

Homology and Derived
Series of Groups
&
Rank Invariants

Shelly Harvey

MIT & Rice University

In the previous talk, we described how one could get topological invariants

$$\delta_n : H^1(X) \longrightarrow \mathbb{Z}$$

of 3-manifolds (and knots and links).

These invariants were associated to the derived series of $\pi_1(X)$.

However, the δ_n are not invariants of homology cobordism (or knot concordance).

Q. Can we obtain homology cobordism invariants associated to the derived series?

Let M be a closed, oriented 3-manifold.

Def M_1 is homology cobordant to M_2 , $M_1 \sim_{\#} M_2$, if there exists a (smooth) 4-manifold W such that $\partial W = M_1 \cup -M_2$ and $i_j: M_j \rightarrow W$ induces isomorphisms on $H_x(-; \mathbb{Z})$.

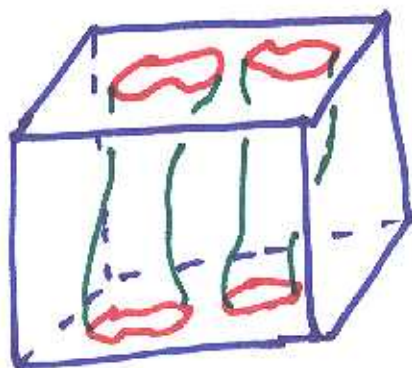


Example: Let L_1, L_2 be links in S^3 .

If L_1 is concordant to L_2 (i.e. the cobound (smoothly) embedded annuli in $S^3 \times I$) then

M_{L_1} is hom. cobordant to M_{L_2}

(0-surgery on L_1) =



We begin to search for our homology cobordism invariants by investigating a classical invariant, Alexander nullity.

Recall if \tilde{X} is the (torsion-free) abelian cover of a 3-mfld X then $H_1(\tilde{X})$ is a module over $\mathbb{Z}[\mathbb{Z}^m] = \mathbb{Z}[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$.

Let $Q(\mathbb{Z}^m)$ be the quotient field of $\mathbb{Z}[\mathbb{Z}^m]$ then

$$H_1(\tilde{X}) \otimes_{\mathbb{Z}[\mathbb{Z}^m]} Q(\mathbb{Z}^m)$$

is a free $Q(\mathbb{Z}^m)$ -module of rank r .

Def: The Alexander nullity of X is

$$r_0(X) = \text{rank}_{Q(\mathbb{Z}^m)} H_1(\tilde{X}) \otimes_{\mathbb{Z}[\mathbb{Z}^m]} Q(\mathbb{Z}^m)$$

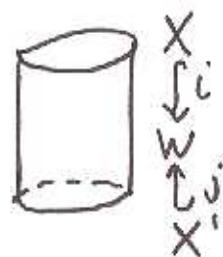
Thm: Let X and X' be homology cobordant 3-manifolds (orientable) then

$$r_0(X) = r_0(X').$$

Proof: Let W be such that

(i) $2W = X \cup X'$,

(ii) i_*, j_* are \cong on $H_*(-; \mathbb{Z})$.



Let $\tilde{W} \xrightarrow{p} W$ be the \mathbb{Z}^m -cover of W

where $\mathbb{Z}^m = H_1(W)/(\mathbb{Z}\text{-torsion})$. Since

$$i_* : H_1(X) \xrightarrow{\cong} H_1(W), \quad p^{-1}(X) \xrightarrow{p} X$$

is the torsion-free abelian cover of X . ($p^{-1}(X) = \tilde{X}$).

(everything same for X')

Since $H_i(X) \xrightarrow{i_*} H_i(W)$ for $i = 0, 1, 2$

$$\Rightarrow H_1(W, X) = H_2(W, X) = 0.$$

Consider long exact sequence of (\tilde{W}, \tilde{X}) and tensor with $Q(\mathbb{Z}^m)$

$$H_2(\tilde{W}, \tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m) \longrightarrow H_1(\tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m) \longrightarrow$$

$$H_1(\tilde{W}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m) \longrightarrow H_1(\tilde{W}, \tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m)$$

• If $H_1(\tilde{W}, \tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m) = H_2(\tilde{W}, \tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m) = 0$

then

$$H_1(\tilde{W}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m) \cong H_1(\tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m).$$

Same is true for X'

$$\Rightarrow \text{rank}_{\mathbb{Q}} H_1(\tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m) =$$

$$\text{rank}_{\mathbb{Q}} H_1(\tilde{X}') \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m)$$

$$\Rightarrow r_0(X) = r_0(X').$$

Goal: Show $H_i(W, X) = 0 \Rightarrow H_i(\tilde{W}, \tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m) = 0$

1) Since $\mathbb{Z}[\mathbb{Z}^m] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$ is Noetherian and W, X are finite CW-complexes,

$H_i(\tilde{W}, \tilde{X})$ is a finitely presented $\mathbb{Z}[\mathbb{Z}^m]$ -mod.

Let M_i be the presentation matrix.
(assume square).

2) M_i has entries in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$. Augment M_i by sending $x_j \mapsto 1$ to get $\mathcal{E}(M_i) = M_i \otimes \mathbb{Z}$ which presents $H_i(W, X)$.

3) Since $H_i(W, X) = 0$, $\det(\mathcal{E}(M_i)) = \pm 1$

$$\Rightarrow (\det M_i)(1, \dots, 1) = \pm 1$$

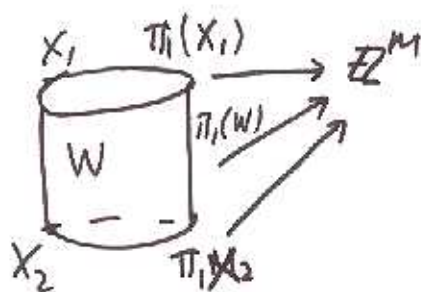
↑ Laurent polynomial in x_1, \dots, x_m

4) Since $(\det M_i)(1, \dots, 1) \neq 0$, $\det M_i \neq 0$

$$\Rightarrow H_i(\tilde{W}, \tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m) = 0 \quad (\text{rank is } 0).$$

Note: The two key points to this proof are

(1) the maps $\pi_i(X_i) \longrightarrow \mathbb{Z}^m$ extend over the 4-manifold W



(2) $\mathbb{Z}[\mathbb{Z}^m]$ ~~embeds~~ embeds in a quotient field so that we can take a rank.

\therefore To define homology cobordism invariants we look for group Γ_i w/ $\mathbb{Z}\Gamma_i \hookrightarrow$ skew field and s.t. $\pi_i(X_i) \longrightarrow \Gamma_i$ extend over W .

Homology Equivalence and Fundamental Gp.

A homology cobordism gives maps

$$z_j: M_j \rightarrow W$$

which induce \cong on H_* (ie. z_j is homology equivalence)

Q. If $f: X \rightarrow Y$ is a homology equivalence what is preserved under

$$f_*: \pi_1(X) \rightarrow \pi_1(Y)?$$

Example 1:

$$f_*: \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]} \xrightarrow{\cong} \frac{\pi_1(Y)}{[\pi_1(Y), \pi_1(Y)]}$$

$H_1(X)$ $H_1(Y)$

Example 2: Let $G = \pi_1(X)$, $E = \pi_1(Y)$ then Stallings shows that for each $n \geq 0$,

$$f_*: G/G_n \xrightarrow{\cong} E/E_n$$

$\{G_n\}$ = lower central series of G :

$$G_1 = G, \quad G_n = [G_{n-1}, G].$$

Theorem (Stallings): Let $\phi: G \rightarrow E$ be a homomorphism of groups that induces \cong on H_1 and an epimorphism on H_2 . Then for any finite n , ϕ induces

$$\text{iso } \phi_*: \frac{G}{G_n} \xrightarrow{\cong} \frac{E}{E_n}.$$


Recall by $H_x(G)$ we mean $H_x(K(G, 1))$

Q. What about the derived series?

Recall, derived series:

$$G^{(0)} = G$$
$$G^{(n+1)} = [G^{(n)}, G^{(n)}]$$

A. $G/G^{(n)}$ is not necessarily preserved under homology equivalence!

Example: Let K be a knot in S^3 ,
with $\Delta_K \neq 1$, $G = \pi_1(S^3 - K)$ 

$\phi: G \longrightarrow \mathbb{Z}$ abelianization

(1) $\phi_* \cong$ on homology

(2) [Cochran] $G/G^{(n)}$ is "large" ($G^{(n)}/G^{(n+1)} \neq 1$)

(3) $\mathbb{Z}/\mathbb{Z}^{(n)} = \mathbb{Z} \quad \forall n \geq 1$.

$\therefore \phi_*: \frac{G}{G^{(n)}} \longrightarrow \mathbb{Z}/\mathbb{Z}^{(n)} = \mathbb{Z}$ not mono!

Also,

meridian: $\mathbb{Z} \longrightarrow G$

\Rightarrow (meridian) $_*$: $\mathbb{Z} \longleftarrow G/G^{(n)}$ not
surjective!

However, we have the following:

Theorem (Cochran-H)

If $\phi: F \rightarrow B$ induces a monomorphism on $H_1(-; \mathbb{Q})$ and an epimorphism on $H_2(-; \mathbb{Q})$ (2-connected or rationally 2-connected), F is a free group, B is finitely related then for all $n \geq 1$,

$$\phi_*: F/F^{(n)} \hookrightarrow B/B^{(n)}$$

is a monomorphism.

Note: F does not have to be finitely generated

One application of this theorem is that the higher-order ranks of a slice link are "maximal".

Let $G = \pi_1(X^3)$, $\Gamma_n = G/G_r^{(n+1)}$ where $G_r^{(n+1)}$ is the $(n+1)^{\text{st}}$ term of the (rational) derived series and $\mathcal{K}_n = \text{quotient field of } \mathbb{Z}\Gamma_n$.

Recall, an element of \mathcal{K}_n is ab^{-1} where $a, b \in \mathbb{Z}\Gamma_n$, $b \neq 0$.

Def: We define the n^{th} -order rank $r_n \in \mathbb{Z}$ to be

$$r_n(x) = \text{rank}_{\mathcal{K}_n} H_1(x; \mathcal{K}_n)$$

Note: $r_n(x) = \text{rank}_{\mathcal{K}_n} H_1(X_n) \otimes_{\mathbb{Z}\Gamma_n} \mathcal{K}_n$

||
covering space of X
with $\pi_1(X_n) = G_r^{(n+1)}$

Properties of r_n

1) Let $\Psi \in H^1(X)$ and $|K_n^\Psi[t^{\pm 1}]|$ as before then

$$H_1(X; |K_n^\Psi[t^{\pm 1}]|) \cong (|K_n^\Psi[t^{\pm 1}]|)^{r_n(X)} \oplus \frac{|K_n^\Psi[t^{\pm 1}]|}{\langle p_1(t) \rangle} \oplus \dots \oplus \frac{|K_n^\Psi[t^{\pm 1}]|}{\langle p_m(t) \rangle}$$

(dim of free summand is independent of Ψ)

2) If K is a knot then $r_n(S^3 - K) = 0$.

More generally, $r_n(X) \leq \beta_1(X) - 1$ (Cochran-Orr-Teichner)

3) If $X_\bullet = X_1 \# X_2$ with $\beta_1(X_i) \geq 1$ ($i=1,2$)

then $r_n(X) = r_n(X_1) + r_n(X_2) + 1$

(i.e. r_{n-1} is additive under connected sum)

4) The r_n are non-decreasing ^{increasing} functions of n :

Theorem (H): For all $n \geq 0$,

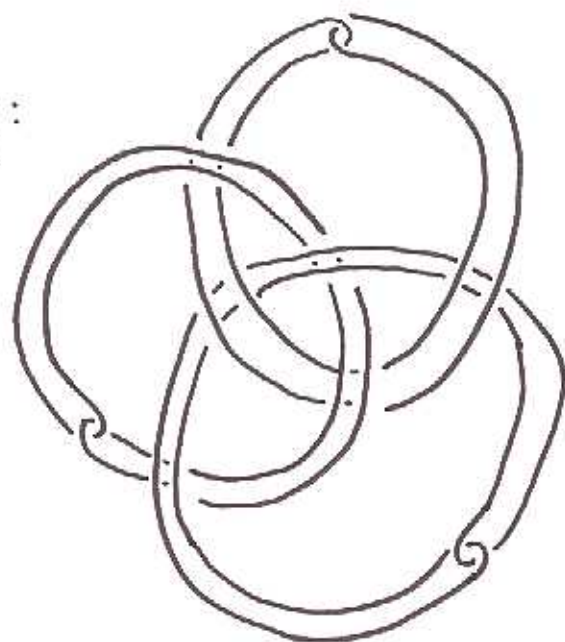
$$\boxed{r_{n+1}(X) \leq r_n(X)}$$

5) A good boundary link L has

$$r_n(S^3 - L) = m - 1$$

where m is the number of components of L .

Example:



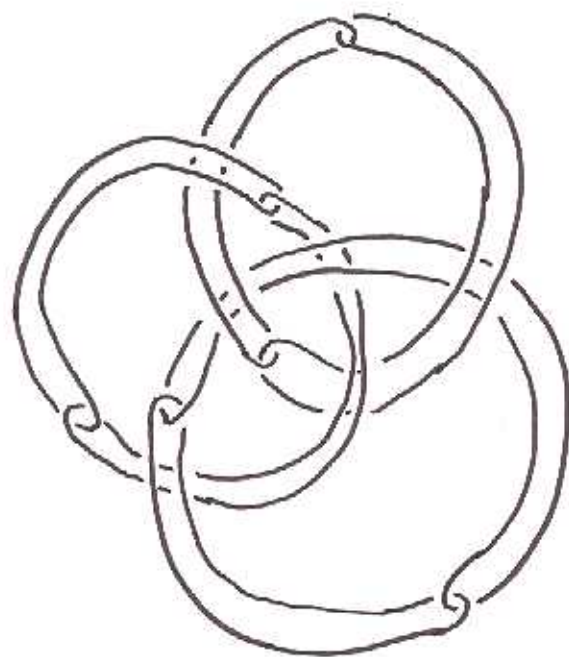
$L =$ Whitehead
double of
Borromean
rings

$$r_n(S^3 - L) = 2$$

Good boundary link $L \Leftrightarrow$

$$H_1(\text{Free cover of } S^3 - L) = 0$$

Example of link without maximal ranks.



$L =$ Bing double
of Borromean
rings

$$r_0(S^3 - L) = 5 \\ = r_0(S^3 - \text{trivial link})$$

$$r_1(S^3 - L) < 5 = r_1(S^3 - \text{trivial link})$$

Theorem (H) If a link L in S^3 is slice
(bounds embedded disjoint disks in B^4)
then for all $n \geq 0$,

$$r_n(S^3 - L) = \beta_1(S^3 - L) - 1 = r_n(S^3 - \text{trivial link})$$

\uparrow with $\beta_1(S^3 - L)$ comps.

\Rightarrow Bing double of Borromean rings is not slice.

Recall

Theorem (Cochran-H): If $\phi: F \rightarrow B$ induces a mono. on $H_1(-; \mathbb{Q})$ and an epi. on $H_2(-; \mathbb{Q})$ with F a free group, B finitely related then $\forall n \geq 1$

$$\phi_*: F/F^{(n)} \hookrightarrow B/B^{(n)}$$

is a monomorphism.

In fact, if we slightly "vary the derived series" we can prove a more general version of this theorem.

To do this, we will need to define the torsion-~~free~~ free derived series of a group: $G_H^{(n)}$

Theorem (Cochran-H):

If $\phi: A \rightarrow B$ is a monomorphism on $H_1(-; \mathbb{Q})$ and an epimorphism on $H_2(-; \mathbb{Q})$ with A finitely generated and B finitely related then for each $n \geq 1$,

$$\phi_*: \frac{A}{A_H^{(n)}} \hookrightarrow \frac{B}{B_H^{(n)}}$$

is a monomorphism.

If ϕ is onto as well then

ϕ_* is an isomorphism.

Torsion-Free Derived Series: $G_H^{(n)}$

(1) $G_H^{(0)} := G$

- (2) Assume (i) $G_H^{(n)}$ has been defined,
 (ii) $G_H^{(n)} \triangleleft G$ (iii) $\mathbb{Z}[G/G_H^{(n)}]$ is an Ore domain

\Rightarrow By (2) $\mathbb{Z}[G/G_H^{(n)}] \longleftrightarrow \mathcal{K}_n =$ classical right ring of quotients

Consider:

$$G_H^{(n)} \xrightarrow{\alpha_n} \underbrace{\frac{G_H^{(n)}}{[G_H^{(n)}, G_H^{(n)}]}}_{\text{right } \mathbb{Z}[G/G_H^{(n)}] \text{ module by } \sigma g = g' \sigma g \text{ for } g \in G} \xrightarrow{B_n} \frac{G_H^{(n)}}{[G_H^{(n)}, G_H^{(n)}]} \otimes_{\mathbb{Z}[G/G_H^{(n)}]} \mathcal{K}_n$$

right $\mathbb{Z}[G/G_H^{(n)}]$ module by
 $\sigma g = g' \sigma g$ for $g \in G$.

(3) $G_H^{(n+1)} := \ker(B_n \circ \alpha_n) = \alpha_n^{-1}(\text{Torsion submodule of } G_H^{(n)}/[G_H^{(n)}, G_H^{(n)}])$

\Rightarrow Immediate that $G_H^{(n+1)} \triangleleft G_H^{(n)}$

Can show $G_H^{(n+1)} \triangleleft G$ and

$\mathbb{Z}[G/G_H^{(n+1)}]$ ORE ring.

$\star G_H^{(n)}/G_H^{(n+1)} = H_1(G_H^{(n)}; \mathbb{Z}) / \mathbb{Z}G_H^{(n)}\text{-torsion.}$

Torsion-Free Derived Series via Group Theory

Let G be a group.

Define $G_H^{(0)} := G$

$G_H^{(n+1)}$ = set of $g \in G_H^{(n)}$ such that
 $\exists (0 \neq) \sum_{i=0}^{\ell} k_i \gamma_i \in \mathbb{Z}[G/G_H^{(n)}]$

and

$$\prod_{i=0}^{\ell} \gamma_i^{-1} g^{k_i} \gamma_i \in [G_H^{(n)}, G_H^{(n)}].$$

Note: $G^{(n)} \subset G_H^{(n)}$ for all $n \geq 0$

* $\{G_H^{(n)}\}$ is a characteristic but not totally invariant series of G !

However, we have the following:

Proposition (Cochran-H): If $\phi: A \rightarrow B$ induces a monomorphism on

$$\frac{A}{A_H^{(n)}} \hookrightarrow \frac{B}{B_H^{(n)}} \quad \text{then}$$

$$\phi(A_H^{(n+1)}) \subset B_H^{(n+1)}.$$

In particular, we have homomorphism

$$\phi_*: \frac{A}{A_H^{(n+1)}} \longrightarrow \frac{B}{B_H^{(n+1)}}.$$

Examples:

(1) F free group

Since $F^{(n)}/F^{(n+1)}$ is torsion-free
as $\mathbb{Z}[F/F^{(n)}]$ -module,

$$\boxed{F_H^{(n)} = F^{(n)}} \quad \forall n \geq 0.$$

(2) K knot in S^3 , $G = \pi_1(S^3 - K)$

Since $G^{(1)}/G^{(2)}$ = Alex. module is
a $\mathbb{Z}[G/G^{(1)}]$ -torsion module,

$G_H^{(n)} = [G, G] \quad \forall n \geq 1$. Hence

$$\boxed{G/G_H^{(n)} \cong \mathbb{Z}} \quad \forall n \geq 1.$$

Theorem (Cochran-H):

If $\phi: A \rightarrow B$ is a monomorphism on $H_1(-; \mathbb{Q})$ and an epimorphism on $H_2(-; \mathbb{Q})$ with A finitely generated and B finitely related then for each $n \geq 1$,

$$\phi_*: \frac{A}{A_H^{(n)}} \hookrightarrow \frac{B}{B_H^{(n)}}$$

is a monomorphism. If ϕ is onto as well, then ϕ_* is an isomorphism.

Corollary: If L is a boundary link then $G \twoheadrightarrow F$ induces

$$G/G_H^{(n)} \xrightarrow{\cong} F/F^{(n)}$$

for all $n \geq 0$.

Defn Let $\mathcal{K}(G/G_H^{(n+1)})$ be the quotient field of $\mathbb{Z}G/G_H^{(n+1)}$. Define

$$R_n(X) = \text{rank}_{\mathcal{K}(G/G_H^{(n+1)})} H_1(X; \mathcal{K}(G/G_H^{(n+1)}))$$

where $G = \pi_1(X)$.

Corollary: R_n is a concordance invariant of links (and an invariant of homology cobordism).

Note: $R_n(X) \neq r_n(X)$ (in general).

Sketch of proof of Theorem:

Theorem:

If $\phi: A \rightarrow B$ is a monomorphism on $H_1(-; \mathbb{Q})$ and an epimorphism on $H_2(-; \mathbb{Q})$, A finitely generated, B finitely related then $\forall n \geq 1$,

$$\phi_*: \frac{A/A_H^{(n)}}{A_H^{(n+1)}} \hookrightarrow \frac{B/B_H^{(n)}}{B_H^{(n+1)}}$$

is a monomorphism

By induction and the following diagram, it suffices to show $\frac{A_H^{(n)}}{A_H^{(n+1)}} \xrightarrow{\phi_*} \frac{B_H^{(n)}}{B_H^{(n+1)}}$

$$\begin{array}{ccccccc} | \rightarrow & A_H^{(n)} / A_H^{(n+1)} & \longrightarrow & A / A_H^{(n+1)} & \longrightarrow & A / A_H^{(n)} & \rightarrow | \\ & \downarrow \phi_* & & \downarrow \phi_* & & \downarrow & \\ | \rightarrow & B_H^{(n)} / B_H^{(n+1)} & \longrightarrow & B / B_H^{(n+1)} & \longrightarrow & B / B_H^{(n)} & \rightarrow | \end{array}$$

suppose $A_H^{(n)}/A_H^{(n+1)} \rightarrow B_H^{(n)}/B_H^{(n+1)}$ is not a monomorphism.

Then $\exists [a]$ s.t.

(1) $a \in (A_H^{(n)})^{ab}$ is not $\mathbb{Z}[A/A_H^{(n)}]$ -torsion

(2) $[\phi(a)]$ is $\mathbb{Z}B_H^{(n)}/B_H^{(n)}$ -torsion
 $\pi(B_H^{(n)})$ abelianization.

i.e.

$$\frac{A_H^{(n)}}{[A_H^{(n)}, A_H^{(n)}]} \otimes_{A_H^{(n)}} \mathcal{K}(A/A_H^{(n)}) \xrightarrow{\alpha} \frac{B_H^{(n)}}{[B_H^{(n)}, B_H^{(n)}]} \otimes_{B_H^{(n)}} \mathcal{K}(B/B_H^{(n)})$$

is not injective.

But α is the composition:

$$\frac{A_H^{(n)}}{[A_H^{(n)}, A_H^{(n)}]} \otimes_{A_H^{(n)}} \mathcal{K}(A/A_H^{(n)}) \xrightarrow{\quad} \frac{A_H^{(n)}}{[A_H^{(n)}, A_H^{(n)}]} \otimes_{A_H^{(n)}} \mathcal{K}(B/B_H^{(n)})$$

$\downarrow (*)$
 $\frac{B_H^{(n)}}{[B_H^{(n)}, B_H^{(n)}]} \otimes_{\mathbb{Z}/B_H^{(n)}} \mathcal{K}(B/B_H^{(n)})$

injective since by
induction $A/A_H^{(n)} \hookrightarrow B/B_H^{(n)}$

$\Rightarrow \mathbb{Z}B_H^{(n)}/B_H^{(n)}$ is a free $\mathbb{Z}A/A_H^{(n)}$ -module

(*) is equivalent to

$$\begin{array}{ccc}
 H_1(A; \mathcal{K}(B/B_H^{(n)})) & \longrightarrow & H_1(B; \mathcal{K}(B/B_H^{(n)})) \\
 \parallel & & \parallel \\
 H_1(A_H^{(n)}) \otimes_{A/A_H^{(n)}} \mathcal{K}(B/B_H^{(n)}) & & H_1(B_H^{(n)}) \otimes_{B/B_H^{(n)}} \mathcal{K}(B/B_H^{(n)})
 \end{array}$$

which is a monomorphism by the following.

Proposition (H-Cochran): If $\phi: A \rightarrow B$ induces a mono. on $H_1(-; \mathbb{Q})$ and an epi. on $H_2(-; \mathbb{Q})$ (A finitely generated, B finitely related) then for any coefficient system $B \rightarrow \Gamma$, \mathcal{P} PTFA,

$$H_1(A; \mathcal{K}(\Gamma)) \hookrightarrow H_1(B; \mathcal{K}(\Gamma)).$$

Note: This proposition is a generalization of the statement we proved in the beginning:

$$\text{for } \Gamma = \mathbb{Z}^m, \quad H_1(\tilde{X}) \otimes \mathbb{Q}(\mathbb{Z}^m) \xrightarrow{\cong} H_1(\tilde{W}) \otimes \mathbb{Q}(\mathbb{Z}^m)$$

if $i: X \rightarrow W$ induces an \cong on H_1 , and $H_2, \mathbb{Z}^m = H_1(X) / \text{torsion}$