Noncommutative Knot Theory

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<u>Reference</u>: "Noncommutative Knot Theory"
Alg. + Geom. Topology
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Objects of study:

1. 3-dim. manifolds

   a. Knot or link in S^3

exterior
$X = S^3 - K$

   b. general compact 3-manifold X
   with $\beta_1(X) \geq 1$

2. 4-dimensional manifolds $\beta_1 \geq 1$

3. Homeomorphisms of surfaces
   $f: X \longrightarrow X$

4. Non-abelian groups whose abelianization $G/[G,G]$ is infinite

How noncommutative rings arise:

1. Homology of covering spaces with Deck translations G are modules over $\mathbb{Z}G$.

2. If $f: G \to G$ is a group isomorphism and $H \lhd G$ is characteristic then

$$H/[H,H] \xrightarrow{f} H/[H,H]$$

is an isomorphism of $\mathbb{Z}[G/H]$-modules if f induces identity on $G/H \to G/H$

Commutative Knot Theory

Alexander module $A_0(K)$

" polynomial $\Delta_0(K)$

Blanchfield linking form on $A_0(K)$

Signatures σ_θ $\|\theta\|=1$

Arf Invariant $\in \mathbb{Z}_2$

Seifert Matrix (S-equiv. class)

genus of K

unknotting #

crossing #

bridge #

Alexander Module via covering spaces.

Let $X = S^3 \setminus K$ $\quad G = \pi_1(S^3 \setminus K)$

$$\begin{tikzcd}
\tilde{X}_{\mathbb{Z}} \arrow[d, "p"] \\
X
\end{tikzcd}$$

$\pi_1 \cong G^{(1)} \equiv [G,G]$

(integral) $A_0(K) \equiv H_1(\tilde{X}_{\mathbb{Z}})$ as $\mathbb{Z}[t^{\pm 1}]$ module

(rational) $A_0(K) \equiv H_1(\tilde{X}_{\mathbb{Z}}; \mathbb{Q})$ as $\mathbb{Q}[t, t^{-1}]$ module

Alexander Module - via group theory

$$A_0(K) = \frac{[G,G]}{[[G,G],[G,G]]} = \frac{G^{(1)}}{G^{(2)}}$$

as a $\underbrace{\mathbb{Z}\left[\underset{\uparrow\ \text{abelian group}}{G/G^{(1)}}\right]}_{\text{commutative ring}}$ -module

Derived Series of G: $G^{(1)} \equiv [G,G]$

$$G^{(n+1)} \equiv [G^{(n)}, G^{(n)}]$$

Properties of classical Alexander Module

1. it is a TORSION module
2. DUALITY: ∃ Blanchfield Linking FORM

$$Bl_0 : \mathcal{A}_0 \times \mathcal{A}_0 \longrightarrow Q(t) \mod \mathbb{Z}[t,t^{-1}]$$

non-singular for "rational" Alexander mod.

3. after localizing to $\mathbb{Q}[t,t^{-1}]$, canonical decomposition

$$\mathcal{A}_0 \cong \oplus \frac{\mathbb{Q}[t,t^{-1}]}{\langle p_i(t) \rangle}$$

Alexander polynomial $\equiv \Delta_0 \equiv \prod_i p_i(t)$

4. can be computed from loops on a Seifert surface
5. can be computed from presentation of π_1
6. Has special behavior for fibered Knots, can be used to estimate genus (K)

Higher-Order Alexander Modules

$X = S^3 \setminus K$ $\quad \widetilde{X}_\Gamma$ = regular cover with covering group Γ

$$\begin{tikzcd}\widetilde{X}_\Gamma \arrow[d, "p"] \\ X\end{tikzcd}$$

$H_1(\widetilde{X}_\Gamma)$ is a $\mathbb{Z}\Gamma$-module

called a higher-order Alexander module.

$(\mathbb{Z}\Gamma = \{\sum n_i \gamma_i \mid \gamma_i \in \Gamma \; n_i \in \mathbb{Z}\})$

Problem: If Γ is not abelian then $\mathbb{Z}\Gamma$ is a noncommutative ring.

Need special techniques to study modules over noncommutative rings.

Difficult to get numerical invariants.

Focus on a family of modules associated to the derived series of $G = \pi_1(X)$

$$\begin{tikzcd}
\widetilde{X} \arrow[d, phantom, "\vdots"] & \pi_1 = G^{(n+1)} & H_1 \cong G^{(n+1)}/G^{(n+2)} \equiv \mathcal{A}_n & \\
{} \arrow[d] & & & \\
\widetilde{X} \arrow[dd, "\mathbb{Z}"] & \pi_1 = G^{(1)} & H_1 \cong G^{(1)}/[\ ,\ ] \cong & G^{(1)}/G^{(2)} \arrow[d, phantom, sloped, "\equiv"] \\
& & & \mathcal{A}_0 \\
X & \pi_1 = G & H_1 = G/[G,G] \cong \mathbb{Z} &
\end{tikzcd}$$

$$\mathcal{A}_n(K) \equiv H_1\left(\widetilde{X}_{G/G^{(n+1)}}\right) \cong G^{(n+1)}/G^{(n+2)}$$

n^{th} higher-order Alexander module, module over $\mathbb{Z}[G/G^{(n+1)}]$

<u>Main Theorem</u>: The higher-order Alexander Modules satisfy most of the properties of the classical Alexander module and can be used to

- distinguish Knots with same classical Alexander module
- new obstructions to fibered Knot
- new lower bound for genus(K)
- new obstructions to amphicheiral

The first higher-order Alexander polynomial can be algorithmically be calculated!

Thm: If K is a non-trivial knot then

$$\delta_0(K) \le \delta_1 + 1 \le \delta_2 + 1 \le \dots \le 2\,\text{genus}(K)$$

↖degree of higher-order

↑

degree of classical Alexander polynomials

and there exist knots where ~~these are~~

$$\delta_0 < \delta_1 + 1 < \delta_2 + 1 \dots$$

Other RESULTS :

- higher-order Blanchfield form

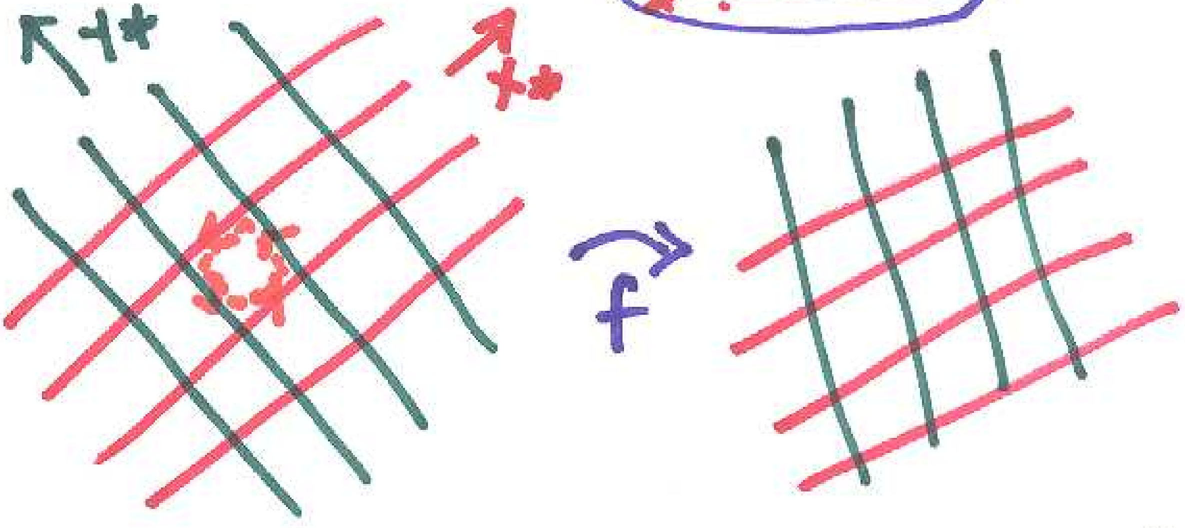
$$\mathcal{B}\ell_n : \mathcal{A}_n \times \mathcal{A}_n \longrightarrow \mathcal{K}_{\Gamma_n} \bmod \mathbb{K}_n[t^{\pm}]$$

C. Leidy : stronger than $\mathcal{A}_n$ alone

- higher-order Arf invariants
- " " signatures
- If K does not have classical Alexander polynomial 1 then $\mathcal{A}_n \neq 0$ for any n (all higher-order modules non-trivial)

Example: Trefoil Knot

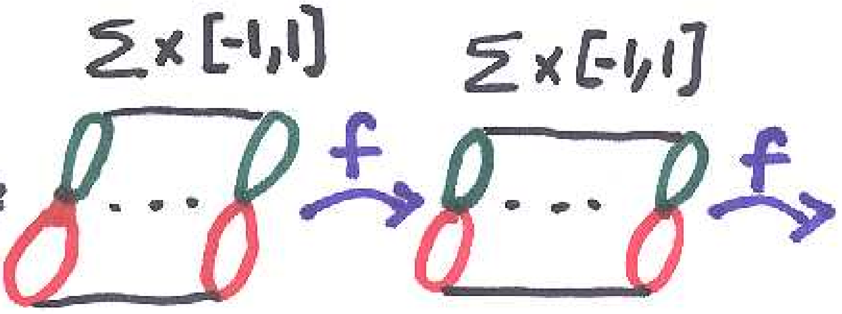
$\text{Lattice} \times \mathbb{R} \cong$



LATTICE × [-1,1] Lattice × [-1,1]

$\Big\downarrow$ $\mathbb{Z}\times\mathbb{Z}$ -cover

$\Sigma \times \mathbb{R} \cong$



$H_1 \cong \underset{x}{\mathbb{Z}} \times \underset{y}{\mathbb{Z}}$

$\Big\downarrow$ ∞-cyclic cover

$S^3 \setminus K$

$\mathcal{A}_1 \equiv H_1$ of this covering space

$\cong \infty$-gen. free abelian gp.

action of deck translations? This should be a module over $\mathbb{Z}[G/G''] = \mathbb{Z}\Gamma$

$$\begin{tikzcd}
1 \arrow[r] & G'/G'' \arrow[r] \arrow[d, phantom, sloped, "\cong"] & G/G'' \arrow[r, two heads] & G/G' \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 1 \\
& \mathbb{Z}*\mathbb{Z} = \{x_*, y\} & & \mathbb{Z} = \langle t \rangle &
\end{tikzcd}$$

Forgetting t action, $\mathcal{A}_1$ is a a free module since x_*, y_* introduce no relations. So $\mathcal{A}_1$ is a cyclic $\mathbb{Z}\Gamma$-module with one generator C and one relation

$$t_* C = f_* C$$

$$C = xyx^{-1}y^{-1}$$

but C is the longitude and we can assume $f_*(C) = C$ so

$$\mathcal{A}_1 \cong \frac{\mathbb{Z}\Gamma}{(t-1)\mathbb{Z}\Gamma}$$

Note: $\exists$ t-1 torsion unlike classical module

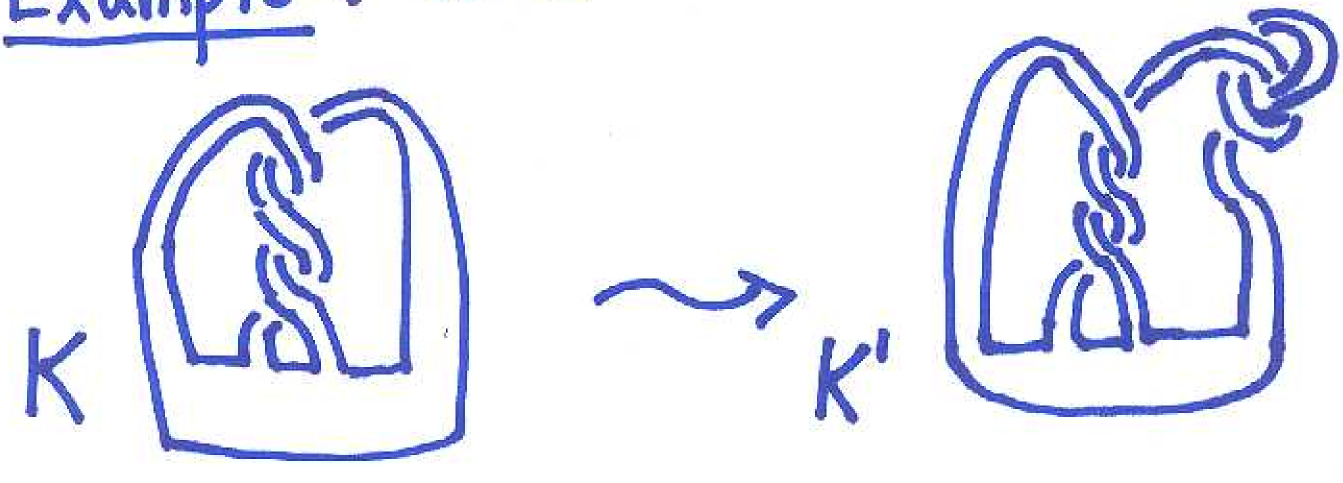
In general, higher-order modules of fibered knots reflect action of monodromy f_* on

$$\frac{G^{(1)}}{G^{(n+2)}} \xrightarrow{f_*} \frac{G^{(1)}}{G^{(n+2)}}$$

How to construct knots with same $\mathcal{A}_0, \dots, \mathcal{A}_{n-1}$ but different $\mathcal{A}_n$:

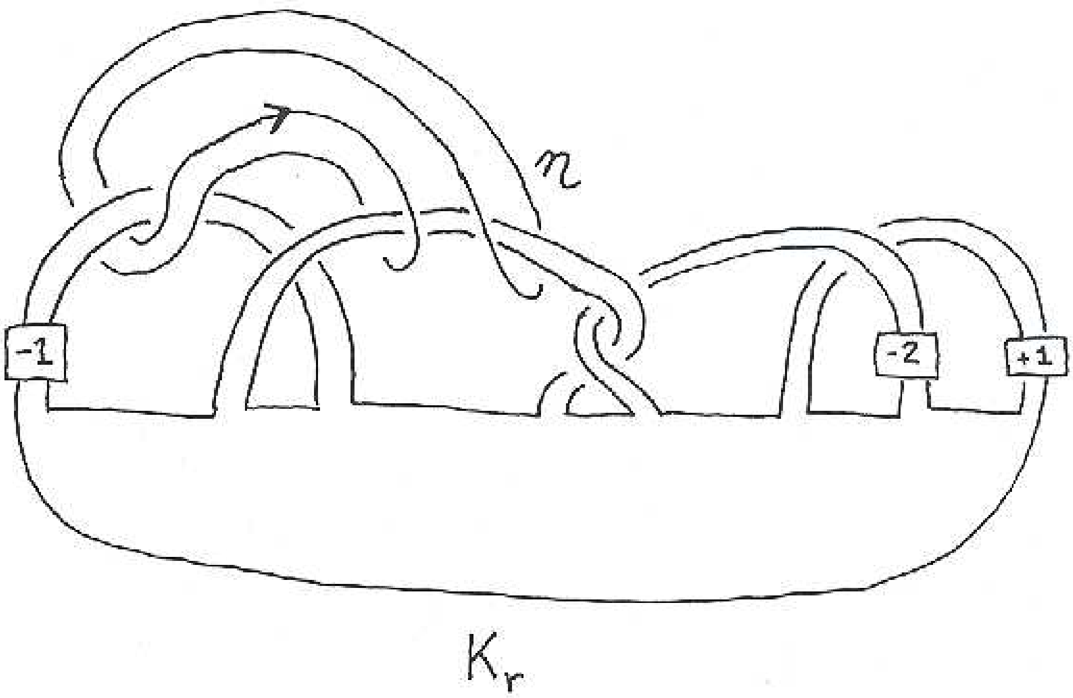
1. Find $\eta \in G^{(n)}$ $\quad G = \pi_1(S^3 \setminus K)$
2. Represent η by unknotted circle

Example : $n=1$



Have same classical Alexander module $\mathcal{A}_0$ but different $\mathcal{A}_1$

Example: $n=2$



Infection on this η does not change $\mathcal{A}_0, \mathcal{A}_1$ but changes $\mathcal{A}_2$

Algebraic Tools for studying modules over $\mathbb{Z}\Gamma$ where Γ not abelian:

It would help if:

- $\mathbb{Z}\Gamma$ is integral domain (no 0-divisors)
- $\mathbb{Z}\Gamma$ is an **Ore domain**

Theorem (Ore): If R is an Ore Domain then R embeds in a (noncommutative) field of fractions $\mathcal{K}_R$, in which every element is of the form rs^{-1}, $r, s \in R$, $s \neq 0$.

These are useful technical tools

Def: A domain R is a right Ore domain if for all r, s in R $\exists\, r', s'$

$$rr' = ss'$$

Example of usefullness:

Prop: If M is a right R module where R is a right Ore Domain then the torsion elements of M are a submodule.

$m \in M$ torsion if $m \cdot r = 0$ some $r \neq 0$

Proof: Suppose m_1, m_2 are torsion elements, so [redacted] $m_1 \cdot r = 0$ $m_2 \cdot s = 0$. Is $m_1 + m_2$ torsion element? Note:

$$(m_1+m_2)rs = m_1 rs + m_2 rs = 0 + \overset{\text{No!}}{m_2 \cdot rs}$$

But using Ore condition above

$$(m_1+m_2)rr' = m_1 rr' + m_2 rr' = 0 + m_2 ss'$$

$$= (m_2 s)s' = 0$$

In our case $\Gamma_n = G/G^{(n+1)}$ is a solvable group and $G^{(i)}/G^{(i+1)}$ is known to be a $\mathbb{Z}$-torsion free abelian group so Γ is a poly-torsion-free-abelian group (PTFA) meaning it is iterated extensions by torsion-free abelian groups.

Thm (Lewin) If Γ is a torsion-free solvable group then $\mathbb{Z}\Gamma$ is an Ore Domain.

so

$$\mathbb{Z}\Gamma \hookrightarrow \mathcal{K}_\Gamma$$

'' skew field

- any $\mathcal{K}$-module is a $\mathcal{K}$-vector space so has a rank r '' $\mathcal{K}^r$
- $\mathcal{K}_\Gamma$ is a flat $\mathbb{Z}\Gamma$-module so

$$H_*(\quad ; \mathcal{K}_\Gamma) \cong H_*(\quad ; \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathcal{K}_\Gamma$$

Even more useful:

$$\mathbb{Z}\Gamma_n \subseteq \begin{matrix}\text{principal ideal}\\ \text{domain}\end{matrix} \subseteq \mathcal{K}_{\Gamma_n}$$

Eg. n=1
$\Gamma_1 \cong \mathbb{Z}$

$$\mathbb{Z}[t,t^{-1}] \subseteq \mathbb{Q}[t,t^{-1}] \subseteq \mathbb{Q}(t)$$

$$\begin{tikzcd}\mathbb{Z}[t,t^{-1}] \arrow[r, bend right, "\text{Localization}"'] & \mathbb{Q}[t,t^{-1}]\end{tikzcd}$$

In fact:

$$\mathbb{Z}\Gamma_n \subseteq \mathbb{K}_n[t^{\pm 1}] \subseteq \mathcal{K}_{\Gamma_n}$$

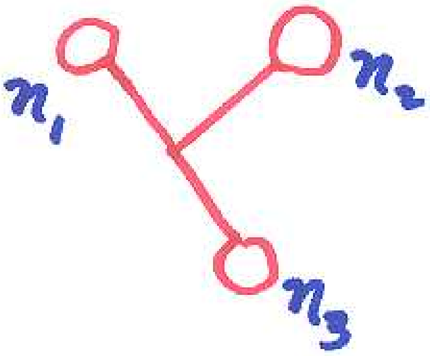
" twisted Laurent polynomial ring

Define: n^{th} localized Alexander module is $\mathcal{A}_n \otimes_{\mathbb{Z}\Gamma_n} \mathbb{K}_n[t^{\pm 1}]$

- decomposes $\bigoplus_i \dfrac{\mathbb{K}_n[t^{\pm 1}]}{\langle p_i(t)\rangle}$
- higher-order Alexander polyn $= \prod p_i(t)$
- Let $\delta_n =$ degree of higher-order Alexander polynomial.

OPEN PROBLEMS

1. characterize the modules and polynomials that can arise as higher-order Alexander
2. Find "higher-order Seifert form"
3. Find numerical invariants besides the degrees δ_n
4. Find geometric moves that preserve $\mathcal{A}_n$, $\mathcal{B}\ell_n$ example: clasper surgery



$\eta_i \in G^{(n+1)}$

generalizes Doubled Delta Move

5. Write a computer program to calculate δ_n.
6. How to compute higher-order signatures
7. Are there noncommutative elementary ideals?
8. Discover more about Higher-order Arf Invariants.

Construction of PID:

$$\mathbb{Z}\Gamma \subseteq \mathbb{K}[t^{\pm 1}] \subseteq \mathcal{K}_\Gamma$$

Step 1:

$$\begin{tikzcd}
1 \arrow[r] & \Gamma' \arrow[r] \arrow[d, equal] & \Gamma \arrow[r, "\pi"] & \Gamma/\Gamma' \arrow[l, bend left, dotted, "s"] \arrow[r, phantom, "\cong"] & \mathbb{Z} \arrow[r] & 1 \\
& {[\Gamma,\Gamma]} & & & \langle \bar{t} \rangle &
\end{tikzcd}$$

∴ any elt. of Γ can be written ct^m
$c \in \Gamma'$, $\pi(t) = \bar{t}$

∴ any elt. of $\mathbb{Z}\Gamma$ can be written as
$\sum_i c_i t^{m_i}$ $\quad c_i \in \mathbb{Z}\Gamma'$

∴ $\mathbb{Z}\Gamma$ is a [illegible] twisted polynomial ring $(\mathbb{Z}\Gamma')[t,t^{-1}]$ with coefficients in $\mathbb{Z}\Gamma'$

called twisted since $ct\,bt \neq cbt^2$
since $tb \neq bt$

Step 2: Let $\mathbb{K}$ be quotient field of $\mathbb{Z}\Gamma'$

$$\mathbb{Z}\Gamma \cong (\mathbb{Z}\Gamma')[t,t^{-1}] \subseteq \mathbb{K}[t,t^{-1}] \subseteq \mathcal{K}_\Gamma$$

EASY TO SEE THIS IS ↑ P.I.D. using "degree"

Higher-order bordism invariants generalizing Arf invariant:

$M =$ 0-surgery on K $\quad G = \pi_1(M)$

consider $\left(M^3, \phi: G \twoheadrightarrow G/G^{(n+1)} \equiv \Gamma_n\right)$

in $\Omega_3^{\text{Spin}}(K(\Gamma_n, 1))$ [redacted]

Arf invariant is case $n = 0$, $\Gamma_0 \cong \mathbb{Z}$

$\Omega_3^{\text{Spin}}(S^1) \cong \mathbb{Z}_2$

Higher-order signatures:

Cheeger-Gromov von Neumann ρ-invt.

associated to any $(M^3, \phi: \pi_1(M^3) \to \Gamma)$

$$\rho_n(K) \equiv \rho\left(M; \phi: G \twoheadrightarrow \Gamma_n\right) \in \mathbb{R}$$

(discuss more in later talks)