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Homology and Derived Series of Groups & Rank Invariants

Shelly Harvey

MIT & Rice University

In the previous talk, we described how one could get topological invariants

$$\delta_n : H^1(X) \longrightarrow \mathbb{Z}$$

of 3-manifolds (and knots and links).

These invariants were associated to the derived series of $\pi_1(X)$.

However, the δ_n are not invariants of homology cobordism (or knot concordance).

Q. Can we obtain homology cobordism invariants associated to the derived series?

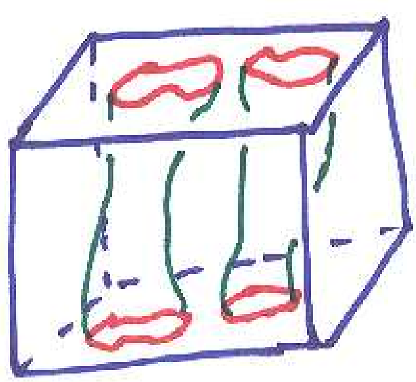
Let M be a closed, oriented 3-manifold.

Def M_1 is <u>homology cobordant</u> to M_2, $M_1 \sim_H M_2$, if there exists a (smooth) 4-manifold W such that $\partial W = M_1 \cup -M_2$ and $i_j : M_j \longrightarrow W$ induces isomorphisms on $H_*(-;\mathbb{Z})$.

<u>Example:</u> Let L_1, L_2 be links in S^3. If L_1 is concordant to L_2 (i.e. they cobound (smoothly) embedded annuli in $S^3 \times I$) then

M_{L_1} is hom. cobordant to M_{L_2}

(0-surgery on L_1) =



We begin to search for our homology cobordism invariants by investigating a classical invariant, Alexander nullity.

Recall if $\tilde{X}$ is the (torsion-free) abelian cover of a 3-mfld X then $H_1(\tilde{X})$ is a module over $\mathbb{Z}[\mathbb{Z}^m] = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$.

Let $Q(\mathbb{Z}^m)$ be the quotient field of $\mathbb{Z}[\mathbb{Z}^m]$ then

$$H_1(\tilde{X}) \otimes_{\mathbb{Z}[\mathbb{Z}^m]} Q(\mathbb{Z}^m)$$

is a free $Q(\mathbb{Z}^m)$-module of rank r.

Def: The Alexander nullity of X is

$$r_0(X) = \mathrm{rank}_{Q(\mathbb{Z}^m)} H_1(\tilde{X}) \otimes_{\mathbb{Z}[\mathbb{Z}^m]} Q(\mathbb{Z}^m)$$

Thm: Let X and X' be homology cobordant 3-manifolds (orientable) then

$$r_0(X) = r_0(X').$$

Proof: Let W be such that

$X \xhookrightarrow{i} W \xhookleftarrow{j} X'$

(i) $\partial W = X \cup X'$,

(ii) i_*, j_* are $\cong$ on $H_*(-;\mathbb{Z})$.

Let $\tilde{W} \xrightarrow{p} W$ be the $\mathbb{Z}^m$-cover of W where $\mathbb{Z}^m = H_1(W)/(\mathbb{Z}\text{-torsion})$. Since $i_* : H_1(X) \xrightarrow{\cong} H_1(W)$, $p^{-1}(X) \xrightarrow{p} X$ is the torsion-free abelian cover of X ($p^{-1}(X) = \tilde{X}$).

(everything same for X')

Since $H_i(X) \xrightarrow[i_*]{\cong} H_i(W)$ for $i = 0,1,2$

$\Longrightarrow$ $H_1(W,X) = H_2(W,X) = 0$.

Consider long exact sequence of $(\tilde{W}, \tilde{X})$ and tensor with $Q(\mathbb{Z}^m)$

$$H_2(\tilde{W}, \tilde{X}) \otimes_{\mathbb{Z}^m} \mathbb{Q}(\mathbb{Z}^m) \longrightarrow H_1(\tilde{X}) \otimes_{\mathbb{Z}^m} \mathbb{Q}(\mathbb{Z}^m) \longrightarrow$$
$$H_1(\tilde{W}) \otimes_{\mathbb{Z}^m} \mathbb{Q}(\mathbb{Z}^m) \longrightarrow H_1(\tilde{W}, \tilde{X}) \otimes_{\mathbb{Z}^m} \mathbb{Q}(\mathbb{Z}^m)$$

• <u>If</u> $H_1(\tilde{W}, \tilde{X}) \otimes_{\mathbb{Z}^m} \mathbb{Q}(\mathbb{Z}^m) = H_2(\tilde{W}, \tilde{X}) \otimes_{\mathbb{Z}^m} \mathbb{Q}(\mathbb{Z}^m) = 0$

then
$$H_1(\tilde{W}) \otimes_{\mathbb{Z}^m} \mathbb{Q}(\mathbb{Z}^m) \cong H_1(\tilde{X}) \otimes_{\mathbb{Z}^m} \mathbb{Q}(\mathbb{Z}^m).$$

Same is true for X'

$$\Rightarrow \operatorname{rank}_{\mathbb{Q}} H_1(\tilde{X}) \otimes_{\mathbb{Z}^m} \mathbb{Q}(\mathbb{Z}^m) = \operatorname{rank}_{\mathbb{Q}} H_1(\tilde{X}') \otimes_{\mathbb{Z}^m} \mathbb{Q}(\mathbb{Z}^m)$$

$$\Rightarrow \quad r_0(X) = r_0(X').$$

Goal: Show $H_i(W,X)=0 \Rightarrow H_i(\tilde{W},\tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m)=0$

1) Since $\mathbb{Z}[\mathbb{Z}^m]=\mathbb{Z}[x_1^{\pm 1},\dots,x_m^{\pm 1}]$ is Noetherian and W,X are finite CW-complexes, $H_i(\tilde{W},\tilde{X})$ is a <u>finitely presented</u> $\mathbb{Z}[\mathbb{Z}^m]$-module.

Let M_i be the presentation matrix. (assume square).

2) M_i has entries in $\mathbb{Z}[x_1^{\pm 1},\dots x_m^{\pm 1}]$. Augment M_i by sending $x_j \longmapsto 1$ to get $\varepsilon(M_i)=M_i \otimes \mathbb{Z}$ which presents $H_i(W,X)$.

3) Since $H_i(W,X)=0$, $\det(\varepsilon(M_i))=\pm 1$

$$\Rightarrow (\det M_i)(1,\dots,1)=\pm 1$$

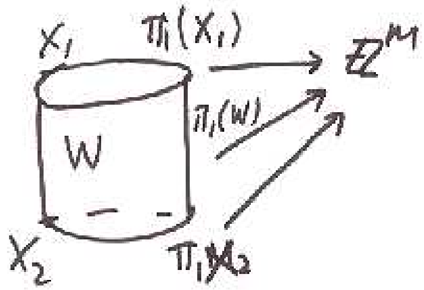
↑Laurent polynomial in $x_1,\dots,x_m$

4) Since $(\det M_i)(1,\dots,1)\neq 0$, $\det M_i \neq 0$

$$\Rightarrow H_i(\tilde{W},\tilde{X}) \otimes_{\mathbb{Z}^m} Q(\mathbb{Z}^m)=0 \quad \text{(rank is 0).}$$

<u>Note:</u> The two key points to this proof are

(1) the maps $\pi_1(X_i) \longrightarrow \mathbb{Z}^m$ extend over the 4-manifold W



(2) $\mathbb{Z}[\mathbb{Z}^m]$ embeds in a quotient field so that we can take a rank.

$\therefore$ To define homology cobordism invariants we look for groups Γ_i w/ $\mathbb{Z}\Gamma_i \hookrightarrow$ skew field and s.t. $\pi_1(X_i) \longrightarrow \Gamma_i$ extend over W.

Homology Equivalence and Fundamental Gp.

A homology cobordism gives maps

$$i_j : M_j \longrightarrow W$$

which induce $\cong$ on H_* (ie. i_j is homology equivalence)

Q. If $f : X \longrightarrow Y$ is a homology equivalence what is preserved under

$$f_* : \pi_1(X) \longrightarrow \pi_1(Y) \; ?$$

Example 1:

$$f_* : \underset{\substack{\| \\ H_1(X)}}{\frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}} \xrightarrow{\cong} \underset{\substack{\| \\ H_1(Y)}}{\frac{\pi_1(Y)}{[\pi_1(Y), \pi_1(Y)]}}$$

Example 2: Let $G = \pi_1(X)$, $E = \pi_1(Y)$ then Stallings shows that for each $n \geq 0$,

$$f_* : G/G_n \xrightarrow{\cong} E/E_n$$

$\{G_n\}$ = lower central series of G:

$$G_1 = G \quad , \quad G_n = [G_{n-1}, G].$$

7

Theorem (Stallings): Let $\phi: G \to E$ be a homomorphism of groups that induces $\cong$ on H_1 and an epimorphism on H_2. Then for any finite n, ϕ induces iso $\phi_*: \dfrac{G}{G_n} \xrightarrow{\cong} \dfrac{E}{E_n}$.

Recall by $H_*(G)$ we mean $H_*(K(G,1))$

Q. What about the derived series?

Recall, derived series:

$$G^{(0)} = G$$
$$G^{(n+1)} = [G^{(n)}, G^{(n)}]$$

A. $G/G^{(n)}$ is not necessarily preserved under homology equivalence!

<u>Example</u>: Let K be a knot in S^3, with $\Delta_K \neq 1$, $G = \pi_1(S^3 - K)$

$$\phi : G \twoheadrightarrow \mathbb{Z} \quad \text{abelianization}$$

(1) ϕ_* $\cong$ on homology

(2) [Cochran] $G/G^{(n)}$ is "large" $\left(G^{(n)}/G^{(n+1)} \neq I \right)$

(3) $\mathbb{Z}/\mathbb{Z}^{(n)} = \mathbb{Z} \quad \forall\, n \geq 1.$

$$\therefore \quad \phi_* : \frac{G}{G^{(n)}} \twoheadrightarrow \mathbb{Z}/\mathbb{Z}^{(n)} = \mathbb{Z} \quad \text{not mono!}$$

Also,

$$\text{meridian} : \mathbb{Z} \longrightarrow G$$

$$\Rightarrow (\text{meridian})_* : \mathbb{Z} \hookrightarrow G/G^{(n)} \quad \text{not surjective!}$$

However, we have the following:

<u>Theorem</u> (Cochran-H)

If $\phi: F \longrightarrow B$ induces a monomorphism on $H_1(-;\mathbb{Q})$ and an epimorphism on $H_2(-;\mathbb{Q})$ (2-connected or rationally 2-connected), F is a free group, B is finitely related then for all $n \geq 1$,

$$\phi_*: F/F^{(n)} \hookrightarrow B/B^{(n)}$$

is a monomorphism.

<u>Note</u>: F does <u>not</u> have to be finitely generated

One application of this theorem is that the higher-order <u>ranks</u> of a slice link are "maximal".

Let $G = \pi_1(X^3)$, $\Gamma_n = G/G_r^{(n+1)}$ where $G_r^{(n+1)}$ is the $(n+1)^{st}$ term of the (rational) derived series and $\mathcal{K}_n$ = quotient field of $\mathbb{Z}\Gamma_n$.
Recall, an element of $\mathcal{K}_n$ is ab^{-1} where $a, b \in \mathbb{Z}\Gamma_n$, $b \neq 0$.

<u>Def</u>: We define the <u>n^{th}-order rank</u> <u>r_n</u> $\in \mathbb{Z}$ to be

$$\boxed{r_n(x) = \operatorname{rank}_{\mathcal{K}_n} H_1(x; \mathcal{K}_n)}$$

<u>Note:</u> $r_n(x) = \operatorname{rank}_{\mathcal{K}_n} H_1(X_n) \otimes_{\mathbb{Z}\Gamma_n} \mathcal{K}_n$

$\parallel$

covering space of X with $\pi_1(X_n) = G_r^{(n+1)}$

Properties of r_n

1) Let $\psi \in H^1(X)$ and $\mathbb{K}_n^\psi[t^{\pm 1}]$ as before then

$$H_1(X;\mathbb{K}_n^\psi[t^{\pm1}]) \cong \left(\mathbb{K}_n^\psi[t^{\pm1}]\right)^{r_n(X)} \oplus \frac{\mathbb{K}_n^\psi[t^{\pm1}]}{\langle p_1(t)\rangle} \oplus \cdots \oplus \frac{\mathbb{K}_n^\psi[t^{\pm1}]}{\langle p_m(t)\rangle}$$

(dim of free summand is independent of ψ)

2) If K is a knot then $r_n(S^3-K)=0$.
More generally, $r_n(X) \le \beta_1(X) - 1$ (Cochran-Orr-Teichner)

3) If $X = X_1 \# X_2$ with $\beta_1(X_i) \ge 1$ $(i=1,2)$
then $r_n(X) = r_n(X_1) + r_n(X_2) + 1$
(i.e. $r_n - 1$ is additive under connected sum)

4) The r_n are non-~~decreasing~~ increasing functions of n:

Theorem (H): For all $n \ge 0$,

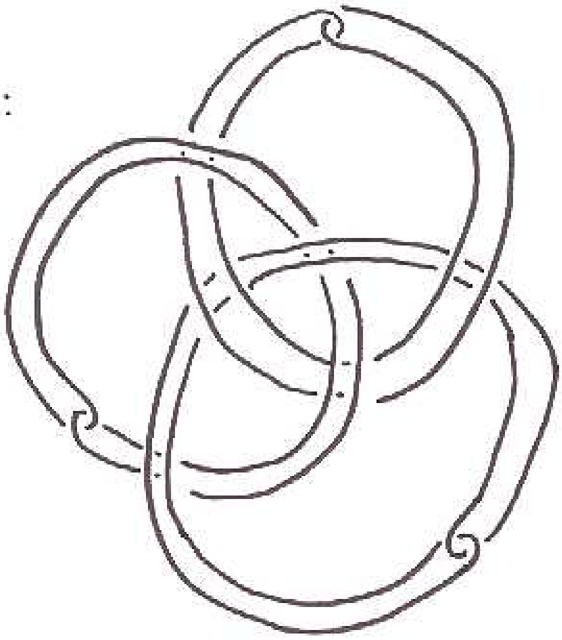
$$\boxed{r_{n+1}(X) \le r_n(X)}.$$

5) A <u>good</u> <u>boundary</u> <u>link</u> L has

$$r_n(S^3 - L) = m - 1$$

where m is the number of components of L.

<u>Example</u>:



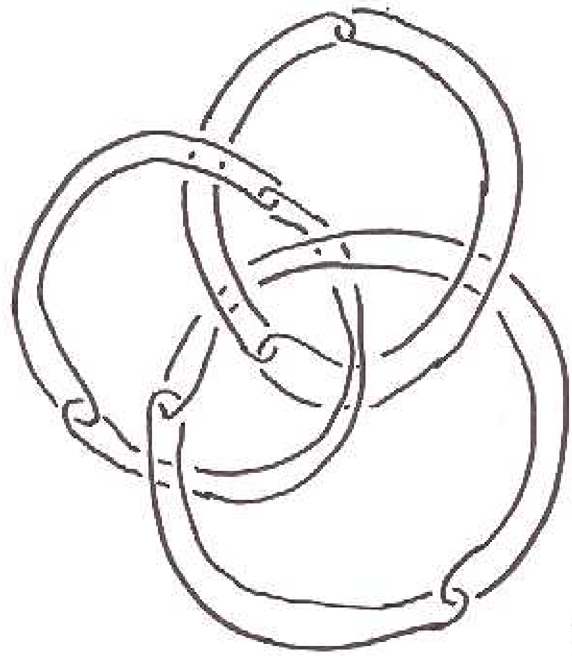
L = Whitehead double of Borromean rings

$r_n(S^3 - L) = 2$

Good boundary link $L \Leftrightarrow$

$H_1(\text{Free cover of } S^3 - L) = 0$

Example of link without maximal ranks.



L = Bing double of Borromean rings

$$r_0(S^3-L) = 5 = r_0(S^3 - \text{trivial link})$$

$$r_1(S^3-L) < 5 = r_1(S^3 - \text{trivial link})$$

Theorem (H) If a link L in S^3 is slice (bounds embedded disjoint disks in B^4) then for all $n \geq 0$,

$$r_n(S^3-L) = \beta_1(S^3-L) - 1 = r_n(S^3 - \underset{\text{link}}{\text{trivial}})$$

with $\beta_1(S^3-L)$ comps.

$\Rightarrow$ Bing double of Borromean rings is not slice.

:3

Recall

Theorem (Cochran-H): If $\phi: F \longrightarrow B$ induces a mono. on $H_1(-;\mathbb{Q})$ and an epi. on $H_2(-;\mathbb{Q})$ with F a free group, B finitely related then $\forall n \geq 1$

$$\phi_* : F/F^{(n)} \hookrightarrow B/B^{(n)}.$$

is a monomorphism.

In fact, if we slightly "vary the derived series" we can prove a more general version of this theorem.

To do this, we will need to define the torsion-~~der~~ free derived series of a group: $G_H^{(n)}$

Theorem (Cochran-H):

If $\phi : A \longrightarrow B$ is a monomorphism on $H_1(-;\mathbb{Q})$ and an epimorphism on $H_2(-;\mathbb{Q})$ with A finitely generated and B finitely related then for each $n \geq 1$,

$$\phi_* : \frac{A}{A_H^{(n)}} \hookrightarrow \frac{B}{B_H^{(n)}}$$

is a monomorphism.

If ϕ is onto as well then ϕ_* is an isomorphism.

Torsion-Free Derived Series: $G_H^{(n)}$

(1) $G_H^{(0)} := G$

(2) Assume (i) $G_H^{(n)}$ has been defined,
(ii) $G_H^{(n)} \triangleleft G$ (iii) $\mathbb{Z}[G/G_H^{(n)}]$ is an Ore domain

$\Rightarrow$ By (2) $\mathbb{Z}[G/G_H^{(n)}] \hookrightarrow \mathcal{K}_n =$ classical right ring of quotients

Consider:

$$G_H^{(n)} \xrightarrow{\alpha_n} \underbrace{\frac{G_H^{(n)}}{[G_H^{(n)}, G_H^{(n)}]}}_{\substack{\text{right } \mathbb{Z}[G/G_H^{(n)}] \text{ module by} \\ \sigma g = g^{-1}\sigma g \text{ for } g \in G.}} \xrightarrow{B_n} \frac{G_H^{(n)}}{[G_H^{(n)}, G_H^{(n)}]} \otimes_{\mathbb{Z}G/G_H^{(n)}} \mathcal{K}_n$$

(3) $G_H^{(n+1)} := \ker(B_n \circ \alpha_n) = \alpha_n^{-1}\left(\text{Torsion submodule of } G_H^{(n)}/[G_H^{(n)}, G_H^{(n)}]\right)$

$\Rightarrow$ Immediate that $G_H^{(n+1)} \triangleleft G_H^{(n)}$.

Can show $G_H^{(n+1)} \triangleleft G$ and $\mathbb{Z}[G/G_H^{(n+1)}]$ ORE ring.

★ $G_H^{(n)}/G_H^{(n+1)} = H_1(G_H^{(n)};\mathbb{Z}) / \mathbb{Z}\frac{G}{G_H^{(n)}}\text{-torsion}.$

Torsion-Free Derived Series via Group Theory

Let G be a group.

Define $G_H^{(0)} := G$

$$G_H^{(n+1)} = \text{set of } g \in G_H^{(n)} \text{ such that}$$

$$\exists \ (0 \neq) \sum_{i=0}^{\ell} k_i \gamma_i \in \mathbb{Z}[G/G_H^{(n)}]$$

and

$$\prod_{i=0}^{\ell} \gamma_i^{-1} g^{k_i} \gamma_i \in [G_H^{(n)}, G_H^{(n)}].$$

<u>Note:</u> $G^{(n)} \subset G_H^{(n)}$ for all $n \geq 0$

* $\{G_H^{(n)}\}$ is a characteristic but <u>not</u> <u>totally</u> <u>invariant</u> series of G!

However, we have the following:

<u>Proposition</u> (Cochran-H): If $\phi : A \longrightarrow B$ induces a monomorphism on

$$\frac{A}{A_H^{(n)}} \hookrightarrow \frac{B}{B_H^{(n)}} \quad \text{then}$$

$$\phi(A_H^{(n+1)}) \subset B_H^{(n+1)}.$$

In particular, we have homomorphism

$$\phi_* : \frac{A}{A_H^{(n+1)}} \longrightarrow \frac{B}{B_H^{(n+1)}}.$$

Examples:

(1) F free group

Since $F^{(n)}/F^{(n+1)}$ is torsion-free as $\mathbb{Z}[F/F^{(n)}]$-module,

$$\boxed{F_H^{(n)} = F^{(n)}} \quad \forall\, n \geq 0.$$

(2) K knot in S^3, $G = \pi_1(S^3 - K)$

Since $G^{(1)}/G^{(2)}$ = Alex. module is a $\mathbb{Z}[G/G^{(1)}]$-torsion module, $G_H^{(n)} = [G, G]$ $\forall\, n \geq 1$. Hence

$$\boxed{G/G_H^{(n)} \cong \mathbb{Z}} \quad \forall\, n \geq 1.$$

Theorem (Cochran-H):

If $\phi: A \longrightarrow B$ is a monomorphism on $H_1(-;\mathbb{Q})$ and an epimorphism on $H_2(-;\mathbb{Q})$ with A finitely generated and B finitely related then for each $n \geq 1$,

$$\phi_*: \frac{A}{A_H^{(n)}} \hookrightarrow \frac{B}{B_H^{(n)}}$$

is a monomorphism. If ϕ is onto as well, then ϕ_* is an isomorphism.

Corollary: If L is a boundary link then $G \twoheadrightarrow F$ induces

$$G/G_H^{(n)} \xrightarrow{\cong} F/F^{(n)}$$

for all $n \geq 0$.

Def: Let $\mathcal{K}(G/G_H^{(n+1)})$ be the quotient field of $\mathbb{Z}G/G_H^{(n+1)}$. Define

$$R_n(X) = \operatorname{rank}_{\mathcal{K}(G/G_H^{(n+1)})} H_1(X; \mathcal{K}(G/G_H^{(n+1)}))$$

where $G = \pi_1(X)$.

Corollary: R_n is a concordance invariant of links (and an invariant of homology cobordism).

Note: $R_n(X) \neq r_n(X)$ (in general).

Sketch of proof of Theorem:

Theorem: If $\phi: A \longrightarrow B$ is a monomorphism on $H_1(-;\mathbb{Q})$ and an epimorphism on $H_2(-;\mathbb{Q})$, A finitely generated, B finitely related then $\forall\, n \geq 1$,

$$\phi_*: A/A_H^{(n)} \hookrightarrow B/B_H^{(n)}$$

is a monomorphism

By induction and the following diagram, it suffices to show $\dfrac{A_H^{(n)}}{A_H^{(n+1)}} \overset{\phi_*}{\hookrightarrow} \dfrac{B_H^{(n)}}{B_H^{(n+1)}}$.

$$\begin{array}{ccccccccc}
1 \to & A_H^{(n)}/A_H^{(n+1)} & \longrightarrow & A/A_H^{(n+1)} & \longrightarrow & A/A_H^{(n)} & \longrightarrow & 1 \\
 & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} & & \downarrow & & \\
1 \to & B_H^{(n)}/B_H^{(n+1)} & \longrightarrow & B/B_H^{(n+1)} & \longrightarrow & B/B_H^{(n)} & \longrightarrow & 1
\end{array}$$

suppose $A_H^{(n)}/A_H^{(n+1)} \longrightarrow B_H^{(n)}/B_H^{(n+1)}$ is <u>not</u> a monomorphism.

Then $\exists\ [a]$ s.t.

(1) $a \in (A_H^{(n)})_{ab}$ is not $\mathbb{Z}[A/A_H^{(n)}]$-torsion

(2) $[\phi(a)]$ is $\mathbb{Z}B/B_H^{(n)}$-torsion

$\in (B_H^{(n)})_{\text{abelianization}}$.

i.e.

$$\frac{A_H^{(n)}}{[A_H^{(n)}, A_H^{(n)}]} \otimes_{A/A_H^{(n)}} K(A/A_H^{(n)}) \xrightarrow{\ \alpha\ } \frac{B_H^{(n)}}{[B_H^{(n)}, B_H^{(n)}]} \otimes_{B/B_H^{(n)}} K(B/B_H^{(n)})$$

is <u>not</u> injective.

But α is the composition:

$$\begin{tikzcd}
\dfrac{A_H^{(n)}}{[A_H^{(n)}, A_H^{(n)}]} \otimes_{A/A_H^{(n)}} K(A/A_H^{(n)}) \arrow[r] & \dfrac{A_H^{(n)}}{[A_H^{(n)}, A_H^{(n)}]} \otimes_{A/A_H^{(n)}} K(B/B_H^{(n)}) \arrow[d, "\circledast"] \\
 & \dfrac{B_H^{(n)}}{[B_H^{(n)}, B_H^{(n)}]} \otimes_{B/B_H^{(n)}} K(B/B_H^{(n)})
\end{tikzcd}$$

injective since by induction $A/A_H^{(n)} \hookrightarrow B/B_H^{(n)}$

$\Rightarrow \mathbb{Z}B/B_H^{(n)}$ is a free $\mathbb{Z}A/A_H^{(n)}$-module

⊛ is equivalent to

$$\begin{tikzcd}
H_1(A;\mathcal{K}(B/B_H^{(n)})) \arrow[r] \arrow[d, equal] & H_1(B;\mathcal{K}(B/B_H^{(n)})) \arrow[d, equal] \\
H_1(A_H^{(n)})\otimes_{A/A_H^{(n)}}\mathcal{K}(B/B_H^{(n)}) & H_1(B_H^{(n)})\otimes_{B/B_H^{(n)}}\mathcal{K}(B/B_H^{(n)})
\end{tikzcd}$$

which a monomorphism by the following.

<u>Proposition</u> (H-Cochran): If $\Phi: A \longrightarrow B$ induces a mono. on $H_1(-;\mathbb{Q})$ and an epi. on $H_2(-;\mathbb{Q})$ (A finitely generated, B finitely related) then for any coefficient system $B \longrightarrow \Gamma$, Γ PTFA,

$$H_1(A;\mathcal{K}(\Gamma)) \hookrightarrow H_1(B;\mathcal{K}(\Gamma)).$$

<u>Note</u>: This proposition is a generalization of the statement we proved in the beginning: for $\Gamma = \mathbb{Z}^m$, $H_1(\tilde{X})\otimes \mathbb{Q}(\mathbb{Z}^m) \xrightarrow{\cong} H_1(\tilde{W})\otimes \mathbb{Q}(\mathbb{Z}^m)$ if $i: X \to W$ induces an $\cong$ on H_1, and H_2, $\mathbb{Z}^m \cong H_1(X)/\text{torsion}$