 1 \) so \( \text{Wh}(J) \) is topologically slice.

- \( \text{RHT} \) = right handed trefoil = \( \bigcirc \)
  Known \( \text{Wh}(\text{RHT}) \) is not (smoothly) slice.

\[ \Rightarrow \text{Wh}(\text{RHT}) \text{ is top but not smoothly slice.} \]
Conjecture: \( \text{Wh}(J) \) is smoothly slice \( \iff J \) is smoothly slice
Denote by \( \mathcal{T} \), the (smooth) concordance classes of topologically slice knots.

\[ \Rightarrow \{0\} \not= \mathcal{T} \leq C \]

Endo showed that \( \mathbb{R}^\infty \subset \mathcal{T} \) and Hedden-Livingston-Ruberman showed that \( \mathcal{T}/\Delta \cong \mathbb{R}^\infty \) where \( \Delta = \text{knots smoothly concordant to knots w/ Alex. poly 1} \).
However, \[ \gamma \subseteq \bigcap_{n=0}^{\infty} \mathcal{F}_n \]

We refine the \( n \)-solvable filtration to get a non-trivial filtration on \( \gamma \).
subgroup of topologically slice knots

$O c \cdots c T_n \cdots c T_1 \cdots c T_0 \subset G \subset \prod_{n=0}^{\infty} \mathcal{F}_n \subset \mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{C}$

new filtration

smoothly slice knots

Smooth concordance group

COT filtration
A knot $K$ is $n$-positive if there exists a smooth 4-manifold $W$ with $\partial W = S^3$ and $K = \partial D$, with $D \subset W$ a smoothly embedded disk s.t.

- $H_1(W) = 0$

- There exist disjointly embedded surfaces $S_1, ..., S_j$ with $S_i \cdot S_i = +1$ and \{ $S_i$ \} is a basis for $H_2(W)$.

- $S_i \cap D = \emptyset \ \forall \ i$

- $\pi_1(S_i) \subset \pi_1(W - D)^{(n)}$ (derived series) $\forall i$
Similar for n-negative but $S_i \cdot S_i = -1$. (Euler class of normal bundle is -1).

Note: If $K$ is smoothly slice $\Rightarrow W = B^4$ and $H_2(W) = 0$ so $K$ is $n$-positive and $n$-negative for all $n$.

\[ P_n = \{ n\text{-positive knots} \} \subset \mathbb{C} \]
\[ N_n = \{ n\text{-negative knots} \} \subset \mathbb{C} \]
\[ NP_n = N_n \cap P_n \text{ is a filtration by subgroups of } \mathbb{C}. \]

\[ \cdots \subset NP_2 \subset NP_1 \subset NP_0 \subset \mathbb{C} \]

Thm(CHT): \[ NP_n \subset \mathfrak{F}_{\text{no spin}}^{\text{no spin}} \quad \forall \quad n \quad \text{ and} \]

\[ \Theta(\mathbb{Z}^\infty \oplus \mathbb{Z}/2) \subset NP_n / NP_{n+1} \quad \forall \quad n. \]
Let \( T_n = T_n \setminus N P_n \) then

\[ \ldots < T_1 < T_0 < T < C \]

We are interested in \( T_n/T_{n+1} \).
Prop(CT/H): If $K$ can be changed to a slice knot by changing positive crossings to negative crossings then $K \in \mathcal{P}_0$.

Ex: $\text{RHT} = \overset{\circ}{\circ} \Rightarrow \overset{\circ}{\circ} = \mathcal{O}$

$\Rightarrow \text{RHT} \in \mathcal{P}_0$. 
Ex: Twist knots

$T_{w_n} =$ <Diagram of twist knot>

Can change a positive or a negative crossing to get to the unknot

$\Rightarrow T_{w_n} \in NP_0$

Can show certain twist knots $\notin NP_1$. 
Properties:

(1) If $K \in \mathcal{P}_o \Rightarrow \sigma(K) \leq 0$ ($K \in \mathcal{N}_o \Rightarrow \sigma(K) \geq 0$)

$\Rightarrow$ If $K \in \mathcal{NP}_o \Rightarrow \sigma(K) = 0$

$\Rightarrow$ If $K \in \mathcal{NP}_o \Rightarrow \tau(K) = 0$

(2) If $K \in \mathcal{P}_o \Rightarrow \tau(K) \leq 0$ ($K \in \mathcal{N}_o \Rightarrow \tau(K) \leq 0$)

If $K \in \mathcal{NP}_o \Rightarrow \tau(K) = 0$
Theorem (Cochran-H-Horn) Suppose $K \in P_k$ and $Y$ is the $p^r$-fold cyclic branched cover of $S^3$ branched over $K$. There is a subgroup $G \subset H^2(Y)$ with $|G|^2 = |H^2(Y)|$ and a spin$^c$-structure $\xi$ on $Y$ s.t. the Ozsváth-Szabó correction terms

$$d(Y, 5+g) \leq 0$$

$\forall g \in G$. 
Note: * $S = \text{spin}^c$ structure on $Y$ that comes from a spin structure on $Y$ and $G \leftrightarrow \text{Poincaré dual of classes in } \ker (H_2(Y) \xrightarrow{i_*} H_2(W))$.

* If $K \in N_1$, get $d(Y, 5+h) > 0$ for $h \in H$, some a priori different subgroup of $H^2(Y)$. 
Using Casson-Gordon invariants and Ozsváth-Szabó d-inuts we show:

**Theorem (Cochran-H-Horn):**

\[ \mathcal{T}_1 / \mathcal{T}_2 \subset \mathbb{Z} \]

**Ex:**

\[ T_5 \quad \text{and} \quad \text{Wh}(RHT) \]